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A NOTE ON C-ULTRADISTRIBUTION SEMIGROUPS AND C-ULTRADISTRIBUTION COSINE FUNCTIONS

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A b s t r a c t. In this paper, we reconsider the recent research studies of $(degenerate)$ C-ultradistribution semigroups and $(degenerate)$ C-ultradistribution cosine functions in locally convex spaces. We follow Komatsu's approach to the theory of ultradistributions, introduce a new condition on the sequence $(M_p)_{p\in\mathbb{N}_0}$ of strictly positive real numbers such that $M_0 = 1$, and slightly improve a great number of structural results obtained in some of author's recent papers.

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1. *Introduction and preliminaries*

Let $(M_p)_{p\in\mathbb{N}_0}$ be any sequence of strictly positive real numbers such that $M_0 = 1$. The associated function of sequence (M_p) is defined by

$$M(\rho) := \sup_{p\in\mathbb{N}} \log \frac{\rho^p}{M_p},\ \rho > 0;\ M(0) := 0,\ M(\lambda) := M(|\lambda|),\ \lambda \in \mathbb{C} \setminus [0, \infty).$$

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In a series of recent research articles, S. Pilipović, D. Velinov and M. Kostić have analyzed various classes of (degenerate) C-ultradistribution semigroups and (degenerate) C-ultradistribution cosine functions in locally convex spaces, provided that the sequence $(M_p)_{p\in\mathbb{N}_0}$ satisfies the following conditions (see [9]–[12] and [16], as well as the research monographs [7]–[8]):

(M.1): [Logarithmic convexity]

$$M_p^2 \leq M_{p+1}M_{p-1}, \quad p \in \mathbb{N},$$

(M.2): [Stability under ultra-differential operators]

$$M_p \leq lH^p \sup_{0\leq i\leq p} M_i M_{p-i}, \quad p \in \mathbb{N}, \text{ for some finite } l,\ H > 1,$$

(M.3)': [Non-quasi-analyticity]

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

Sometimes we have used a slightly stronger condition than (M.3)':

(M.3): [Strong non-quasi-analyticity]

$$\sup_{p\in\mathbb{N}} \sum_{q=p+1}^{\infty} \frac{M_{q-1}M_{p+1}}{pM_pM_q} < \infty.$$

The main aim of this paper is to reconsider the obtained results for a general sequence $(M_p)_{p\in\mathbb{N}_0}$ of strictly positive real numbers such that $M_0 = 1$, (M.1) and (M.3)' hold; with the exception of our considerations given in Section 2, this will be our standing assumptions henceforth. Our intention is to slightly improve a great number of the structural results from [9]–[11] and [16] by assuming condition

(M.2)': [Stability under differential operators]

$$M_{p+1} \leq lH^p M_p, \quad p \in \mathbb{N}, \qquad \text{for some finite } l,\ H > 1,$$

in place of a slightly stronger condition (M.2).

It is worth noting that we use, a priori, condition (M.1) in our analysis since Gorny's theorem holds in our framework (see e.g., [4, pp. 49–50] and the third section in the doctoral dissertation of I. Voulis [17]). On the other hand, the famous Denjoy-Carleman-Mandelbrojt theorem (see e.g., [4, Theorem 4.2, p. 56]) states that, if condition (M.1) holds for (M_p), then the non-triviality of space $\mathcal{D}_K^*$ for some non-empty compact set $K \subseteq \mathbb{R}$ implies the validity of condition (M.3)'; conversely, if (M_p) satisfies (M.1) and (M.3)', then there exists a function $\rho \in \mathcal{D}_K^*$ such that $\rho \geq 0$ and $\int_K \rho(t)\, \mathrm{d}t = 1$ (see the next section for the notion and more details).

This almost imposes the use of condition (M.3)' in our analysis and justifies the use of our standing assumptions (M.1) and (M.3)' henceforth.

For more details about distribution semigroups, ultradistribution semigroups and their applications, we also refer the reader to the research monograph [15] by I. V. Melnikova and A. I. Filinkov. Concerning the notion and basic properties of multivalued linear operators (MLOs), we refer the reader to the research monograph [8] and references cited therein.

Notation and terminology. Unless specified otherwise, we assume henceforth that E is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short. We assume that the state space E is barreled. By $L(E)$ we denote the space consisting of all continuous linear mappings from E into E. The symbol $\circledast_E$ ($\circledast$, if there is no risk for confusion) denotes the fundamental system of seminorms which defines the topology of E.

Let X be an SCLCS, let $\mathcal{B}$ be the family of bounded subsets of E, and let

$$p_B(T) := \sup_{x\in B} p(Tx), \quad p \in \circledast_X, \quad B \in \mathcal{B}, \quad T \in L(E, X).$$

Then $p_B(\cdot)$ is a seminorm on $L(E, X)$ and the calibration $(p_B)_{(p,B)\in\circledast_X\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(E, X)$. By E^* we denote the dual space of E. The Hausdorff locally convex topology on E^* defines the calibration $(|\cdot|_B)_{B\in\mathcal{B}}$ of seminorms on E^*, where

$$|x^*|_B := \sup_{x\in B} |\langle x^*, x\rangle|, \quad x^* \in E^*, \quad B \in \mathcal{B}.$$

The polars of nonempty sets $M \subseteq E$ and $N \subseteq E^*$ are defined as follows

$$M^\circ := \{y \in E^* : |y(x)| \leq 1 \text{ for all } x \in M\}$$

and $N^\circ := \{x \in E : \ |y(x)| \leq 1 \text{ for all } y \in N\}$. If A is a linear operator acting on E, then the domain, kernel space and range of A will be denoted by $D(A)$, $N(A)$ and $R(A)$, respectively. Since E is barreled, the spaces $L(E)$ and E^* are sequentially complete.

