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DISCRETE PROBABILITY MODELS USING TAYLOR SERIES

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A b s t r a c t. *We briefly consider basic properties of power series distribution and derive generating functions, Laplace transforms and combinations of such distributions. In the main section of the paper, we also study compound processes for sums and for extremes. We then provide new Panjer type of relationships and study the asymptotic behaviour of the compound processes. This analysis shows that these distributions may be used for modeling discrete data with light as well as heavy tails.*

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1. *Introduction and definition*

In the literature, for a long time, there has been attention for discrete probability distributions that are linked to power series. Among others we mention the works of Joshi (1975), Kosambi (1949) or Noack (1950). Patil (1962) and Kemp (1968, 1970) made great advances in the theory of generalized power series. A lot of information can be found in Johnson et al. (1992).

In this paper, we assume that the Taylor series expansion:

$$f(a+\lambda) = f(a) + f'(a)\frac{\lambda}{1!} + \cdots + f^{(k)}(a)\frac{\lambda^k}{k!} + \cdots = \sum_{n=0}^{\infty} f^{(n)}(a)\frac{\lambda^n}{n!}. \quad (1.1)$$

converges for values $\lambda \in A \subseteq (-\infty, \infty)$. If $f^{(n)}(a) \geq 0,\ \forall\, n \geq 0$ and $f(a+\lambda) > 0$, then, we have

$$\frac{1}{f(a+\lambda)} \sum_{n=0}^{\infty} f^{(n)}(a)\frac{\lambda^n}{n!} = 1. \quad (1.2)$$

Without loss of generality, we assume throughout the paper that $a = 0$ and then (1.2) leads to the following definition.

1.1. Definition

The probability distribution (p.d.) $(p_n(\lambda; f), n \geq 0, \lambda \in A)$ generated by f as above is given by

$$p_n(\lambda; f) = \frac{f^{(n)}(0)}{n!}\frac{\lambda^n}{f(\lambda)} = \Theta_n(\lambda; f)\lambda^n, \quad n \geq 0. \quad (1.3)$$

In what follows, we let $X(\lambda; f)$ denote a random variable (r.v.) with probability density function (p.d.f.)

$$P(X(\lambda; f) = n) = p_n(\lambda; f), \qquad n \geq 0.$$

When it is clear from the context, we will write $p_n(\lambda; f) = p_n(\lambda)$ and $X(\lambda; f) = X(\lambda)$.

Probability distributions of the type (1.3) are called power series distributions. They were studied, among others, by Joshi (1934), Noack (1950), Johnson et al. (1992), Momeni (2011).

1.2. Remarks

1) Taking logarithms in (1.3) we find that

$$\log p_n(\lambda; f) = \log f^{(n)}(0) - \log n! - \log f(\lambda) + n \log \lambda.$$

Taking the derivative with respect to (w.r.t.) λ, we find that

$$\frac{\mathrm{d}p_n(\lambda; f)}{\mathrm{d}\lambda} = \left(\frac{n}{\lambda} - \frac{f^{(1)}(\lambda)}{f(\lambda)}\right) p_n(\lambda; f).$$

2) Using (1.3) we easily see that

$$p_{n+1}(\lambda; f) = \frac{\lambda}{n+1} \frac{f^{(n+1)}(0)}{f^{(n)}(0)} p_n(\lambda; f).$$

3) We can easily construct a family of probability distributions. Using (1.1) for $f^{(k)}(\cdot)$, $k = 1, 2, \ldots$, we have

$$f^{(k)}(\lambda) = \sum_{n=0}^{\infty} f^{(n+k)}(0) \frac{\lambda^n}{n!}, \quad \lambda \in A.$$

If for each $n \geq 0$, $f^{(n+k)}(0) \geq 0$ and $f^{(k)}(\lambda) > 0$, then it follows that (with obvious notation)

$$p_n(\lambda; f^{(k)}) = \frac{f^{(n+k)}(0)}{f^{(k)}(\lambda)} \frac{\lambda^n}{n!}, \quad n \geq 0. \tag{1.4}$$

From (1.4) if follows that

$$p_n(\lambda; f^{(k)}) = \frac{n+1}{\lambda} \frac{f^{(k-1)}(\lambda)}{f^{(k)}(\lambda)} p_{n+1}(\lambda, f^{(k-1)}), n \geq 0,$$

or equivalently that

$$p_{n+1}(\lambda, f^{(k-1)}) = \frac{\lambda}{n+1} \frac{f^{(k)}(\lambda)}{f^{(k-1)}(\lambda)} p_n(\lambda; f^{(k)}).$$

4) Modified power series distributions are obtained by studying power series of the form $f(\lambda) = \sum_{n=0}^{\infty} a_n (u(\lambda))^n$ for suitable functions $u(\lambda)$. The p.d. generated by f is given by

$$P(X = n) = \frac{a_n (u(\lambda))^n}{f(\lambda)}.$$

1.3. Examples

1) For $f(x) = \mathrm{e}^{\alpha x}$, we have

$$f^{(m)}(x) = \alpha^m f(x), \quad f^m(x) = f(mx), \quad m \geq 1,$$

and $f(\lambda) = \sum_{n=0}^{\infty} (\alpha\lambda)^n / n!$. Using (1.3) we find that

$$p_n(\lambda; f) = \mathrm{e}^{-\lambda} \frac{(\alpha\lambda)^n}{n!} = \Theta_n \lambda^n, \quad n \geq 0,$$

and $X(\lambda; f)$ has a Poisson distribution with parameter $\alpha\lambda$. Note that we have $\Theta_{n+1}/\Theta_n \to 0$ as $n \to \infty$. We also have

$$X(\lambda; f^{(m)}) \triangleq X(\lambda; f) \quad \text{and} \quad X(\lambda; f^m) \triangleq X(m\lambda; f).$$

2) Assume that $f(x) = (1-x)^{-\alpha},\ \alpha > 0$. For $|\lambda| < 1$ we have

$$f(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \frac{\lambda^n}{n!},$$

and then we find

$$p_n(\lambda; f) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}(1-\lambda)^\alpha \lambda^n = \Theta_n \lambda^n, \quad n \geq 0.$$

In this example, we have

$$\frac{\Theta_{n+1}}{\Theta_n} = \frac{\alpha+n}{n+1} \to 1,$$

and

$$n\left(\frac{\Theta_{n+1}}{\Theta_n} - 1\right) \to \alpha - 1.$$

It follows that (Θ_n) is a regularly varying sequence or short $(\Theta_n) \in RV(\alpha-1)$. For properties of regularly varying sequences, we refer to Bojanic and Seneta (1973) of Bingham et al. (1987).

Clearly $p_n(\lambda; f)$, $p_n(\lambda; f^m)$ and $p_n(\lambda; f^{(m)})$ are all of the same negative binomial type.

3) For $f(x) = -\log(1-x)$ we have

$$f(x) = \sum_{n=1}^{\infty} n^{-1} x^n, \quad |x| < 1,$$

and the p.d. generated by f is given by:

$$p_n(\lambda; f) = \frac{1}{(-\log(1-\lambda))n} \lambda^n = \Theta_n \lambda^n, \quad n \geq 1, \quad 0 < \lambda < 1.$$

Here Θ_n satisfies

$$n\left(\frac{\Theta_{n+1}}{\Theta_n} - 1\right) \to -1$$

so that $(\Theta_n) \in RV(-1)$.

