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INCOMPLETE KLOOSTERMAN SUMS TO PRIME POWER MODULES

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To the Blessed Memory of Professor Aleksandar Pavle Ivić (6.3.1949 – 27.12.2020)

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A b s t r a c t. We prove that for prime p, $p \to +\infty$, integer $r \geqslant 4$ and $q = p^r$ an incomplete Kloosterman sum of length N to modulus q can be estimated non-trivially (with power-saving factor) for very small N, namely, for $N \gg (q \log q)^{1/(r-1)}$.

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1. *Introduction*

Let q, a, b be integers, $q \geqslant 3$, $(a, q) = 1$, and let $1 < N < N_1 < q$, $N_1 \leqslant 2N$. An incomplete Kloosterman sum is an exponential sum of the type

$$S_q(N) = S_q(a, b; N, N_1) = \sideset{}{'}\sum_{N<n\leqslant N_1} e_q(a\overline{n} + bn).$$

Here, as usual, $e_q(u) = e(u/q)$, $e(\alpha) = e^{2\pi i\alpha}$, prime sign means the summation over n coprime to q and $\overline{n} = 1/n$ denotes the inverse residue to n modulo q, that is, the solution of the congruence $n\overline{n} \equiv 1 \pmod q$. Both the classical A. Weil's bound

$$\left|\sum_{n=1}^{p-1} e_p(a\overline{n} + bn)\right| \leqslant 2\sqrt{p}$$

for prime p (see [1] and [2, Appendix V, 11]) and "multiplicative" property of complete Kloosterman sums

$$S_q(a,b) = {\sum_{n=1}^{q}}' e_q(a\overline{n} + bn)$$

together with the precise expressions for $S_q(a,b)$, $q = p^r$, $r \geqslant 2$ (see, for example, [3, Lemma 4b], [4, formulas (50)]) lead to the estimate

$$|S_q(N)| \ll \sqrt{q}\,\tau(q)\log q,$$

where $\tau(q)$ denotes divisor function. Since $\tau(q) \ll q^{\varepsilon}$ for any fixed $\varepsilon > 0$, this estimate in non-trivial for $N \gg q^{0.5+\varepsilon}$.

In the case $N \leqslant \sqrt{q}$, the problem of estimating the sum $S_q(N)$ was open until the early 1990s, when A.A. Karatsuba invented a new method (see [5]–[8]) that allows one to get the bound of the type

$$|S_q(N)| \ll \frac{N}{(\log q)^c}.$$

In the case of an arbitrary modulus q, this bound holds for any $N \gg q^{\varepsilon}$ with some c, $0 < c < 0.5$. For prime $q = p$, Karatsuba's method allows one to obtain a non-trivial bound of $S_q(N)$ for very small N, namely, for $N \gg \mathrm{e}^{(\log q)^{2/3+\varepsilon}}$ (see [9], [10])[1].

The estimate of $S_q(N)$ for $N \leqslant \sqrt{q}$ in general case with power-saving factor is still open problem. However, such estimate is possible for some special modules, namely, for $q = p^r$, where p is prime and r is large integer. In such case, A.A. Karatsuba noted [6, p. 722] that the identity

$$\frac{1}{1+px} \equiv 1 - px + p^2x^2 - \cdots + (-1)^{r-1}p^{r-1}x^{r-1} \pmod q$$

allows one to reduce the incomplete Kloosterman sums to the exponential sums with polynomials. These sums can be treated by I.M. Vinogradov's method[2]. The implementation of such idea allows to the first author [14, Theorem 4] to prove the following theorem:

[1]A.A. Karatsuba himself expected to obtain non-trivial estimates for the case $N \gg \mathrm{e}^{(\log q)^{2/3+\varepsilon}}$; see the facsimile of his manuscript dated by October 19, 2007 on page 356 of [11].

[2]The idea of such reduction to the exponential sums with polynomials belongs to A.G. Postnikov who applied I.M. Vinogradov's method to short character sums. For the details, see [12], [13].