2. *Condition* (L) *and a few important observations*

We would like to note that condition (M.1) always implies condition (L), where:

(L) There exist two finite real constants $l > 0$ and $H > 0$ such that

$$\sum_{j=0}^{p-1} M_{p-1-j}M_j \leq lH^p M_p \quad \text{for all} \quad p \in \mathbb{N}.$$

Proposition 2.1. *Suppose that (M_p) is a sequence of positive real numbers satisfying $M_0 = 1$ and* (M.1). *Then (M_p) satisfies* (L).

PROOF. Since $M_0 = 1$ and (M.1) holds, it is well known that $M_{p+q} \geq M_p M_q$ for all $p,\ q \in \mathbb{N}_0$; see e.g. [1, Lemma 2.1.1]. Therefore,

$$\sum_{j=0}^{p-1} M_{p-1-j} M_j \leq p M_{p-1} = p M_1^{-1} M_{p-1} M_1$$
$$\leq 2^p M_1^{-1} M_{p-1} M_1 \leq M_1^{-1} 2^p M_p, \quad p \in \mathbb{N}.$$

As we will see a bit later, condition (L) does not imply any of the above-mentioned conditions.

Let us consider now the Gevrey sequence $M_p = p!^s$, where $s > 0$. In the following illustrative example, which can be viewed of some independent interest, we will prove that (L) holds with $H = 1$:

Example 2.1. The sequence $M_p = p!^s$, where $s > 0$, satisfies condition (L) with $H = 1$. The argumentation is trivial and well known in the case that $s \geq 1$. Suppose now that $0 < s < 1$; then there exists $s_1 \in \mathbb{N}$ such that $ss_1 > 1$. We need to prove the existence of a finite real constant $a > 0$ such that

$$p^{-s} \sum_{j=0}^{p-1} \binom{p-1}{j}^{-s} \leq a, \quad p \in \mathbb{N}. \tag{2.1}$$

It is clear that (2.1) holds if we prove that, for every $p \in \mathbb{N}$ with $\lfloor (p-1)/2 \rfloor \geq s_1+1$, we have

$$p^{-s} \sum_{j=s_1}^{p-s_1-1} \binom{p-1}{j}^{-s} \leq a; \tag{2.2}$$

here, $\lfloor (p-1)/2 \rfloor$ denotes the integer part of $(p-1)/2$. The elementary inequality between means gives

$$\sum_{j=s_1}^{p-s_1-1} \left(\frac{\binom{p-1}{j}^{-s}}{p-2s_1} \right)^{1/s} \leq \sum_{j=s_1}^{p-s_1-1} \left(\frac{\binom{p-1}{j}^{-1}}{p-2s_1} \right)^{1/1},$$

which simply implies

$$\sum_{j=s_1}^{p-s_1-1} \binom{p-1}{j}^{-s} \leq (p-2s_1)^{1-s} \left(\sum_{j=s_1}^{p-s_1-1} \binom{p-1}{j}^{-1} \right)^{s}, \quad p \in \mathbb{N}.$$

Since $\binom{p-1}{j} = \binom{p-1}{p-1-j}$ for all integers $j \in [0, p-1]$, the last estimate implies the existence of a finite real constant constant $d > 0$, independent of p, such that

$$\sum_{j=s_1}^{p-s_1-1} \binom{p-1}{j}^{-s} \leq \bigl(p-2s_1\bigr)^{1-s} \left(\sum_{j=s_1}^{\lfloor (p-1)/2 \rfloor} \binom{p-1}{j}^{-1} + \frac{1}{\binom{p-1}{\lfloor p/2 \rfloor}} \right)^{s}$$
$$= \bigl(p-2s_1\bigr)^{1-s} \left(\frac{s_1!}{(p-1)\cdots(p-s_1)} \left[\sum_{j=s_1}^{\lfloor (p-1)/2 \rfloor} \frac{s_1+1}{p-s_1-1} \cdots \frac{j}{p-j} \right] + \frac{1}{\binom{p-1}{\lfloor p/2 \rfloor}} \right)^{s}$$
$$\leq \bigl(p-2s_1\bigr)^{1-s} \left(\frac{s_1!}{(p-1)\cdots(p-s_1)} \lfloor (p-1)/2 \rfloor + \frac{p}{2^{p-1}} \right)^{s}$$
$$\leq dp^{1-ss_1}.$$

This simply implies (2.2) since $ss_1 > 1$.

The genesis of paper is motivated by the following facts:

(A1) Let $h > 0$, let $K \subseteq \mathbb{R}$ be a non-empty compact set, and let (M_p) be any sequence of positive real numbers such that $M_0 = 1$. Then $(\mathcal{D}_K^{M_p,h}, \|\phi\|_{M_p,h,K})$ is a complex Banach space, where

$$\mathcal{D}_K^{M_p,h} := \bigl\{ \phi \in C^\infty(\mathbb{R}) : \text{supp}(\phi) \subseteq K,\ \|\phi\|_{M_p,h,K} < \infty \bigr\},$$

and

$$\|\phi\|_{M_p,h,K} := \sup \left\{ \frac{h^p \bigl|\phi^{(p)}(t)\bigr|}{M_p} : t \in K,\ p \in \mathbb{N}_0 \right\}.$$

Therefore, we are in a position to define the corresponding spaces of Beurling, respectively, Roumieu ultradifferentiable functions, through

$\mathcal{D}^{(M_p)} := \mathcal{D}^{(M_p)}(\mathbb{R}) := \text{indlim}_{K \Subset \Subset \mathbb{R}} \mathcal{D}_K^{(M_p)}$, respectively,