Since $f^{(1)}(x) = (1-x)^{-1}$, we have

$$f^{(1)}(x) = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

and we find

$$p_n(\lambda; f^{(1)}) = \frac{n+1}{\lambda}\frac{f(\lambda)}{f^{(1)}(\lambda)}p_{n+1}(\lambda; f) = (1-\lambda)\lambda^{n-1}, \quad n \geq 1.$$

This corresponds to a geometric distribution. Using the $k-$th derivative, we find that $f^{(k)}(x) = (k-1)!(1-x)^{-k}$ and then $p_n(\lambda, f^{(k)})$ yields a negative binomial distribution as in Example 2.

4) For $f(x) = -(1-x)^{-1}\log(1-x)$, we have $f(x) = \sum_{n=1}^{\infty} H(n)x^n$, $|x| < 1$, where $H(n) = \sum_{k=1}^{n} 1/k$. In this example, we find

$$p_n(\lambda; f) = \frac{1}{f(\lambda)}H(n)\lambda^n, \quad n \geq 1,\ 0 < \lambda < 1,$$

and here $(\Theta_n) = (H(n)/f(\lambda)) \in RV(0)$.

5) The Gauss hypergeometric function F for $|x| < 1$ is defined by

$$F(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},$$

where a, b, c are real numbers and $c \neq 0$, $c \neq -1$, $c \neq -2, \ldots$. We write

$$p_n(\lambda; F) = \frac{1}{F(a, b, c; \lambda)}\frac{(a)_n(b)_n}{(c)_n}\frac{\lambda^n}{n!} = \Theta_n\lambda^n.$$

We have

$$\frac{\Theta_{n+1}}{\Theta_n} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \to 1,$$

which shows that the series above converges for $|x| < 1$. We also have that

$$\begin{aligned} n\left(\frac{\Theta_{n+1}}{\Theta_n} - 1\right) &= n\frac{(a+n)(b+n) - (c+n)(n+1)}{(c+n)(n+1)} \\ &= \frac{n}{n+1}\frac{ab + (a+b)n - (c+1)n - c}{c+n} \ \to\ a+b-c-1, \end{aligned}$$

and this shows that $(\Theta_n) \in RV(a+b-c-1)$.

6) The generalized hypergeometric function is defined by its power series as follows:

$${}_pF_q(\overline{a}, \overline{b}; x) = {}_pF_q\left(\begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix}\ ; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n}\frac{x^n}{n!}.$$

Here a_i, b_j are real numbers with $b_j \neq 0,\ -1,\ -2,\ \ldots$.

Also, if $a_i > 0$, $1 \le i \le p$, $b_j > 0$, $1 \le j \le q$, then ${}_pF_q(\overline{a}, \overline{b}; x) > 0$ and we find that for $0 < \lambda < 1$,

$$p_n(\overline{a}, \overline{b}; \lambda) = \frac{1}{{}_pF_q(\overline{a}, \overline{b}; \lambda)} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{\lambda^n}{n!} = \Theta_n \lambda^n, \quad n \ge 0,$$

is a p.d. Note that

$$\frac{\Theta_{n+1}}{\Theta_n} = \frac{n(a_1+n)\cdots(a_p+n)}{(n+1)(b_1+n)\cdots(b_q+n)} \sim n^{p-q-1}.$$

If $p \le q+1$, then the series converges for $|x| < 1$.

If $p < q+1$, then the series converges for all values of x.

If $p = q+1$, then $\Theta_{n+1}/\Theta_n \to 1$ and

$$\begin{aligned} n\left(\frac{\Theta_{n+1}}{\Theta_n} - 1\right) &= n\left(\frac{(a_1+n)\cdots(a_p+n)}{(n+1)(b_1+n)\cdots(b_q+n)} - 1\right) \\ &= \frac{n}{n+1}\left(\frac{(a_1+n)\cdots(a_p+n) - (n+1)(b_1+n)\cdots(b_q+n)}{(b_1+n)\cdots(b_q+n)}\right) \\ &\to (a_1 + \cdots + a_p - b_1 - \cdots - b_q - 1), \end{aligned}$$

so that $(\Theta_n) \in RV(a_1 + \cdots + a_p - b_1 - \cdots - b_q - 1)$.

2. *Some basic properties*

The distribution function (d.f.) $F_{X(\lambda;f)}(x)$ of $X(\lambda; f)$ is given by

$$F_{X(\lambda;f)}(x) = \sum_{n=0}^{x} p_n(\lambda; f) = \frac{1}{f(\lambda)} \sum_{n=0}^{x} f^{(n)}(0) \frac{\lambda^n}{n!},$$

and its survival function $S_{X(\lambda:f)}$ is given by

$$S_{X(\lambda;f)}(x) = P(X(\lambda; f) \ge x) = \frac{1}{f(\lambda)} \sum_{n=x}^{\infty} f^{(n)}(0) \frac{\lambda^n}{n!}.$$

The hazard function $h_{X(\lambda;f)}(n)$ is given by

$$h_{X(\lambda;f)}(n) = \frac{p_n(\lambda; f)}{S_{X(\lambda;f)}(n)} = f^{(n)}(0) \frac{\lambda^n}{n!} \cdot \frac{1}{\sum\limits_{k=n}^{\infty} f^{(k)}(0) \frac{\lambda^k}{k!}}.$$

2.1. The generating function

Using (1.1) and (1.3), the moment generating function (MGF) of $X(a, \lambda; f)$ is given by:

$$\varphi_{X(\lambda;f)}(z) = Ez^{X(\lambda;f)} = \frac{f(z\lambda)}{f(\lambda)}, \quad \lambda \in A, \ \ z\lambda \in A. \tag{2.1}$$

Using (2.1) we obtain the following property.

Proposition 2.1. *Suppose that $X(\lambda; f)$ and $X(\lambda; g)$ are independent r.v.s.*

(a) *Then $X(\lambda; fg) \triangleq X(\lambda, f) + X(\lambda; g)$.*

(b) *If $Y_1, Y_2, \ldots, Y_m$ are i.i.d. with $Y_i \triangleq X(\lambda; f)$, then*

$$Y_1 + Y_2 + \cdots + Y_m \triangleq X(\lambda; f^m).$$

PROOF. (a) For independent r.v.s $X(\lambda; f)$ and $X(\lambda; g)$ we find

$$\varphi_{X(\lambda;f)+X(\lambda;g)}(z) = \varphi_{X(\lambda;f)}(z) \times \varphi_{X(\lambda;g)}(z) = \frac{f(z\lambda)g(z\lambda)}{f(\lambda)g(\lambda)}.$$

and conclude that $X(\lambda, f) + X(\lambda; g) \triangleq X(\lambda; fg)$.

(b) If we take independent and identically distributed (i.i.d.) r.v.s $Y_1, Y_2, \ldots, Y_m$ where $Y_i \triangleq X(\lambda; f)$, then part a) shows that $Y_1 + Y_2 + \cdots + Y_m \triangleq X(\lambda; f^m)$.

Examples. 1) For $f(x) = \exp(\alpha x)$, we have $X(\lambda; f) \sim$POISSON($\alpha\lambda$).
We also have $X(\lambda; f^m) \sim$POISSON($m\alpha\lambda$). Using Prop 1(b),we find back the well known property that $X_1(\lambda; f) + \cdots + X_m(\lambda; f) \sim$ POISSON($m\lambda$). This shows that the Poisson distribution is infinitely divisible, cf. Fisz (1962), Feller (1971), Steutel (1979).