Let $p \geqslant 2$ be a fixed prime, $(a, p) = 1$, $r \geqslant 72$, $q = p^r$, and suppose that $p^{24} \leqslant N \leqslant q^{1/3}$. Then, for any $1 \leqslant \mu \leqslant q$, the sum

$$S = \sum_{\mu < n \leqslant \mu + N}{}' e_q(a\overline{n} + bn)$$

obeys the estimate

$$|S| \leqslant 6N \exp\left(-\frac{c(\log N)^3}{(\log q)^2}\right), \quad c = 13^{-4}$$

(see also [15]). One can check that this estimate is non-trivial for very small N, namely, for

$$N \geqslant \exp\left(c_1(\log q)^{2/3}\right),$$

and provides the power saving decreasing factor for $N \asymp q^{\alpha}$ with any fixed α:

$$|S| \ll N^{1-\delta}, \quad \delta \asymp \alpha^3.$$

The first author also obtained a generalization of this theorem to the case of so-called "powerful" modules, that is, to the modules q such that the radical $d(q) = \prod_{p|q} p$ is much smaller that q at logarithmic scale:

Suppose that q is sufficiently large integer, $c_1 = 900$, $c = 160^{-4}$, and suppose that

$$\max\left(d^{15}, \mathrm{e}^{c_1(\log q)^{2/3}}\right) \leqslant N \leqslant \sqrt{q}.$$

Then for any a, b, $(a, q) = 1$ we have:

$$|S| \leqslant N \exp\left(-\frac{c(\log N)^3}{(\log q)^2}\right).$$

Thus, the case $q = p$ is the most complicated for studying, and the case of "very powerful" modulus $q = p^r$, $r \to +\infty$, is the easiest. The purpose of the present paper is to study the "middle case" $q = p^r$, where r is a fixed integer and prime $p \to +\infty$, and to find out when the non-trivial bound for $S_q(N)$, $N \leqslant \sqrt{q}$, is available. It appears that it is possible to do for all $r \geqslant 4$. Namely, the following assertion holds true.

Theorem 1.1. *Suppose that $r \geqslant 4$ is fixed, p is prime and $q = p^r$. Further, let*

$$\left(q \log q\right)^{\frac{1}{r-1}} \leqslant N \leqslant q^{3/5}.$$

Then the following inequality holds:

$$|S_q(N)| \leqslant cN\biggl(\frac{q\log q}{N^{r-1}}\biggr)^{\gamma_r},$$

where $c = c(r) > 0$ *and* $\gamma_r^{-1} = r^2(r-1)^2 + (4r+1)(r-1) + 4$.

Remark 1.1. One can check that

$$\gamma_4^{-1} = 199, \quad \gamma_5^{-1} = 488, \quad \gamma_6^{-1} = 1029, \quad \gamma_7^{-1} = 1942,$$
$$\gamma_8^{-1} = 3371, \quad \gamma_9^{-1} = 5484, \quad \gamma_{10}^{-1} = 8473.$$

In the case $N \geqslant q^{\frac{1}{r-1}+\varepsilon}$, it is possible to prove a slightly better estimate with $\gamma_r^{-1} = (r-1)^2(r-2)^2 + r - 1$. To do this, one should take $k = \frac{1}{2}\,r(r-1)$ instead of $k = \frac{1}{2}\,r(r-1)+1$ in the proof of theorem. If

$$q^{\frac{1}{r-1}+\varepsilon} \leqslant N \leqslant q^{\frac{1}{r-1}+\frac{1}{r(r-1)}},$$

then the application of H. Weyl's method yields to the similar estimate with $\gamma_r = 2^{2-r}$. Such estimate is more precise than the previous for $4 \leqslant r \leqslant 18$.

If $q^{3/5} < N \leqslant q$ then the results from [1]–[4] imply much more precise estimate $|S_q(N)| \ll \sqrt{q}\,\tau(q)\log q$. The same bound holds true in the cases $q = p^2, p^3$ and $\sqrt{q}\log q \ll N \leqslant q$. The problem of estimating $S_q(N)$ for $q = p^2, p^3$ and $N \leqslant \sqrt{q}$ is still open. However, in the case $q = p^3$, we obtain an estimate for $S_q(N)$ which is slightly better than the trivial. Namely, the following assertion is true.

Theorem 1.2. *Suppose that* $q = p^3$ *and* $\sqrt{q} \ll N \leqslant 0.5q^{2/3}$. *Then* $|S_q(N)| \ll \sqrt{q}$*, where the implied constants are absolute.*

2. *Auxilliary assertions*

The proof of the above Theorem 1.1 is quite simple but the assertion seems to be new. It relies on the upper bound for the integral $J_{k,s}(P)$ of I. M. Vinogradov. Suppose that $k, s \geqslant 2$ are integers, $P \geqslant 2$. Then $J_{k,s}(P)$ coincides with the number of integral solutions of the system

$$\begin{cases} x_1 + \cdots + x_k = x_{k+1} + \cdots + x_{2k}, \\ x_1^2 + \cdots + x_k^2 = x_{k+1}^2 + \cdots + x_{2k}^2, \\ \qquad\qquad\vdots \\ x_1^s + \cdots + x_k^s = x_{k+1}^s + \cdots + x_{2k}^s, \end{cases}$$

with $1 \leqslant x_1, \ldots, x_{2k} \leqslant P$. The following inequality holds true.