$\mathcal{D}^{\{M_p\}} := \mathcal{D}^{\{M_p\}}(\mathbb{R}) := \text{indlim}_{K \Subset \Subset \mathbb{R}} \mathcal{D}_K^{\{M_p\}}$, where

$\mathcal{D}_K^{(M_p)} := \text{projlim}_{h \to \infty} \mathcal{D}_K^{M_p,h}$, respectively, $\mathcal{D}_K^{\{M_p\}} := \text{indlim}_{h \to 0} \mathcal{D}_K^{M_p,h}$;

henceforth, the asterisk $*$ stands for the both classes, the Beurling class (M_p) and the Roumieu class $\{M_p\}$ (for more details concerning the topological properties of these spaces, the reader may consult [4]-[6] and [17]). The mappings $f(\cdot) \mapsto f(\cdot + h)$ and $f(\cdot) \mapsto f(a\cdot)$, where $h \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$, are linear and continuous from $\mathcal{D}^*$ into $\mathcal{D}^*$. We set $\mathcal{D}_0^* := \{\varphi \in \mathcal{D}^* : \text{supp}(\varphi) \subseteq [0, \infty)\}$.

If (M.2)' holds, then any differential operator $D : \mathcal{D}^* \to \mathcal{D}^*$ is linear and continuous. Let us recall that an entire function of the form $P(\lambda) = \sum_{p=0}^{\infty} a_p \lambda^p$, is of class (M_p), resp., of class $\{M_p\}$, if and only if there exist $l > 0$ and $M > 0$, resp., for every $l > 0$, there exists a constant $M > 0$, such that $|a_p| \leq Ml^p/M_p$ for all $p \in \mathbb{N}_0$. The corresponding ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$, is of class (M_p), resp., of class $\{M_p\}$. It is well known that (M.2) implies that $P(D) : \mathcal{D}^* \to \mathcal{D}^*$ is a linear continuous mapping (see e.g., [4, Theorem 2.10, pp. 47-48]); basically, we will not use condition (M.2) below.

(A2) Let (M_p) be any sequence of positive real numbers such that $M_0 = 1$. If $\varphi,\ \psi \in \mathcal{D}^*$, then the convolution

$$t \mapsto (\varphi * \psi)(t) := \int_{-\infty}^{+\infty} \varphi(t-s)\psi(s)\,\mathrm{d}s, \quad t \in \mathbb{R}$$

also belongs to $\mathcal{D}^*$.

(A3) Let (M_p) be any sequence of positive real numbers such that $M_0 = 1$ and condition (L) holds. If $\varphi,\ \psi \in \mathcal{D}^*$, then the finite convolution

$$t \mapsto \big(\varphi *_0 \psi\big)(t) := \int_0^t \varphi(t-s)\psi(s)\,\mathrm{d}s, \quad t \in \mathbb{R}$$

also belongs to $\mathcal{D}^*$. To prove this fact, fix a real number $h > 0$ and a compact set $K \subseteq \mathbb{R}$. Then it is only non-trivial to observe that, for every two functions $\varphi,\ \psi \in \mathcal{D}_K^{M_p,h}$, we have the existence of a finite constant $c > 0$ such that $|\phi^{(p)}(t)| + |\phi^{(p)}(t)| \leq cM_p h^{-p}$ for all $p \in \mathbb{N}_0$ and $t \in \mathbb{R}$, which implies along the equality

$$\big(\varphi *_0 \psi\big)^{(p)}(t) = \big(\varphi^{(p)} *_0 \psi\big)(t) + \sum_{j=0}^{p-1} \varphi^{(p-1-j)}(0)\psi^{(j)}(t), \quad t \in \mathbb{R},\ p \in \mathbb{N}_0,$$

and condition (L) that

$$\begin{aligned}\left|(\varphi *_0 \psi)^{(p)}(t)\right| &\leq M_p h^{-p}\big\|\psi\big\|_{L^1(\mathbb{R})} + h^{1-p}\sum_{j=0}^{p-1} M_{p-1-j}M_j \\ &\leq M_p h^{-p}\big\|\psi\big\|_{L^1(\mathbb{R})} + lh^{1-p}H^p M_p, \quad p \in \mathbb{N},\ t \in \mathbb{R}.\end{aligned}$$

(A4) A sequence (M_p) of positive real numbers satisfying $M_0 = 1$ and condition (L) need not satisfy condition (M.1) or condition (M.2)' [(M.2)]. To see that (M.1) is not a consequence of (L), it suffices to construct inductively a strictly increasing sequence (M_p) of positive real numbers satisfying condition (L) with $l = H = 1$ and a sequence $(m_p)_{p\in\mathbb{N}}$ of positive real numbers, tending to plus infinity as $p \to +\infty$, satisfying additionally that the sequence $(m_p \equiv M_p/M_{p-1})_{p\in\mathbb{N}}$ is not increasing. In actual fact, if $M_0, \dots, M_{p-1}$ and $m_0, \dots, m_{p-1}$ are already constructed, it suffices to appropriately choose $M_p \geq \sum_{j=0}^{p-1} M_{p-1-j}M_j$ and set after that $m_p = M_p/M_{p-1}$. To see that (M.2)' is not a consequence of (L), we may choose $M_p \geq \sum_{j=0}^{p-1} M_{p-1-j}M_j$ arbitrarily large in each reiteration by requiring that $M_p > p^p M_{p-1}$ for all $p \in \mathbb{N}$.

It is clear that the use of condition (L) has some obvious drawbacks since we allow the situation in which M_p can rapidly grow compared to $M_0, \dots, M_{p-1}$. From the point of view of the theory of ultradistributions, we must control the growth of M_p by some condition of type (M.2)' or (M.2).