2) Suppose that $f(x) = (1 - x)^{-\alpha}$, $\alpha > 0$. In this case we have

$$p_n(\lambda; f) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)\Gamma(n + 1)}(1 - \lambda)^{\alpha}\lambda^n, \quad n \geq 0.$$

If we take i.i.d. r.v.s $Y_1, Y_2, \ldots, Y_m$ where $Y_i \triangleq X(\lambda; f)$, then $Y_1+Y_2+\cdots+Y_m \triangleq X(\lambda; f^m)$. Among others this shows that the sum of negative binomial distributed r.v.s again has a negative binomial distribution.

3) Assume that $f(x) = (1 + x)^m$. We have $f(\lambda) = \sum_{n=0}^{m} \binom{m}{n}\lambda^n$.and

$$p_n(\lambda; f) = \binom{m}{n}\frac{\lambda^n}{(1 + \lambda)^m}, 0 \leq n \leq m.$$

We find that $X(a, \lambda) \sim BIN(m, \lambda/(1+\lambda))$. Note that we have $f^k(x) = (1+x)^{km}$, and we find back the well known property that

$$X_1(\lambda; f) + \cdots + X_n(\lambda; f) \sim BIN\left(km, \frac{\lambda}{1 + \lambda}\right).$$

2.2. Moments

2.2.1. First approach. Following Kosambi (1949), Noack (1950), we start from (5) and take derivatives w.r.t. z. We find

$$E(X(\lambda;f)z^{X(\lambda;f)-1}) = \lambda\frac{f^{(1)}(z\lambda)}{f(\lambda)},$$

and for $m \geq 1$,

$$E([X(\lambda;f)]_m\, z^{X(\lambda;f)-m}) = \lambda^m \frac{f^{(m)}(z\lambda)}{f(\lambda)},$$

where $[x]_m = x(x-1)\cdots(x-m+1)$. For $m = 2$, we have

$$E(X(\lambda;f)(X(\lambda;f)-1)z^{X(\lambda;f)-2}) = \lambda^2\frac{f^{(2)}(z\lambda)}{f(\lambda)}.$$

Taking $z = 1$, we find the following

Proposition 2.2. (2) (a) *For $m \geq 1$ we have*

$$E([X(\lambda;f)]_m) = \lambda^m f^{(m)}(\lambda)/f(\lambda).$$

(b) *We have $EX(\lambda;f) = \lambda f^{(1)}(\lambda)/f(\lambda)$ and*

$$VarX(\lambda;f) = \lambda^2\frac{f^{(2)}(\lambda)}{f(\lambda)} + EX(\lambda;f) - E^2X(\lambda;f).$$

Remarks. (1) Clearly we have $VarX(\lambda;f) = EX(\lambda;f)$ if and only if

$$\lambda^2\frac{f^{(2)}(\lambda)}{f(\lambda)} = E^2X(\lambda;f) = \lambda^2\left(\frac{f^{(1)}(\lambda)}{f(\lambda)}\right)^2,$$

and this holds if and only if

$$\frac{f^{(2)}(\lambda)}{f^{(1)}(\lambda)} = \frac{f^{(1)}(\lambda)}{f(\lambda)}.$$

It is easy to see that $f(\cdot)$ has to be an exponential function.

(2) Patil (1962) studies the properties of a generalized power series distributions. Among others he studied the case where the generating function is of the form $f(\lambda) = \exp(a + b\lambda)$ or $f(\lambda) = \exp(P(\lambda) + a + b\lambda)$, where a and b are constants and $P(\cdot)$, along with its derivative, is a positive monotone-increasing function of λ.

(3) Also Gupta (1974) and Khatri (1959) studied the properties of the moments that occur in modified power series distributions.

2.2.2. Second approach. Alternatively, by definition, we have

$$EX^r(\lambda; f) = \sum_{n=0}^{\infty} n^r \frac{f^{(n)}(0)}{n!} \frac{\lambda^n}{f(\lambda)}.$$

Taking derivatives w.r.t. λ, we find

$$\begin{aligned}
\lambda \frac{\mathrm{d}EX^r(\lambda; f)}{\mathrm{d}\lambda} &= \sum_{n=0}^{\infty} n^r \frac{f^{(n)}(0)}{n!} \frac{n\lambda^n}{f(\lambda)} - \sum_{n=0}^{\infty} n^r \frac{f^{(n)}(0)}{n!} \frac{\lambda^{n+1}}{f^2(\lambda)} f^{(1)}(\lambda) \\
&= EX^{r+1}(\lambda; f) - \lambda \frac{f^{(1)}(\lambda)}{f(\lambda)} \times EX^r(\lambda; f) \\
&= EX^{r+1}(\lambda; f) - EX(\lambda; f) \times EX^r(\lambda; f).
\end{aligned}$$

Hence we obtain the following recursion:

Proposition 2.3. *For $r \geq 1$ we have*

$$\begin{aligned}
EX^{r+1}(\lambda; f) &= EX(\lambda; f) \times EX^r(\lambda; f) + \lambda \frac{\mathrm{d}EX^r(\lambda; f)}{\mathrm{d}\lambda}, \\
Var(X(\lambda; f) &= \lambda \frac{\mathrm{d}EX(\lambda; f)}{\mathrm{d}\lambda}.
\end{aligned}$$

2.3. Combining power series distributions

Starting from two functions f and g we can find r.v. $X(\lambda; f)$ and $X(\theta; g)$. We can define a modified power series distribution as follows. For $0 \leq \beta \leq 1$ we define the r.v. X by its p.d.:

$$P(X = n) = \beta P(X(\lambda; f) = n) + (1 - \beta) P(X(\theta; g) = n), \quad n \geq 0.$$

The generating function of X is given by

$$\phi_X(z) = \beta \phi_{X(\lambda; f)}(z) + (1 - \beta) \phi_{X(\theta; g)}(z) = \beta \frac{f(\lambda z)}{f(\lambda)} + (1 - \beta) \frac{g(\theta z)}{g(\theta)}.$$

If $f(x) = 1$, we find that

$$P(X = n) = (1 - \beta) P(X(\theta; g) = n), \quad n \geq 1,$$

$$P(X = 0) = \beta + (1 - \beta) P(X(\theta; g) = 0).$$

It means that we pay extra attention to the value $X = 0$. This type of p.d. is often referred as a zero-inflated p.d. cf. Kolev at al. (2000), Minkova (2002a, 2002b).

If $f(x) = x^k$ and $\lambda = 1$, for some positive number k, we find

$$P(X = n) = (1 - \beta)P(X(\theta; g) = n), \quad n \neq k,$$
$$P(X = k) = \beta + (1 - \beta)P(X(\theta; g) = k),$$

and we call the result a non-zero inflated modified power series distribution. For different families of g, this type of pd was studied by Murat and Szynal (1998) and pay special attention to estimating the parameters in these models.