Lemma 2.1. *For any fixed $\varepsilon > 0$, for any fixed $k, s \geqslant 2$ there exists the constant $c = c(\varepsilon, k, s) > 0$ such that the inequality*

$$J_{k,s}(P) \leqslant cP^{\varepsilon}\Big(P^k + P^{2k-\frac{1}{2}s(s+1)}\Big).$$

holds for $P \geqslant 2$. Moreover, if $k > \frac{1}{2}\, s(s+1)$ then the factor P^{ε} can be dropped and the constant c will depend on k, s only.

This estimate was proved partially by T. Wooley and then by J. Bourgain, C. Demeter, L. Guth for an arbitrary $k, s \geqslant 2$ (see [16] for the proof and [17] for short survey and references).

We also need the following simple lemma.

Lemma 2.2. *Let $p \geqslant 3$ be a prime number, $(a, p) = 1$, and suppose that Λ is integer, $1 < \Lambda < p/2$. Then the sum*

$$V = \sum_{\mu=1}^{\Lambda}\Big|\sum_{|\lambda|<\Lambda} e\Big(\frac{a\mu\lambda}{p}\Big)\Big|$$

obeys the estimate $V < 2p\log p$.

PROOF. For any μ under considering, we have $(\mu, p) = 1$. Hence,

$$\Big|\sum_{|\lambda|<\Lambda} e\Big(\frac{a\mu\lambda}{p}\Big)\Big| \leqslant 1 + 2\Big|\sum_{\lambda=1}^{\Lambda-1} e\Big(\frac{a\mu\lambda}{p}\Big)\Big| \leqslant 1 + 2\Big|\sin\frac{\pi a\mu}{p}\Big|^{-1} \leqslant 1 + \Big\|\frac{a\mu}{p}\Big\|^{-1},$$

where, as usual, $\|\xi\|$ means the distance between ξ and closest integer. Since all the numbers $a\mu$, $1 \leqslant \mu \leqslant \Lambda$ are pairwisely noncongruent modulo p and $\Lambda < p/2$ then

$$V \leqslant \sum_{\mu=1}^{\Lambda-1}\Big(1 + \Big\|\frac{a\mu}{p}\Big\|^{-1}\Big) \leqslant \Lambda + \sum_{1\leqslant x\leqslant (p-1)/2}\frac{p}{x} \leqslant \frac{1}{2}\,p + p\Big(\log\frac{p}{2} + 1\Big),$$

i.e., $V < 2p\log p$. □

Lemma 2.3. *There exists an absolute constant $c_0 > 0$ such that for any integers q, a, b and N satisfying the conditions $q \geqslant 2$, $(a, q) = 1$, $1 < N < q$, the following inequality holds:*

$$\Big|\sum_{x=1}^{N} e_q(ax^2 + bx)\Big| \leqslant c_0\sqrt{q}.$$

This is Lemma 4 from [18].

3. *Proof of theorems*

PROOF OF THEOREM 1.1. We may suppose that $N > q^{1/(r-1)} > 10q^{1/r} = 10p$. Let us take an integer X such that $1 < X < N/(10p)$ and apply to $S_q(N)$ a so-called “additive shift”.

Suppose that $1 \leqslant x, y \leqslant X$ are integers. Then we have

$$\begin{aligned} S_q(N) &= \sum_{N<n\leqslant N_1} e_q\left(\frac{a}{n}\right) = \sum_{N<n+pxy\leqslant N_1} e_q\left(\frac{a}{n+pxy}\right) \\ &= \sum_{N-pxy<n\leqslant N_1-pxy} e_q\left(\frac{a}{n+pxy}\right) = \sum_{N<n\leqslant N_1} e_q\left(\frac{a}{n+pxy}\right) + 2\theta_1 pxy \end{aligned}$$