3. *Degenerate C-ultradistribution semigroups in locally convex spaces*

In the remainder of paper, we will always assume that $C \in L(E)$ and $(M_p)_{p\in\mathbb{N}_0}$ is a sequence of strictly positive real numbers satisfying $M_0 = 1$, (M.1) and (M.3)'. Since E is barreled, any $\mathcal{G} \in \mathcal{D}'^*(L(E))$ is boundedly equicontinuous, i.e., for every $p \in ⊛$ and for every bounded subset B of $\mathcal{D}^*$, there exist $c > 0$ and $q \in ⊛$ such that $p(\mathcal{G}(\varphi)x) \leq cq(x),\ \varphi \in B,\ x \in E.$

We recall the following well known definition, which can be introduced even in the case that (M_p) does not satisfy (M.2)':

Definition 3.1. Let $\mathcal{G} \in \mathcal{D}_0'^*(L(E))$ satisfy $C\mathcal{G} = \mathcal{G}C$. Then it is said that $\mathcal{G}$ is a pre-(C-UDS) of $*$-class if and only if the following holds:

$$\mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi), \quad \varphi,\ \psi \in \mathcal{D}^*. \tag{C.S.1}$$

If, additionally,

$$\mathcal{N}(\mathcal{G}) := \bigcap_{\varphi\in\mathcal{D}_0^*} N(\mathcal{G}(\varphi)) = \{0\}, \tag{C.S.2}$$

then $\mathcal{G}$ is called a C-ultradistribution semigroup of $*$-class, (C-UDS) of $*$-class, in short. A pre-(C-UDS) of $*$-class $\mathcal{G}$ is called dense if and only if

$$\mathcal{R}(\mathcal{G}) := \bigcup_{\varphi\in\mathcal{D}_0^*} R(\mathcal{G}(\varphi)) \text{ is dense in } E. \tag{C.S.3}$$

Suppose that $\mathcal{G}$ is a pre-(C-UDS) of $*$-class. Then $\mathcal{G}(\varphi)\mathcal{G}(\psi) = \mathcal{G}(\psi)\mathcal{G}(\varphi)$ for all $\varphi,\ \psi \in \mathcal{D}^*$, and $\mathcal{N}(\mathcal{G})$ is a closed subspace of E.

In the remainder of section, we will always assume that (M_p) additionally satisfies (M.2)'. First of all, we will reconsider several statements proved by P. C. Kunstmann for distribution semigroups ([13]). We recall the notion of a regularizing sequence in $\mathcal{D}^*$: Suppose that $\rho \in \mathcal{D}^*$, supp$(\rho) \subseteq [0,1]$, $\rho \geq 0$ and $\int_{-\infty}^{\infty} \rho(t)\,dt = 1$. Put $\rho_n(t) := n\rho(nt)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$; then (ρ_n) is said to be a regularizing sequence in $\mathcal{D}^*$.

We need the following useful lemma:

Lemma 3.1. *Suppose that (ρ_n) is a regularizing sequence in $\mathcal{D}^*$ as well as that (M_p) additionally satisfies* (M.2)'. *Then for each $\varphi \in \mathcal{D}^*$ we have $\lim_{n\to+\infty}(\varphi * \rho_n) = \varphi$ in $\mathcal{D}^*$.*

PROOF. Let $\varphi \in \mathcal{D}_K^{M_p,h}$, and let $p \in \mathbb{N}$. The conclusion simply follows from the following estimates, valid for any $x \in$ supp$(K) + [0,1]$:

$$\begin{aligned}
&\left|\int_0^x \varphi^{(p)}(x-t)\rho_n(t)\,\mathrm{d}t - \varphi^{(p)}(x)\right| \\
&\quad = \left|\int_0^x \varphi^{(p)}(x-t)\rho_n(t)\,\mathrm{d}t - \int_0^x \varphi^{(p)}(x)\rho_n(t)\,\mathrm{d}t\right| \\
&\quad = \left|\int_0^x \Big[\varphi^{(p)}(x-t) - \varphi^{(p)}(x)\Big]\rho_n(t)\,\mathrm{d}t\right|,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left|\int_0^x \varphi^{(p)}(x-t)\rho_n(t)\,\mathrm{d}t - \varphi^{(p)}(x)\right| \\
&\quad \leq \int_0^1 t\rho_n(t)\,\mathrm{d}t \cdot \left\|\varphi^{(p+1)}\right\|_\infty \leq n^{-1}\int_0^1 t\rho(t)\,\mathrm{d}t \cdot cM_{p+1}h^{-1-p} \\
&\quad \leq n^{-1}\int_0^1 t\rho(t)\,\mathrm{d}t \cdot clH^pM_ph^{-1-p}, \quad n \in \mathbb{N},
\end{aligned}$$

where $c > 0$ is a finite real constant. Here we have used (M.2)' in the last line of computation and the Lagrange mean value theorem in the fourth line of computation.

Using Lemma 3.1 and the proof of [7, Propositon 3.1.2(i)], we may clarify the following result:

Proposition 3.1. *Suppose that E is a Banach space, and $C \in L(E)$ is injective. Let $\mathcal{G}$ be a pre-*(C-UDS), *$F := E/\mathcal{N}(\mathcal{G})$ and let q be the corresponding canonical mapping $q : E \to F$. Further on, let $H \in L(\mathcal{D}^* : L(F))$ be defined by $q\mathcal{G}(\varphi) :=$*

$H(\varphi)q$ for all $\varphi \in \mathcal{D}^*$ and let $\tilde{C}$ be a linear operator in F defined by $\tilde{C}q := qC$. Then $\tilde{C} \in L(F)$ and $\tilde{C}$ is injective. Moreover, H is a ($\tilde{C}$-UDS) in F.