A discrete Lindley p.d. was introduced by Gomez and Calderin-Ojeda (2011) and reinvented by Bakouch et al. (2014). It has the following form:

$$\begin{aligned} P(X = n) &= \frac{(\theta(1 - 2p) + (1 - p)}{1 + \theta} p^n + \frac{\theta(1 - p)}{1 + \theta} np^n \\ &=: Ap^n + Bnp^n. \end{aligned}$$

It is easy to find the generating function:

$$\begin{aligned} \phi_X(z) &= A\frac{1}{1 - zp} + B\frac{zp}{(1 - zp)^2} \\ &= (A - B)\frac{1}{1 - zp} + B\frac{1}{(1 - zp)^2}. \end{aligned}$$

Now let U and V denote i.i.d. r.v.s with $P(U = n) = (1 - p)p^n$, $n \geq 0$. Clearly we have

$$\begin{aligned} \phi_X(z) &= \frac{A - B}{1 - p}\phi_U(z) + \frac{B}{(1 - p)^2}\phi_{U+V}(z) \\ &= \beta\phi_U(z) + (1 - \beta)\phi_{U+V}(z), \end{aligned}$$

where

$$\beta = 1 - \frac{\theta}{(1 - p)(1 + \theta)}.$$

So we find that $P(X = n) = \beta P(U = n) + (1 - \beta)P(U + V = n)$. The authors also study how to estimate all the parameters in this model.

Remark 2.1. We may further generalize and study properties of p.d. of the form

$$P(X = n) = \beta P(X(\lambda; f) = n) + \delta P(X(\theta; g) = n) + (1 - \beta - \delta)P(X(\sigma; h) = n),$$

where $n \geq 0$ and β, δ, $1 - \beta - \delta \geq 0$.

2.4. The class $SHS(\lambda, \delta)$

The examples in the previous section 1.3 show that in many cases we have

$$\frac{\Theta_{n+1}}{\Theta_n} = \frac{f^{(n+1)}(0)}{(n+1)f^{(n)}(0)} \to 1,$$

and then we can take $0 < \lambda < 1$ in (1.1). Note that in this case we have

$$\lim_{n\to\infty} \frac{p_{n+1}(\lambda; f)}{p_n(\lambda; f)} = \lambda.$$

It follows that $\log p_{n+1}(a, \lambda; f) - \log p_n(a, \lambda; f) \to \log \lambda$. By Césaro-Stolz, we have

$$\lim_{n\to\infty} \frac{\log p_n(\lambda; f)}{n} = \log \lambda.$$

Further, it often happens that

$$n\left(\frac{\Theta_{n+1}}{\Theta_n} - 1\right) = n\left(\frac{f^{(n+1)}(0)}{(n+1)f^{(n)}(0)} - 1\right) \to \delta,$$

and, if this is the case, then $(\Theta_n) \in RV(\delta)$.

Probability measures of the form $(p_n(\lambda; f) = \Theta_n \lambda^n)$ with $0 < \lambda < 1$ and $(\Theta_n) \in RV(\delta)$ have been called "semi-heavy tailed sequences", cf Omey et al. (2018) and then we use the notation $(p_n(\lambda; f)) \in SHS(\lambda, \delta)$. In our paper Omey et al. (2018) we have proved that if $(p_n(\lambda; f)) \in SHS(\lambda, \delta)$, then

$$P(X(\lambda; f) \geq n) = \sum_{k=n}^{\infty} p_k(\lambda; f) \in SHS(\lambda, \delta)),$$

and

$$P(X(\lambda; f) \geq n) \sim \frac{1}{1-\lambda} p_n(\lambda; f).$$

It also follows that the hazard rate satisfies:

$$h(n) = \frac{p_n(\lambda; f)}{P(X(\lambda; f) \geq n)} \to 1 - \lambda.$$

Note that $\log p_n(\lambda; f) = n \log \lambda + \log \Theta_n$. If $(\Theta_n) \in RV(\delta)$, then $\log \Theta_n / \log n \to \delta$ and hence

$$\frac{\log p_n(\lambda; f) - n \log \lambda}{\log n} \to \delta.$$

We have the following result:

Proposition 2.4. *If* $(p_n(\lambda; f)) \in SHS(\lambda, \delta)$, $0 < \lambda < 1$, *then*

$$P(X(\lambda; f) \geq n) \sim \frac{1}{1-\lambda} p_n(\lambda; f),$$

and the hazard rate satisfies $h(n) \to 1 - \lambda$. *Moreover we have*

$$\lim_{n\to\infty} \frac{\log p_n(\lambda; f)}{n} = \lim_{n\to\infty} \frac{\log P(X(\lambda; f) \geq n)}{n} = \log \lambda,$$

and

$$\frac{\log p_n(\lambda; f) - n \log \lambda}{\log n} \to \delta,$$

and

$$\frac{\log P(X(\lambda; f) \geq n) - n \log \lambda}{\log n} \to \delta.$$

Now we consider independent r.v. $X(\lambda; f)$ and $X(\lambda; g)$. In this case, we find

$$P(X(\lambda; f) = n) = \Theta_n \lambda^n$$

$$P(X(\lambda; g) = n) = \Delta_n \lambda^n,$$

where

$$\Theta_n = \frac{f^{(n)}(0)}{f(\lambda) n!}; \quad \Delta_n = \frac{g^{(n)}(0)}{g(\lambda) n!}.$$

For the sum we find $P(X(\lambda; f) + X(\lambda; g) = n) = (\Theta * \Delta)_n \lambda^n$, where $(\Theta * \Delta)_n = \sum_{i=0}^{n} \Theta_i \Delta_{n-i}$. In the paper by Omey et al. (2018) we studied the asymptotic of this expression. Among others, Omey et al. (2018) proved the following result.

Proposition 2.5. *Suppose* $X(\lambda; f), X(\lambda; g)$ *as above. Assume that* $(\Theta_n) \in RV(\theta)$ *and* $(\Delta_n) \in RV(\delta)$. *If* $A := \sum_{n=0}^{\infty} \Theta_n < \infty$ *and* $B := \sum_{n=0}^{\infty} \Delta_n < \infty$, *then*

$$P(X(\lambda; f) + X(\lambda; g) = n)$$
$$= (A + o(1)) P(X(\lambda; g) = n) + (B + o(1)) P(X(\lambda; f) = n).$$

3. *Compound Processes*

3.1. Some notations

In what follows $Y, Y_1, Y_2, \ldots$ denote nonnegative i.i.d. r.v.s with d.f. $G(x)$ and Laplace transform $\phi(s) = E(\mathrm{e}^{-sY})$. If Y is discrete and takes values in $\mathbb{N}$, we let $q_n = P(Y = n)$ and denote the generating function of Y by $Q(z) = E(z^Y)$. We will consider partial sums defined as $S(0) = 0$ and $S(n) = \sum_{i=1}^{n} Y_i$, $n \geq 1$, and the extreme values $m(n) = \min(Y_1, \dots, Y_n)$, $M(n) = \max(Y_1, \dots, Y_n)$.

Let $X(\lambda; f)$ denote an r.v. as before and assume that all variables involved are independent.