(here and below θ_j denote some complex numbers such that $|\theta_j| \leqslant 1$). The summation over $1 \leqslant x, y \leqslant X$ yields

$$X^2 S_q(N) = \sum_{x,y=1}^{X} \sum_{N<n\leqslant N_1} e_q\left(\frac{a}{n+pxy}\right) + 2\theta_2 pX^4,$$

$$S_q(N) = \frac{1}{X^2} \sum_{N<n\leqslant N_1} \sum_{x,y=1}^{X} e_q\left(\frac{a}{n+pxy}\right) + 2\theta_2 pX^2.$$

Denote by n_0 the value of integer variable n, $N < n \leqslant N_1$, $(n,p) = 1$, that corresponds to the maximal absolute value of the sum over x, y. Passing to the estimates, we get

$$|S_q(N)| \leqslant \frac{N}{X^2}|W_1| + 2pX^2, \qquad W_1 = \sum_{x,y=1}^{X} e_q\left(\frac{a}{n_0+pxy}\right).$$

Since

$$\begin{aligned} \frac{a}{n_0+pxy} &\equiv \frac{a\overline{n}_0}{1+p\overline{n}_0 xy} \pmod q \\ &\equiv a\overline{n}_0\bigl(1 - p\overline{n}_0 xy + (pn_0)^2(xy)^2 - \cdots \\ &\qquad + (-1)^{r-1}(p\overline{n}_0)^{r-1}(xy)^{r-1}\bigr) \pmod q \\ &\equiv a_0 + pa_1 xy + p^2 a_2 (xy)^2 + \cdots + p^{r-1}a_{r-1}(xy)^{r-1} \pmod q, \end{aligned}$$

where $a_j \equiv (-1)^j a\overline{n}_0^{j+1} \pmod q$, then

$$\begin{aligned} e_q\left(\frac{a}{n_0+pxy}\right) &= \exp\left\{\frac{2\pi \mathrm{i}}{p^r}\bigl(a_0 + pa_1 xy + p^2 a_2 (xy)^2 + \cdots + p^{r-1}a_{r-1}(xy)^{r-1}\bigr)\right\} \\ &= \mathrm{e}\bigl(\alpha_0 + \alpha_1 xy + \alpha_2 (xy)^2 + \cdots + \alpha_{r-1}(xy)^{r-1}\bigr) \end{aligned}$$

and

$$\alpha_j = \frac{a_j}{q_j}, \quad q_j = p^{r-j}, \quad j = 0, 1, \ldots, r-1.$$

Hence, $|W_1| = W$, where

$$\begin{aligned} W &= \left| \sum_{x,y=1}^{X} \mathrm{e}(\alpha_1 xy + \alpha_2 (xy)^2 + \cdots + \alpha_{r-1}(xy)^{r-1}) \right| \\ &\leqslant \sum_{x=1}^{X} \left| \sum_{y=1}^{X} e(\alpha_1 xy + \alpha_2 (xy)^2 + \ldots + \alpha_{r-1}(xy)^{r-1}) \right|. \end{aligned}$$

Now let us choose an integer k with the condition $k \geqslant \frac{1}{2} r(r-1) + 1$ and apply Hölder's inequality to W. Thus we get

$$\begin{aligned} W^{2k} &\leqslant X^{2k-1} \sum_{x=1}^{X} \left| \sum_{y=1}^{X} e(\alpha_1 xy + \alpha_2 (xy)^2 + \cdots + \alpha_{r-1}(xy)^{r-1}) \right|^{2k} \\ &= X^{2k-1} \sum_{x=1}^{X} \sum_{y_1,\ldots,y_{2k}=1}^{X} e\Big(\alpha_1 (y_1 + \cdots - y_{2k}) + \alpha_2 (y_1^2 + \cdots - y_{2k}^2) \\ &\qquad\qquad + \cdots + \alpha_{r-1}(y_1^{r-1} + \cdots - y_{2k}^{r-1})\Big) \\ &= X^{2k-1} \sum_{x=1}^{X} \sum_{\mathbf{\Lambda}} J_{k,r-1}(X; \mathbf{\Lambda}) \mathrm{e}\Big(\alpha_1 \lambda_1 x + \alpha_2 \lambda_2 x^2 + \cdots + \alpha_{r-1}\lambda_{r-1} x^{r-1}\Big), \end{aligned}$$

where $\mathbf{\Lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ denotes a vector with integral components and $J_{k,r-1}(X; \mathbf{\Lambda})$ is the number of integral solutions of the system