The following statements can be deduced in general locally convex spaces, with the help of Lemma 3.1 and the argumentation given in the case that (M_p) satisfies (M.1), (M.2) and (M.3)':

Proposition 3.2. *Let $\mathcal{G}$ be a pre-*(C-UDS) *of $*$-class. Then the following holds:*

(i) $C(\overline{\langle \mathcal{R}(\mathcal{G})\rangle}) \subseteq \overline{\mathcal{R}(\mathcal{G})}$, *where $\langle \mathcal{R}(\mathcal{G})\rangle$ denotes the linear span of $\mathcal{R}(\mathcal{G})$.*

(ii) *Assume $\mathcal{G}$ is not dense and $\overline{C\mathcal{R}(\mathcal{G})} = \overline{\mathcal{R}(\mathcal{G})}$. Put $R := \overline{\mathcal{R}(\mathcal{G})}$ and $H := \mathcal{G}_{|R}$. Then H is a dense pre-*(C_1-UDS) *of $*$-class on R with $C_1 = C_{|R}$.*

(iii) *The dual $\mathcal{G}(\cdot)^*$ is a pre-*(C^*-UDS) *of $*$-class on E^* and $\mathcal{N}(\mathcal{G}^*) = \overline{\mathcal{R}(\mathcal{G})}^{\circ}$.*

(iv) *If E is reflexive, then $\mathcal{N}(\mathcal{G}) = \overline{\mathcal{R}(\mathcal{G}^*)}^{\circ}$.*

(v) *The $\mathcal{G}^*$ is a* (C^*-UDS) *of $*$-class in E^* if and only if $\mathcal{G}$ is a dense pre-*(C-UDS) *of $*$-class. If E is reflexive, then $\mathcal{G}^*$ is a dense pre-*(C^*-UDS) *of $*$-class in E^* if and only if $\mathcal{G}$ is a* (C-UDS) *of $*$-class.*

Now we will state and provide the main details of the proof of the following imortant result, which has been considered for the first time by J. Kisyński for pre-distribution semigroups in [3]:

Proposition 3.3. *Suppose that $\mathcal{G} \in \mathcal{D}_0'^*(L(E))$ and $\mathcal{G}(\varphi)C = C\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}^*$. Then $\mathcal{G}$ is a pre-*(C-UDS) *of $*$-class if and only if*

$$\mathcal{G}(\varphi')\mathcal{G}(\psi) - \mathcal{G}(\varphi)\mathcal{G}(\psi') = \psi(0)\mathcal{G}(\varphi)C - \varphi(0)\mathcal{G}(\psi)C, \quad \varphi,\ \psi \in \mathcal{D}^*.$$

PROOF. Suppose that $\varphi,\ \psi \in \mathcal{D}^{M_p,h}_{[-a,a]}$ for some $a > 0$. The conclusion follows similarly as in the proof of [11, Proposition 3.5]; we will only prove here that

$$\mathcal{G}\int_0^a \varphi(\,\cdot\, - s)\psi(s)\,\mathrm{d}s = \int_0^a \psi(s)\mathcal{G}(\varphi(\,\cdot\, - s))\,\mathrm{d}s.$$

Using the continuity of $\mathcal{G}$, it suffices to show that

$$\lim_{k\to+\infty} \frac{a}{k}\sum_{i=0}^{k-1} \varphi\Big(\,\cdot\, - \frac{ai}{k}\Big)\psi\Big(\frac{ai}{k}\Big) = \int_0^a \varphi(\,\cdot\, - s)\psi(s)\,\mathrm{d}s \quad \text{in } \mathcal{D}^*; \tag{3.1}$$

here we have used the left Riemann sum for $\int_0^a \varphi(\,\cdot\, - s)\psi(s)\,\mathrm{d}s$. In what follows, we will use the well known estimates from the elementary courses of numerical analysis concerning the difference between the left Riemann sum and the exact value of integral $\int_0^a \varphi(\,\cdot\, - s)\psi(s)\,\mathrm{d}s$: For each $x \in \mathbb{R}$, the difference between

$\int_0^a \varphi(x-s)\psi(s)\,\mathrm{d}s$ and the corresponding left Riemann sum of this integral cannot exceed

$$(a/2)(a/k)\Big(\|\psi\|_\infty \cdot \|\varphi'\|_\infty + \|\psi'\|_\infty \cdot \|\varphi\|_\infty\Big).$$

Using this estimate, for every $h > 0$ and $p \in \mathbb{N}$, there exists a finite real number $c > 0$ such that:

$$\begin{aligned}
&\frac{h^p}{M_p} \sup_{x\in[-a,2a]} \left| \left(\frac{a}{k} \sum_{i=0}^{k-1} \varphi\Big(x - \frac{ai}{k}\Big)\psi\Big(\frac{ai}{k}\Big) - \int_0^a \varphi(x-s)\psi(s)\,\mathrm{d}s \right)^{(p)} \right| \\
&\quad = \frac{h^p}{M_p} \sup_{x\in[-a,2a]} \left| \frac{a}{k} \sum_{i=0}^{k-1} \varphi^{(p)}\Big(x - \frac{ai}{k}\Big)\psi\Big(\frac{ai}{k}\Big) - \int_0^a \varphi^{(p)}(x-s)\psi(s)\,\mathrm{d}s \right| \\
&\quad \leq \frac{h^p}{M_p} \frac{a}{2} \frac{a}{k} \Big(\|\psi\|_\infty \cdot \|\varphi^{(p+1)}\|_\infty + \|\psi'\|_\infty \cdot \|\varphi^{(p)}\|_\infty \Big) \\
&\quad \leq \frac{h^p}{M_p} \frac{a}{2} \frac{a}{k} \left(\|\psi\|_\infty \cdot c\frac{M_{p+1}}{h^{p+1}} + \|\psi'\|_\infty \cdot c\frac{M_p}{h^p} \right) \\
&\quad \leq \frac{h^p}{M_p} \frac{a}{2} \frac{a}{k} \left(\|\psi\|_\infty \cdot clH^p\frac{M_p}{h^{p+1}} + \|\psi'\|_\infty \cdot c\frac{M_p}{h^p} \right),
\end{aligned}$$

which simply implies the required.