3.2. Random sums

3.2.1. Compounding. We consider the partial sums $S(n)$, and we replace the fixed index n by the discrete random index $X(\lambda; f)$. We define $S(X(\lambda; f))$ as follows:

$$\begin{aligned} S(X(\lambda; f)) &= 0 \quad \text{if} \quad X(\lambda; f) = 0; \\ S(X(\lambda; f)) &= \sum_{i=1}^{n} Y_i \quad \text{if} \quad X(\lambda; f) = n \geq 1. \end{aligned}$$

For the d.f. of $S(X(\lambda; f))$ we find:

$$\begin{aligned} F_{S(X(\lambda;f))}(x) := &\ P(S(X(\lambda; f)) \leq x) \\ = &\sum_{n=0}^{\infty} P(S(n) \leq x) P(X(\lambda; f) = n) \\ = &\sum_{n=0}^{\infty} G^{*n}(x) p_n(\lambda; f), \end{aligned}$$

where $G^{*0}(x) = \delta_0(x)$ and $G^{*n}(x)$ is the $n-$fold Lebesgue-Stieltjes convolution of G with itself. As a consequence, for the Laplace transform $\psi(s) = E(e^{-sS(X(\lambda;f))})$ we find

$$\begin{aligned} \psi(s) &= \sum_{n=0}^{\infty} E(e^{-sS(n)}) P(X(\lambda; f) = n) \\ &= \sum_{n=0}^{\infty} \phi^n(s) p_n(\lambda; f) \\ &= \varphi_{X(\lambda;f)}(\phi(s)) \\ &= \frac{f(\lambda\phi(s))}{f(\lambda)}. \end{aligned}$$

If Y is discrete then we find that

$$P(S(X(\lambda; f)) = m) = \sum_{n=0}^{\infty} q^{*n}(m) p_n(\lambda; f),$$

and the GF is given by

$$\psi(z) = \varphi_{S(X(\lambda;f))}(z) = E(z^{S(X(\lambda;f))}) = \frac{f(\lambda Q(z))}{f(\lambda)}.$$

Using the notations as above, we conclude.

Proposition 3.1. *The Laplace transform/GF of the compound process is given by*

$$\psi(s) = E(\mathrm{e}^{-sS(X(\lambda;f))}) = \frac{f(\lambda\phi(s))}{f(\lambda)},$$

resp.

$$\psi(z) = E(z^{S(X(\lambda;f))}) = \frac{f(\lambda Q(z))}{f(\lambda)}.$$

The distribution of $S(X(\lambda; f))$ is called a compound distribution with compounder $X(\lambda; f)$. Compound distributions appear in many papers related to finance and insurance. A special interest goes to compound Poisson distributions (where $f(x) = \exp(\alpha x)$). As examples we mention R. M. Adelson (1966), Afuecheta et al. (2020), Panjer (1981), Willmot and Lin (2001), Minkova and Balakrishnan (2013), Schmidli (1999), Momeni (2011), Minkova (2002). Stam (1973) used regular variation to study the tail of subordinated distributions. In Roozegar and Nadaraja (2017), the authors consider Y to have a normal distribution, and they consider different choices of $f(x)$.

Many authors have studied the asymptotic behaviour of $\overline{F}_{S(X(\lambda;f))}(x)$ as $x \to \infty$. Clearly we have

$$\overline{F}_{S(X(\lambda;f))}(x) = \sum_{n=1}^{\infty}(1 - G^{*n}(x))p_n(\lambda; f).$$

If $\overline{G}(x) \in RV(-\beta), \beta > 0$ and $\varphi_{X(\lambda;f)}(z) = f(\lambda z)/f(\lambda)$ is analytic at $z = 1$, one can prove that as $x \to \infty$ we have

$$\overline{F}_{S(X(\lambda;f))}(x) \sim \overline{G}(x)\sum_{n=1}^{\infty} np_n(\lambda; f) = \overline{G}(x)EX(\lambda; f).$$

For more results of this type see for example Foss et al. (2011), Stam (1973).

3.2.2. The p.d. $\psi_n = P(S(X(\lambda; f)) = n)$. It is often very hard to obtain explicit formulas for the p.d. $\psi_n = P(S(X(\lambda; f)) = n)$. As an alternative, one can try to find Panjer type of recursions or other types of recursive relationships. Our approach is as follows. Based on the derivative $f^{(1)}(\cdot)$ we can define the random sum $S(X(\lambda; f^{(1)}))$ as before. The GF of this new compound process is given by

$$\psi_1(z) = \frac{f^{(1)}(\lambda Q(z))}{f^{(1)}(\lambda)}.$$

On the other hand, we have $\psi(z) = f(\lambda Q(z))/f(\lambda)$. Taking the derivative w.r.t. z, we find

$$\psi^{(1)}(z) = \frac{f^{(1)}(\lambda Q(z))}{f(\lambda)} \lambda Q^{(1)}(z).$$

We have the following result.

Proposition 3.2. *We have*

$$\psi^{(1)}(z) = \lambda \frac{f^{(1)}(\lambda)}{f(\lambda)} \psi_1(z) Q^{(1)}(z).$$

This relation can be used to obtain more precise information about ψ_n.

Example 3.1. Take $f(x) = (1-x)^{-\alpha}$. In this case we have

$$\psi(z) = \frac{f(\lambda Q(z))}{f(\lambda)} = \frac{(1-\lambda)^{\alpha}}{(1-\lambda Q(z))^{\alpha}},$$

and

$$\psi_1(z) = \frac{f^{(1)}(\lambda Q(z))}{f(\lambda)} = \frac{(1-\lambda)^{1+\alpha}}{(1-\lambda Q(z))^{1+\alpha}} = \psi(z) \frac{1-\lambda}{1-\lambda Q(z)}.$$

Using the proposition, we see that

$$\psi^{(1)}(z) = \alpha \lambda \psi(z) \frac{1}{1-\lambda Q(z)} Q^{(1)}(z),$$

and it follows that

$$\psi^{(1)}(z) - \lambda Q(z) \psi^{(1)}(z) = \alpha \lambda \psi(z) Q^{(1)}(z).$$

Using ψ_n and q_n, we find that

$$n\psi_n - \lambda \sum_{k=0}^{n} k \psi_k q_{n-k} = \alpha \lambda \sum_{k=0}^{n} (n-k) q_{n-k} \psi_k.$$

It follows that

$$\begin{aligned} n\psi_n - \lambda n \psi_n q_0 &= \lambda \sum_{k=0}^{n-1} k \psi_k q_{n-k} + \alpha \lambda \sum_{k=0}^{n-1} (n-k) q_{n-k} \psi_k \\ &= \lambda \sum_{k=0}^{n-1} (k + \alpha(n-k)) \psi_k q_{n-k}, \end{aligned}$$

or

$$\psi_n = \frac{\lambda}{1-\lambda q_0} \sum_{k=0}^{n-1} \left(\alpha + \frac{(\alpha-1)k}{n} \right) \psi_k q_{n-k}.$$

Example 3.2. Take $f(x) = \exp x$. We have $\psi(z) = \exp[-\lambda(1 - Q(z))]$, and $\psi^{(1)}(z) = \lambda\psi(z)Q^{(1)}(z)$. From here we find that

$$n\psi_n = \lambda \sum_{k=0}^{n-1} (n-k) q_{n-k}\psi_k,$$

and

$$\psi_n = \lambda \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) q_{n-k}\psi_k.$$