$$\begin{cases} y_1 + \cdots + y_k = y_{k+1} + \cdots + y_{2k} + \lambda_1, \\ y_1^2 + \cdots + y_k^2 = y_{k+1}^2 + \cdots + y_{2k}^2 + \lambda_2, \\ \quad\vdots \\ y_1^{r-1} + \cdots + y_k^{r-1} = y_{k+1}^{r-1} + \cdots + y_{2k}^{r-1} + \lambda_{r-1}, \end{cases}$$

with $1 \leqslant y_1, \ldots, y_{2k} \leqslant X$. Obviously, if the last system is solvable then the components satisfy the conditions

$$|\lambda_j| = \left| y_1^j + \cdots + y_k^j - y_{k+1}^j - \cdots - y_{2k}^j \right| < kX^j = \Lambda_j.$$

Changing the order of summation we obtain

$$W^{2k} \leqslant X^{2k-1}\sum_{\mathbf{\Lambda}} J_{k,r-1}(X;\mathbf{\Lambda})\biggl|\sum_{x=1}^{X}\mathrm{e}\bigl(\alpha_1\lambda_1 x+\alpha_2\lambda_2 x^2+\cdots+\alpha_{r-1}\lambda_{r-1}x^{r-1}\bigr)\biggr|.$$

The application of Hölder's inequality together with the identity

$$\sum_{\mathbf{\Lambda}} J_{k,r-1}(X;\mathbf{\Lambda}) = X^{2k}$$

yields

$$\begin{aligned}W^{4k^2} &\leqslant X^{2k(2k-1)}\biggl(\sum_{\mathbf{\Lambda}} J_{k,r-1}(X;\mathbf{\Lambda})\biggr)^{2k-1}\\ &\qquad\times\sum_{\mathbf{\Lambda}} J_{k,r-1}(X;\mathbf{\Lambda})\biggl|\sum_{x=1}^{X} e(\alpha_1\lambda_1 x+\alpha_2\lambda_2 x^2+\cdots+\alpha_{r-1}\lambda_{r-1}x^{r-1})\biggr|^{2k}\\ &= X^{4k(2k-1)}\sum_{\mathbf{\Lambda}} J_{k,r-1}(X;\mathbf{\Lambda})\biggl|\sum_{x=1}^{X}\mathrm{e}(\alpha_1\lambda_1 x+\alpha_2\lambda_2 x^2+\cdots+\alpha_{r-1}\lambda_{r-1}x^{r-1})\biggr|^{2k}.\end{aligned}$$

Since $J_{k,r-1}(X;\mathbf{\Lambda}) \leqslant J_{k,r-1}(X;\mathbf{0}) = J_{k,r-1}(X)$ then

$$W^{4k^2} \leqslant X^{4k(2k-1)}J_{k,r-1}(X)\sum_{\lambda_1}\cdots\sum_{\lambda_r}\biggl|\sum_{x=1}^{X}\mathrm{e}(\alpha_1\lambda_1 x+\alpha_2\lambda_2 x^2+\cdots+\alpha_{r-1}\lambda_{r-1}x^{r-1})\biggr|^{2k},$$

where each variable λ_j runs through the whole segment $-(\Lambda_j-1) \leqslant \lambda_j \leqslant \Lambda_j - 1$.

Further,

$$\begin{aligned}&W^{4k^2} \leqslant X^{4k(2k-1)}J_{k,r-1}(X)\\ &\times\sum_{\lambda_1}\cdots\sum_{\lambda_r}\sum_{x_1,\ldots,x_{2k}=1}^{X}\mathrm{e}\bigl(\alpha_1\lambda_1(x_1+\cdots-x_{2k})+\cdots+\alpha_{r-1}\lambda_{r-1}(x_1^{r-1}+\cdots-x_{2k}^{r-1})\bigr)\\ &= X^{4k(2k-1)}J_{k,r-1}(X)\sum_{\lambda_1}\cdots\sum_{\lambda_r}\sum_{\mathbf{M}} J_{k,r-1}(X;\mathbf{M})\mathrm{e}(\alpha_1\lambda_1\mu_1+\cdots+\alpha_{r-1}\lambda_{r-1}\mu_{r-1})\\ &= X^{4k(2k-1)}J_{k,r-1}(X)\sum_{\mathbf{M}} J_{k,r-1}(X;\mathbf{M})\sum_{\lambda_1}\mathrm{e}(\alpha_1\lambda_1\mu_1)\cdots\sum_{\lambda_{r-1}}\mathrm{e}(\alpha_{r-1}\lambda_{r-1}\mu_{r-1}),\end{aligned}$$