Since (M_p) satisfies (M.2)', we are in a position to define the integral generator

$$\mathcal{A} := \Big\{ (x,y) \in E \times E : \mathcal{G}(-\varphi')x = \mathcal{G}(\varphi)y \ \text{ for all } \varphi \in \mathcal{D}_0^* \Big\}$$

of a pre-(C-UDS) of $*$-class $\mathcal{G}$; it is very easy to show that $\mathcal{A}$ is a closed MLO in E as well as that $\mathcal{A} = A$ is a closed linear operator provided that $\mathcal{G}$ is a (C-UDS) of $*$-class. Moreover, $\mathcal{N}(\mathcal{G}) \times \mathcal{N}(\mathcal{G}) \subseteq \mathcal{A}$ and $\mathcal{N}(\mathcal{G}) = \mathcal{A}0$, and therefore, $\mathcal{A}$ is single-valued if and only if (C.S.2) holds. If this is the case, the operator C must be injective.

Applying Proposition 3.3 and a simple argumentation, we get:

Proposition 3.4. *Let $\mathcal{G}$ be a pre-*(C-UDS) *of $*$-class generated by $\mathcal{A}$, let $\psi \in \mathcal{D}^*$ and $x \in E$. Then we have:*

(i) $(\mathcal{G}(\psi)x, \mathcal{G}(-\psi')x - \psi(0)Cx) \in \mathcal{A}$.

(ii) *If $\mathcal{G}$ is dense, then its generator is densely defined.*

Further on, Proposition 3.3 and the argumentation used in the proof of [11, Theorem 3.6(i)–(ii)] immediately imply the validity of the following result (see also the conclusions established in [11, Remark 3.7], which can be slightly improved using the approach followed in this paper):

Theorem 3.1. *Suppose that $\mathcal{G} \in \mathcal{D}_0'^{*}(L(E))$, $\mathcal{G}(\varphi)C = C\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}^*$ and $\mathcal{A}$ is a closed* MLO *on E satisfying that $\mathcal{G}(\varphi)\mathcal{A} \subseteq \mathcal{A}\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}^*$ and*

$$\mathcal{G}(-\varphi')x - \varphi(0)Cx \in \mathcal{A}\mathcal{G}(\varphi)x, \quad x \in E,\ \varphi \in \mathcal{D}\ \ (\varphi \in \mathcal{D}^*). \tag{3.2}$$

Then we have the following:

(i) *If $\mathcal{A} = A$ is single-valued, then $\mathcal{G}$ satisfies* (C.S.1).

(ii) *If $\mathcal{G}$ satisfies* (C.S.2)*, C is injective and $\mathcal{A} = A$ is single-valued, then $\mathcal{G}$ is a* (C-UDS) *of $*$-class generated by $C^{-1}AC$.*

Using Proposition 3.3 and the argumentation given in [11, Proposition 3.8], we get:

Proposition 3.5. *Suppose that C is injective. Then every C-ultradistribution semigroup of $*$-class is uniquely determined by its generator.*

The statement of [11, Proposition 3.3(i)–(ii)] (see also [9, Proposition 4.1(i)–(ii)]) cannot be formulated if condition (M.2) is neglected since the convolution $T * \varphi$, where T is a scalar-valued ultradistribution of $*$-class with compact support and $\varphi \in \mathcal{D}^*$, need not belong to the space $\mathcal{D}^*$, in general; if (M_p) satisfies (M.2)', then [4, Theorem 6.10, p. 71] states that $T * \varphi$ is an infinitely differentiable function with compact support, only. The use of condition (M.2) is mandatory in [9, Theorem 4.1] since we essentially need [6, Theorem 4.8, p. 691] for the proof of this statement.

For any $\psi \in \mathcal{D}^*$, we set $\psi_+(t) := \psi(t)H(t)$, $t \in \mathbb{R}$, where $H(\cdot)$ is the Heaviside function. Fortunately, for every $\psi \in \mathcal{D}^*$, we can define a closed MLO $G(\psi_+)$ by

$$G\big(\psi_+\big) := \big\{(x, y) \in E \times E\,;\, \mathcal{G}(\psi_+ * \varphi)x = \mathcal{G}(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0^*\big\};$$

the reason is quite simple because $\psi_+ * \varphi = \psi *_0 \varphi$ for all $\varphi \in \mathcal{D}_0^*$ (see also [7, Proposition 3.1.3, Lemma 3.1.6]). This implies the following:

Proposition 3.6. *Suppose that $\mathcal{G}$ is a pre-*(C-UDS) *of $*$-class. Then we have*

(d_5) $(Cx, \mathcal{G}(\psi)x) \in G(\psi_+)$, $\psi \in \mathcal{D}^*$, $x \in E$.