3.2.3. The special case where $f(x) = \exp x$. In the special case where $f(x) = \exp x$, earlier we proved that $\psi^{(1)}(z) = \lambda\psi(z)Q^{(1)}(z)$. In this section we consider special choices for $Q(z)$ to obtain Panjer type of recursions. We extend the results of Minkova (2002). We assume that $Q(z)$ is of the following rational form:

$$Q(z) = C + \frac{A_0 + A_1 z}{B_0 + B_1 z + B_2 z^2}.$$

For the derivative we find

$$\begin{aligned} Q^{(1)}(z) &= \frac{(B_0 + B_1 z + B_2 z^2)A_1 - (A_0 + A_1 z)(B_1 + 2B_2 z)}{(B_0 + B_1 z + B_2 z^2)^2} \\ &= \frac{B_0 A_1 - A_0 B_1 - 2A_0 B_2 z - A_1 B_2 z^2}{B_0^2 + 2B_0 B_1 z + (B_1^2 + 2B_0 B_2)z^2 + 2B_1 B_2 z^3 + B_2^2 z^4} \\ &= \frac{\sum\limits_{i=0}^{2} \alpha_i z^i}{\sum\limits_{i=0}^{4} \beta_i z^i}. \end{aligned}$$

Using $\psi^{(1)}(z) = \lambda\psi(z)Q^{(1)}(z)$, it follows that

$$\psi^{(1)}(z) \sum_{i=0}^{4} \beta_i z^i = \lambda\psi(z) \sum_{i=0}^{2} \alpha_i z^i.$$

Hence

$$\sum_{i=0}^{\infty} n\psi_n \sum_{i=0}^{4} \beta_i z^{n+i-1} = \lambda \sum_{i=0}^{\infty} \psi_n \sum_{i=0}^{2} \alpha_i z^{n+i}.$$

Equating the coefficients of z^m, we find

$$\beta_0(m+1)\psi_{m+1} + \sum_{i=1}^{4} \beta_i(m+1-i)\psi_{m+1-i} = \lambda \sum_{i=0}^{2} \alpha_i \psi_{m-i}.$$

It follows that

$$\begin{aligned}\beta_0(m+1)\psi_{m+1} &= -\sum_{j=0}^{3}\beta_{j+1}(m-j)\psi_{m-j} + \lambda\sum_{i=0}^{2}\alpha_i\psi_{m-i}\\ &= \sum_{i=0}^{2}(\lambda\alpha_i - \beta_{i+1}(m-i))\psi_{m-i} - \beta_4(m-3)\psi_{m-3},\end{aligned}$$

and then also that

$$\begin{aligned}\psi_{m+1} &= \frac{1}{\beta_0}\left\{\sum_{i=0}^{2}\left(\frac{\lambda\alpha_i + i\beta_{i+1} - \beta_{i+1}m}{(m+1)}\right)\psi_{m-i} - \frac{\beta_4(m-3)}{(m+1)}\psi_{m-3}\right\}\\ &= \frac{1}{\beta_0}\left\{\sum_{i=0}^{2}\left(-\beta_{i+1} + \frac{\lambda\alpha_i + (i+1)\beta_{i+1}}{(m+1)}\right)\psi_{m-i}\right.\\ &\qquad\qquad\left. + \left(-\beta_4 + \frac{4\beta_4}{m+1}\right)\psi_{m-3}\right\}.\end{aligned}$$

3.2.4. Special cases. 1) Minkova (2002) and Momeni (2011) study examples of the geometric type where

$$Q(z) = \frac{(1-\rho)z}{1-\rho z}.$$

Clearly functions of this type can be rewritten as

$$Q(z) = C + \frac{A_0}{B_0 + B_1 z}.$$

We have

$$Q^{(1)}(z) = \frac{-A_0B_1}{B_0^2 + 2B_0B_1z + B_1^2z^2} = \frac{\alpha_0}{\sum_0^2 \beta_i z^i}.$$

Using $\psi^{(1)}(z) = \lambda\psi(z)Q^{(1)}(z)$ straightforward calculations show that

$$\psi_{n+1} = \frac{1}{\beta_0}\left(\left(-\beta_1 + \frac{\lambda\alpha_0 + \beta_1}{n+1}\right)\psi_n + \left(-\beta_2 + \frac{4\beta_2}{n+1}\right)\psi_{n-1}\right).$$

2) Consider the following modified power distribution with

$$Q(z) = \frac{(1-\alpha)z}{1-\alpha z} \times \frac{(1-\beta)z}{1-\beta z}.$$

Clearly we have

$$Q(z) = \frac{(1-\alpha)(1-\beta)}{\alpha\beta}\left(1 + \frac{-1+(\alpha+\beta)z}{1-(\alpha+\beta)z+\alpha\beta z^2}\right).$$

This is of the form that we studied above and we can apply the same formulas.

3) Earlier we discussed a discrete Lindley distribution with generating function

$$Q(z) = \beta\frac{1-p}{1-pz} + (1-\beta)\frac{(1-p)^2}{(1-pz)^2},$$

where

$$\beta = 1 - \frac{\theta}{(1-p)(1+\theta)}.$$

It is easy to see that $Q(z)$ can be rewritten as

$$\begin{aligned} Q(z) &= (1-p)\frac{(1-\beta)(1-p)+\beta-\beta pz}{1-2pz+p^2z^2} \\ &= \frac{A_0+A_1z}{B_0+B_1z+B_1z^2}, \end{aligned}$$

where

$$A_0 = (1-p)((1-\beta)(1-p)+\beta), \quad A_1 = -\beta p(1-p),$$

$$B_0 = 1, \quad B_1 = -2p, \quad B_2 = p^2.$$

Now the previous procedures lead to a Panjer type of recursion for ψ_n.

3.3. Extreme values

3.3.1. Univariate case. In this section, we consider random variables $X(\lambda; f)$ introduced before, i.e., the extreme values $m(n) = \min(Y_1, Y_2, \ldots, Y_n)$ and $M(n) = \max(Y_1, Y_2, \ldots, Y_n)$, $n \geq 1$, with $m(0) = M(0) = 0$.

Clearly for $x > 0$ we have $P(m(0) > x) = P(M(0) > x) = 0$, and for $n \geq 1$:

$$P(m(n) > x) = (1-G(x))^n = \overline{G}^n(x),$$

$$P(M(n) \leq x) = G^n(x).$$

We agree to set $G^0(x) = 1$ and $\overline{G}^0(x) = 0$ and $G(0) = 0$, $\overline{G}(0) = 1$.

Now we replace the index n by the random index $X(\lambda; f)$. For $0 < \overline{G}(x)$ we find

$$P(m(X(\lambda; f)) > x) = \sum_{n=0}^{\infty} \overline{G}^n(x) p_n(\lambda; f) = \varphi_{X(\lambda;f)}(\overline{G}(x))$$

$$= \frac{f(\lambda\overline{G}(x))}{f(\lambda)},$$

so that

$$F_m(x) = P(m(X(\lambda; f) \leq x) = 1 - \frac{f(\lambda\overline{G}(x))}{f(\lambda)}.$$

For $\overline{G}(x) = 0$ we have $P(m(X(\lambda; f) \leq x) = 1$.