Besides condition (d_5), we can consider the following conditions introduced already by J. L. Lions in his pioneering paper [14]:

(d_1): $\mathcal{G}(\varphi * \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi)$, φ, $\psi \in \mathcal{D}_0^*$,

(d_3): $\mathcal{R}(\mathcal{G})$ is dense in E,

(d_4): for every $x \in \mathcal{R}(\mathcal{G})$, there exists a function $u_x \in C([0, \infty) : E)$ so that $u_x(0) = Cx$ and $\mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)u_x(t)\,dt$, $\varphi \in \mathcal{D}^*$.

Then [11, Theorem 3.18] and all established conclusions stated on [9, p. 11, l. -4-l. -1; p. 118, p. 119, l.1-l.2] hold if we replace condition (M.2) with (M.2)'.

Suppose, finally, that there exist $l > 0$, $\beta > 0$ and $k > 0$, in the Beurling case, resp., for every $l > 0$ there exists $\beta_l > 0$, in the Roumieu case, such that the assumptions of [9, Theorem 4.6] hold with the exponential region $E(a,b)$ replaced with the region $\Omega^{(M_p)}_{l,\beta} := \{\lambda \in \mathbb{C} : \Re\lambda \geq M(l|\lambda|) + \beta\}$, resp. $\Omega^{\{M_p\}}_{l,\beta_l} := \{\lambda \in \mathbb{C} : \Re\lambda \geq M(l|\lambda|) + \beta_l\}$; see also [10, Theorem 2.9, Theorem 2.11]. Define $\mathcal{G}$ as in the formulation of this theorem. Then $\mathcal{G}$ is a pre-(CUDS) of the Beurling, resp. Roumieu class, whose integral generator is an extension of $\mathcal{A}$; see also the proof of [7, Theorem 3.1.27] and the Paley-Wiener type results [4, Theorem 9.1, Theorem 9.4; pp. 81–84], which requires only condition (M.2)'. This enables one to see that certain multiplication operators in L^p-spaces can serve as the integral generators of ultradistribution semigroups of (M_p)-class, where (M_p) satisfies (M.1), (M.2)' and (M.3)'.

We feel it is our duty to say that the use of conditions (M.2) and (M.3) is sometimes inevitable. For example, the structural results obtained by I. Cioranescu in [2], concerning the abstract Beurling spaces of class (M_p) and ultradistribution semigroups of class (M_p), cannot be established if (M.2) or (M.3) is neglected.

Exponential C-ultradistribution semigroups and quasi-equicontinous exponential C-ultradistribution semigroups can be also introduced and analyzed if condition (M.2) is replaced by (M.2)'; see [12] for more details. In the results concerning regularization of various classes of quasi-equicontinous exponential C-ultradistribution semigroups, the use of conditions (M.2) and (M.3) is mandatory. Before we move ourselves to the next section, let us note that the conclusions established in [12, Example 3.3] hold if the sequence (M_p) satisfies (M.1), (M.2)' and (M.3)'.

4. *Degenerate C-ultradistribution cosine functions in locally convex spaces*

Throughout this section, we assume that E is a barreled SCLCS, $C \in L(E)$ is not necessarily injective operator as well as that the sequence (M_p) satisfies our standing assumptions and (M.2)'.

We need some preliminaries concerning the first anti-derivatives of vector-valued ultradistributions. Let $\eta \in \mathcal{D}^*_{[-2,-1]}$ be a fixed test function satisfying $\int_{-\infty}^{\infty} \eta(t)\,\mathrm{d}t = 1$. Then, for every fixed $\varphi \in \mathcal{D}^*$, we define $I(\varphi)$ by

$$[I(\varphi)](x) := \int_{-\infty}^{x} \left[\varphi(t) - \eta(t) \int_{-\infty}^{+\infty} \varphi(u)\,\mathrm{d}u\right] \mathrm{d}t, \quad x \in \mathbb{R}.$$

It can be simply shown that, for every $\varphi \in \mathcal{D}^*$ and $n \in \mathbb{N}$, we have $I(\varphi) \in \mathcal{D}^*$, $I^n(\varphi^{(n)}) = \varphi$, $\frac{d}{dx} I(\varphi)(x) = \varphi(x) - \eta(x) \int_{-\infty}^{\infty} \varphi(u)\,\mathrm{d}u$, $x \in \mathbb{R}$ as well as that, for every $\varphi \in \mathcal{D}^*_{[a,b]}$, where $-\infty < a < b < \infty$, we have: $\mathrm{supp}(I(\varphi)) \subseteq [\min(-2,a), \max(-1,b)]$. Define G^{-1} by $G^{-1}(\varphi) := -G(I(\varphi))$, $\varphi \in \mathcal{D}^*$. Then

$G^{-1} \in \mathcal{D}'^{*}(L(E))$ and $(G^{-1})' = G$; more precisely, $-G^{-1}(\varphi') = G(I(\varphi')) = G(\varphi)$, $\varphi \in \mathcal{D}^{*}$. It can be simply proved that condition (M.2)' is sufficient to ensure that, for every $h > 0$ and for every non-empty compact subset K of $\mathbb{R}$, the convergence $\varphi_n \to \varphi$, $n \to \infty$ in $\mathcal{D}_K^{M_p,h}$ implies the convergence $I(\varphi_n) \to I(\varphi)$, $n \to \infty$ in $\mathcal{D}_{K'}^{M_p,h}$, where $K' = [\min(-2, \inf(K)), \max(-1, \sup(K))]$. Also, $\mathrm{supp}(G) \subseteq [0,\infty) \Rightarrow \mathrm{supp}(G^{-1}) \subseteq [0,\infty)$.

We recall the following well known notion in our new framework:

Definition 4.1. An element $\mathbf{G} \in \mathcal{D}_0'^{*}(L(E))$ is called a pre$-(C - UDCF)$ of $*$-class if and only if $\mathbf{G}(\varphi)C = C\mathbf{G}(\varphi)$, $\varphi \in \mathcal{D}^{*}$ and

$$(CCF_1) : \mathbf{G}^{-1}(\varphi *_0 \psi)C = \mathbf{G}^{-1}(\varphi)\mathbf{G}(\psi) + \mathbf{G}(\varphi)\mathbf{G}^{-1}(\psi), \quad \varphi,\ \psi \in \mathcal{D}^{*};$$

if, additionally,

$$(CCF_2) : \qquad x = y = 0 \text{ if and only if } \mathbf{G}(\varphi)x + \mathbf{G}^{-1}(\varphi)y = 0, \quad \varphi \in \mathcal{D}_0^{*},$$

then $\mathbf{G}$ is called a C-ultradistribution cosine function of $*$-class, in short $(C - UDCF)$ of $*$-class. A pre-$(C - UDCF)$ of $*$-class $\mathbf{G}$ is called dense if and only if the set $\mathcal{R}(\mathbf{G}) := \bigcup_{\varphi \in \mathcal{D}_0^{*}} R(\mathbf{G}(\varphi))$ is dense in E.

Clearly, (CCF_2) implies $\bigcap_{\varphi \in \mathcal{D}_0^{*}} N(\mathbf{G}(\varphi)) = \{0\}$ and $\bigcap_{\varphi \in \mathcal{D}_0^{*}} N(\mathbf{G}^{-1}(\varphi)) = \{0\}$, and the preassumption $\mathbf{G} \in \mathcal{D}_0'^{*}(L(E))$ implies $\mathbf{G}(\varphi) = 0$, $\varphi \in \mathcal{D}^{*}_{(-\infty,0]}$. Moreover, $\varphi * \psi_+ = \varphi *_0 \psi \in \mathcal{D}_0^{*}$ for any $\varphi \in \mathcal{D}_0^{*}$.

The following proposition is essential; see the proofs of [7, Proposition 3.4.3] and [16, Proposition 2.2]:

Proposition 4.1. *Let $\mathbf{G} \in \mathcal{D}_0'^{*}(L(E))$ and $\mathbf{G}(\cdot)C = C\mathbf{G}(\cdot)$. Then $\mathbf{G}$ is a pre-(C-UDCF) of $*$-class in E if and only if*

$$\mathcal{G} \equiv \begin{pmatrix} \mathbf{G} & \mathbf{G}^{-1} \\ \mathbf{G}' - \delta \otimes C & \mathbf{G} \end{pmatrix}$$

is a pre-($\mathcal{C}$-UDS) *of $*$-class in $E \oplus E$, where*

$$\mathcal{C} \equiv \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}.$$

Moreover, $\mathcal{G}$ is a ($\mathcal{C}$-UDS) *of $*$-class if and only if $\mathbf{G}$ is a pre-(C-UDCF) of $*$-class which satisfies (CCF_2).*

The proof of [16, Proposition 2.3] can be repeated verbatim if condition (M.2) is replaced by (M.2)', as well:

Proposition 4.2. *Let $\mathbf{G} \in \mathcal{D}_0'^{*}(L(E))$ and $\mathbf{G}(\cdot)C = C\mathbf{G}(\cdot)$. Then the following holds:*

(i) *If* $\mathbf{G}$ *is a pre-(C-UDCF) of* $*$*-class, then*

$$\mathbf{G}^{-1}\big(\varphi*\psi_+\big)C = \mathbf{G}^{-1}(\varphi)\mathbf{G}(\psi)+\mathbf{G}(\varphi)\mathbf{G}^{-1}(\psi),\ \varphi \in \mathcal{D}_0^*,\ \psi \in \mathcal{D}^*. \quad (4.1)$$

(ii) *If* (CCF_2) *and* (4.1) *hold, then* $\mathbf{G}$ *is a (C-UDCF) of* $*$*-class.*

Due to Proposition 3.3 and Proposition 4.1, we have the following slight extension of [16, Proposition 2.4]:

Proposition 4.3. *Suppose that* $\mathbf{G} \in \mathcal{D}_0'^*(L(E))$ *and* $\mathbf{G}(\cdot)C = C\mathbf{G}(\cdot)$. *Then* $\mathbf{G}$ *is a pre-(C-UDCF) of* $*$*-class if and only if, for every* $\varphi,\ \psi \in \mathcal{D}^*$*, we have:*

$$\mathbf{G}^{-1}(\varphi)\mathbf{G}'(\psi) - \mathbf{G}'(\varphi)\mathbf{G}^{-1}(\psi) = \psi(0)\mathbf{G}^{-1}(\varphi)C - \varphi(0)\mathbf{G}^{-1}(\psi)C.$$

If $\mathbf{G}$ is a pre-$(C - UDCF)$ of $*$-class, then we define the (integral) generator $\mathbf{A}$ of $\mathbf{G}$ by

$$\mathbf{A} := \Big\{(x,y) \in E \oplus E : \mathbf{G}^{-1}\big(\varphi''\big)x = \mathbf{G}^{-1}(\varphi)y \ \text{ for all } \ \varphi \in \mathcal{D}_0^*\Big\}.$$

It can be easily shown that $\mathbf{A}$ is a closed MLO and $\mathbf{A} \subseteq C^{-1}\mathbf{A}C$, with the equality if C is injective. Furthermore, if (CCF_2) holds, then $\mathbf{A} = A$ is a closed single-valued linear operator.

In order to avoid any plagiarism, we only want to note at the end that all statements concerning (degenerate) C-ultradistribution cosine functions, considered in [16, Section 2, pp. 3080–3083], remain true if condition (M.2) is replaced by (M.2)'.

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