For the maximum, in a similar way, we find: for $x \geq 0$,

$$\begin{aligned} F_M(x) &= P(M(X(\lambda; f) \leq x) = \sum_{n=0}^{\infty} p_n(\lambda; f) G^n(x) \\ &= \phi_{X(\lambda;f)}(G(x)) = \frac{f(\lambda G(x))}{f(\lambda)}. \end{aligned}$$

In the case of the maximum, we can calculate

– The survival function:

$$S_M(x) = 1 - \frac{f(\lambda G(x))}{f(\lambda)} = \frac{f(\lambda) - f(\lambda G(x))}{f(\lambda)}.$$

– The density (here we use $G'(x) = g(x)$):

$$f_M(x) = \lambda \frac{f^{(1)}(\lambda G(x))}{f(\lambda)} g(x).$$

– The hazard rate:

$$\begin{aligned} h_M(x) &= \lambda \frac{f^{(1)}(\lambda G(x))}{f(\lambda) - f(\lambda G(x))} g(x) \\ &= \lambda \frac{f^{(1)}(\lambda G(x))}{(f(\lambda) - f(\lambda G(x)))/(1 - G(x))} \frac{g(x)}{\overline{G}(x)} \\ &= \lambda \frac{f^{(1)}(\lambda G(x))}{(f(\lambda) - f(\lambda G(x)))/(1 - G(x))} h_Y(x). \end{aligned}$$

For special cases of G and f, we find back many specific examples that appear in the literature.

Examples. 1) Let $f(x) = \exp \alpha x,\ \alpha > 0$. We have

$$F_M(x) = P(M(X(\lambda; f) \leq x) = \exp[-\alpha\lambda\overline{G}(x)],$$

which is a compound Poisson distribution. Note that as $x \to \infty$, we have

$$\overline{F}_M(x) = 1 - \exp[-\alpha\lambda\overline{G}(x)] \sim \alpha\lambda\overline{G}(x).$$

We also have

– Density: $f_M(x) = F_M(x)\alpha\lambda g(x)$.

– Hazard rate:

$$h_M(x) = \frac{f_M(x)}{1 - F_M(x)} = \frac{F_M(x)}{1 - F_M(x)} \alpha\lambda g(x).$$

As $x \to \infty$, we have

$$h_M(x) \sim \frac{g(x)}{\overline{G}(x)} = h_Y(x).$$

2) Mahmoudi and Jafari (2012) study the case where $G(x)$ is a generalized exponential distribution:

$$G(x) = (1 - e^{-\beta x})^{\alpha}, \quad x > 0.$$

Note that we have $G(x) \sim (\beta x)^{\alpha}$ and $\log G(x) \sim \alpha \log x$ as $x \downarrow 0$. As $x \to \infty$, we have

$$\overline{G}(x) = 1 - (1 - \mathrm{e}^{-\beta x})^{\alpha} \sim \alpha \mathrm{e}^{-\beta x},$$

and $\log \overline{G}(x) \sim -\beta x$. For this function $G(x)$ we find

$$P(m(X(\lambda; f) > x) = \frac{f(\lambda(1 - (1 - e^{-\beta x})^{\alpha})))}{f(\lambda)},$$

and

$$P(M(X(\lambda; f) \leq x) = \frac{f(\lambda(1 - \mathrm{e}^{-\beta x})^{\alpha})}{f(\lambda)}.$$

This distribution is called a generalized exponential power series distribution. Note that the density is given by

$$f_{M(X(\lambda;f))}(x) = \lambda\alpha\beta \frac{f^{(1)}(\lambda(1 - \mathrm{e}^{-\beta x})^{\alpha})}{f(\lambda)} (1 - \mathrm{e}^{-\beta x})^{\alpha - 1} \mathrm{e}^{-\beta x}.$$

If $\alpha = 1$, we find

$$P(m(X(\lambda; f) \leq x) = 1 - \frac{f(\lambda \mathrm{e}^{-\beta x})}{f(\lambda)},$$

$$P(M(X(\lambda; f) \leq x) = \frac{f(\lambda(1 - \mathrm{e}^{-\beta x}))}{f(\lambda)},$$

which has been studied by Chakhandi and Ganjali (2009).

3) Rashid et al. (2017) and Warahena-Liyanage and Paravai (2015) consider Lindley distributions with density function

$$g(x) = \frac{\alpha^2}{1 + \alpha}(1 + x)\mathrm{e}^{-\alpha x}, \quad x > 0, \quad \alpha > 0,$$

and d.f.

$$G(x) = 1 - \left(1 + \frac{\alpha x}{1+\alpha}\right) \mathrm{e}^{-\alpha x}.$$

Using our notation, in the paper the authors study $P(M(X(\lambda; f) \leq x)$ and obtain various properties of this distribution. Note that in this case we have

$$\overline{G}(x) = \left(1 + \frac{\alpha x}{1+\alpha}\right) \mathrm{e}^{-\alpha x} \sim \frac{\alpha}{1+\alpha} x \mathrm{e}^{-\alpha x}, \quad \text{as} \quad x \to \infty,$$

and

$$G(x) \sim \frac{\alpha^2}{1+\alpha} x, \quad \text{as} \quad x \to 0.$$

4) Elbatal et al. (2017) studied an exponential Pareto power distributions where

$$G(x) = 1 - \exp(-\alpha x^{\lambda}), \quad x,\ \alpha,\ \lambda > 0,$$

and they study $M(X(\lambda; f))$ and $m(X(\lambda; f))$ for a variety of choices of f. Note that in this example we have

$$-\log \overline{G}(x) = \alpha x^{\lambda}.$$

As $x \downarrow 0$, we have $G(x) = 1 - \exp(-\alpha x^{\lambda}) \sim \alpha x^{\lambda}$, and $\log G(x) \sim \lambda \log x$.

3.3.2. On $P(m(X(\lambda; f)) > x)$. Consider the following relationship that has been proved before:

$$\overline{F}_m(x) = \frac{f(\lambda \overline{G}(x))}{f(\lambda)}.$$

Now we analyze the behavior of $\overline{F}_m(x)$ as $x \to \infty$. We have $\log f(\lambda)\overline{F}_m(x) = \log f(\lambda \overline{G}(x))$ and

$$\log f(\lambda)\overline{F}_m(x) = \frac{\log f(\lambda \overline{G}(x))}{r(\lambda \overline{G}(x))} r(\lambda \overline{G}(x))$$

If

$$\lim_{z \downarrow 0} \frac{\log f(z)}{r(z)} = \alpha,$$

we conclude that $\log f(\lambda)\overline{F}_m(x) \sim \alpha r(\lambda \overline{G}(x))$, as $x \uparrow \infty$.

The examples below show that $r(x) = x$ often does the job.

Examples. 1) If $f(x) = \exp(\alpha x)$ we have $\log f(x) = \alpha x$ and $\log f(\lambda)\overline{F}_m(x) = \alpha \lambda \overline{G}(x)$.

2) If $f(x)=(1-x)^{-\alpha}$, we have $\log f(x)=(-\alpha)\log(1-x)\sim \alpha x$ as $x\to 0$. Now we have

$$\log f(\lambda)\overline{F}_m(x)=\log f(\lambda\overline{G}(x))\sim \alpha\lambda\overline{G}(x).$$

3) If $f(x)=-(1-x)^{-1}\log(1-x)$, we have

$$\log f(x)=-\log(1-x)+\log(-\log(1-x)).$$

As $x\to 0$, we have $-\log(1-x)\sim x$ and $\log(-\log(1-x))/\log x\to 1$ and then $\log f(x)\sim x$. Now we have

$$\log f(\lambda)\overline{F}_m(x)=\log f(\lambda\overline{G}(x))\sim \alpha\lambda\overline{G}(x).$$

Next we study

$$F_m(x)=1-\frac{f(\lambda\overline{G}(x))}{f(\lambda)},$$

and analyze the behavior of $F_m(x)$ as $x\to 0$. As $x\downarrow 0$ we have $\overline{G}(x)\to\overline{G}(0)=1$, and $F_m(x)\to 0$. Using the mean value theorem, we have

$$F_m(x)=\frac{f(\lambda)-f(\lambda\overline{G}(x))}{f(\lambda)}=\frac{f^{(1)}(\lambda\theta(x))}{f(\lambda)}\lambda\overline{G}(x),$$

where $\overline{G}(x)\le\theta(x)\le 1$ (so that $\theta(x)\to 1$ as $x\downarrow 0$). We obtain that

$$F_m(x)=\frac{f^{(1)}(\lambda\theta(x))}{f^{(1)}(\lambda)}\frac{\lambda f^{(1)}(\lambda)}{f(\lambda)}\overline{G}(x).$$

If we assume that

$$\lim_{z\to 1}\frac{f^{(1)}(\lambda z)}{f^{(1)}(\lambda)}\to 1,$$

we find that

$$F_m(x)\sim\frac{\lambda f^{(1)}(\lambda)}{f(\lambda)}\overline{G}(x),\quad \text{as}\quad x\downarrow 0.$$

Examples. 1) If $f(x)=\exp(\alpha x)$ we have $f^{(1)}(x)=\alpha f(x)$ and

$$f^{(1)}(\lambda z)/f^{(1)}(\lambda)=\exp\lambda(z-1)\to 1,\quad \text{as}\ \ z\to 1,$$

so that we find that

$$F_m(x)\sim\lambda\alpha\overline{G}(x),\quad as\ \ x\downarrow 0.$$

2) If $f(x) = (1-x)^{-\alpha}$, we have $f^{(1)}(x) = \alpha(1-x)^{-\alpha-1}$ and we find that

$$F_m(x) \sim \frac{\lambda\alpha}{1-\lambda}\overline{G}(x), \quad as\ x \downarrow 0.$$

3) If $f(x) = -(1-x)^{-1}\log(1-x)$, we find that

$$F_m(x) \sim \lambda\frac{(1-\log(1-\lambda))}{(1-\lambda)(-\ln(1-\lambda))}\overline{G}(x), \text{ as } x \downarrow 0.$$

3.3.3. On $P(M(X(\lambda; f) > x)$. Earlier we have proved that $F_M(x) = f(\lambda G(x))/f(\lambda)$, and

$$\overline{F}_M(x) = 1 - \frac{f(\lambda G(x))}{f(\lambda)}.$$

First we study the asymptotic behaviour of $\overline{F}_M(x)$ as $x \to \infty$. Clearly we have

$$\overline{F}_M(x) = \frac{f(\lambda) - f(\lambda G(x))}{f(\lambda)} = \int_{G(x)}^{1} \frac{f^{(1)}(\lambda z)}{f^{(1)}(\lambda)}\frac{\lambda f^{(1)}(\lambda)}{f(\lambda)}\,\mathrm{d}z.$$

Now we assume that

$$\lim_{u\uparrow 1}\frac{f^{(1)}(\lambda u)}{f^{(1)}(\lambda)} = 1.$$

Then it follows that as $x \to \infty$,

$$\overline{F}_M(x) \sim \frac{\lambda f^{(1)}(\lambda)}{f(\lambda)}\overline{G}(x).$$

Now we study the asymptotic behaviour of $F_M(x)$ as $x \to 0$. We have $f(\lambda)F_M(x) = f(\lambda G(x))$. If

$$\lim_{z\downarrow 0}\frac{\log f(z)}{r(z)} = \alpha,$$

we find that $\log f(\lambda)F_M(x) \sim \alpha\lambda G(x)$, as $x \downarrow 0$.

3.3.4. Multivariate case. Now let (X,Y), (X_1,Y_1), …, (X_n,Y_n) denote i.i.d. random vectors with d.f. $F(x,y)$ and assume that all variables involved are independent. We consider $M^{(1)}(0) = M^{(2)}(0) = 0$ and for $n \geq 1$, we consider the random vector

$$(M_n^{(1)}(n), M_n^{(2)}(n)) = (\max(X_1,\ldots,X_n), \max(Y_1,\ldots,Y_n)).$$

Clearly we have $P(M^{(1)}(n) \leq x, M^{(2)}(n) \leq y) = F^n(x,y)$. Replacing the index n with the random index $X(\lambda; f)$, we find that

$$\begin{aligned} P(M^{(1)}(X(\lambda; f)) &\leq x,\ M^{(2)}(X(\lambda; f) \leq y) = \sum_{n=0}^{\infty} F^n(x,y) P(X(\lambda; f) = n) \\ &= \frac{f(\lambda F(x,y))}{f(\lambda)}. \end{aligned}$$

This model has been studied by Jafari and Tahmasebi (2016).

4. *Concluding remarks*

1) In Macci et al. (2021) the authors consider $X(\lambda; f)$ with p.d.

$$P(X(\lambda; f) = n) = \frac{f^{(n)}(0)}{n! f(\lambda)} \lambda^n, \quad n \geq 0,$$

and generating function $\phi_{X(\lambda;f)}(z) = f(\lambda z)/f(\lambda)$. Then they consider a function $\delta(t)$ such that $\delta(t) \uparrow \infty$ and consider a family of r.v. $N(t)$ defined as $N(t) = X(\delta(t); f)$ and with generating function

$$\phi_{N(t)}(z) = \frac{f(\delta(t) z)}{f(\delta(t))}.$$

In their paper the authors study large and moderate deviations of $N(t)$. One of the basic assumptions is that there exist functions $A(\cdot)$ and $B(\cdot)$ so that $A(t) \to \infty$ and $B(\cdot)$ is differentiable and

$$(\forall u > 0) \quad \lim_{t \to \infty} \frac{\log f(ut)}{v(t)} = B(u).$$

Note that this implies that

$$\lim_{t \to \infty} \frac{\log \phi_{N(t)}(z)}{v(t)} = B(z) - B(1).$$

2) Earlier, we have proved that

$$P(m(X(\lambda; f)) > x) = \frac{f(\lambda \overline{G}(x))}{f(\lambda)}.$$

In the exponential case ($f(x) = \exp x$) we have $P(m(X(\lambda; f)) > x) = \exp(-\lambda G(x))$. If $G(x)$ has density g, we find that

$$\frac{\mathrm{d}}{\mathrm{d}x} P(m(X(\lambda; f)) > x) = -\lambda P(m(X(\lambda; f)) > x) g(x).$$

Now assume that

$$P(m(X(\lambda;f)) > x) = \sum_{n=0}^{\infty} \psi_n x^n \quad \text{and} \quad g(x) = \sum_{i=0}^{\infty} A_i x^i \Big/ \sum_{j=0}^{\infty} B_j x^j.$$

The above relation shows that

$$\sum_{j=0}^{\infty} B_j x^j \sum_{n=1}^{\infty} n\psi_n x^{n-1} = -\lambda \sum_{n=0}^{\infty} \psi_n x^n \sum_{i=0}^{\infty} A_i x^i,$$

and

$$\sum_{j=0}^{\infty} \sum_{m=0}^{\infty} B_j (m+1)\psi_{m+1} x^{m+j} = -\lambda \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \psi_n A_i x^{n+i}.$$

This type of relation may be used to obtain recursions for (ψ_n).

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