where $\mathbf{M} = (\mu_1,\ldots,\mu_{r-1})$ denotes a vector with integral components, $|\mu_j| < \Lambda_j$, $j = 1,\ldots,r-1$. Passing to the estimates, we get

$$W^{4k^2} \leqslant X^{4k(2k-1)}J_{k,r-1}^2(X)\sum_{\mu_1}\biggl|\sum_{\lambda_1}\mathrm{e}(\alpha_1\lambda_1\mu_1)\biggr|\cdots\sum_{\mu_{r-1}}\biggl|\sum_{\lambda_{r-1}}\mathrm{e}(\alpha_{r-1}\lambda_{r-1}\mu_{r-1})\biggr|.$$

By Lemma 2.2,

$$\begin{aligned}\sum_{\mu_{r-1}}\Big|\sum_{\lambda_{r-1}}\mathrm{e}(\alpha_{r-1}\lambda_{r-1}\mu_{r-1})\Big| &\leqslant \sum_{\mu_{r-1}=-\Lambda_{r-1}}^{\Lambda_{r-1}}\Big|\sum_{|\lambda_{r-1}|<\Lambda_{r-1}}\mathrm{e}\Big(\frac{a_{r-1}}{p}\lambda_{r-1}\mu_{r-1}\Big)\Big| \\ &\leqslant 2\Lambda_{r-1}+2\sum_{\mu_{r-1}=1}^{\Lambda_{r-1}}\Big|\sum_{|\lambda_{r-1}|<\Lambda_{r-1}}\mathrm{e}\Big(\frac{a_{r-1}}{p}\lambda_{r-1}\mu_{r-1}\Big)\Big| \\ &\leqslant 2\Lambda_{r-1}+4p\log p<5p\log p=\frac{5}{r}\,p\log q<2p\log q.\end{aligned}$$

Estimating the sums over $\lambda_j,\mu_j,\ j=1,\ldots,r-2$, trivially by the quantity $(2\Lambda_j)^2\leqslant(2k)^2X^{2j}$, we obtain

$$W^{4k^2}<X^{4k(2k-1)}(2k)^{2(r-2)}X^{(r-1)(r-2)}\cdot 2p(\log q)J^2_{k,r-1}(X).$$

The application of Lemma 2.1 yields

$$W^{4k^2}\leqslant c_1X^{4k(2k-1)+(r-1)(r-2)}X^{4k-r(r-1)}p(\log q)=c_1X^{8k^2-2(r-1)}p\log q,$$

where $c_1=c_1(k,r)>0$. Therefore,

$$W\leqslant c_2X^2\bigg(\frac{p\log q}{X^{2(r-1)}}\bigg)^{\frac{1}{4k^2}},\quad c_2^{4k^2}=c_1.$$

Thus we get

$$|S_q(N)|\leqslant\frac{N}{X^2}\cdot c_2X^2\bigg(\frac{p\log q}{X^{2(r-1)}}\bigg)^{\frac{1}{4k^2}}+2pX^2\leqslant N\bigg\{c_2\bigg(\frac{p\log q}{X^{2(r-1)}}\bigg)^{\frac{1}{4k^2}}+\frac{2pX^2}{N}\bigg\}.$$

Now we define X_1 by the equation

$$\bigg(\frac{p\log q}{X_1^{2(r-1)}}\bigg)^{\frac{1}{4k^2}}=\frac{pX_1^2}{N},\quad\text{that is,}\quad X_1=\bigg(\frac{N^{4k^2}\log q}{p^{4k^2-1}}\bigg)^{\frac{1}{8k^2+2(r-1)}}$$

and set $X=[X_1]$. Then

$$\frac{pX^2}{N}\leqslant\frac{pX_1^2}{N}=\bigg(\frac{q\log q}{N^{r-1}}\bigg)^{\frac{1}{4k^2+r-1}}.$$

Since $N > q^{\frac{1}{r-1}}$, we have

$$N > p^{\frac{r}{r-1}} = p^{1+\frac{1}{r-1}} > 10^{\frac{8k^2+2(r-1)}{4k^2+2(r-1)}} (\log q)^{\frac{1}{4k^2+2(r-1)}} p^{1+\frac{1}{4k^2+2(r-1)}}$$
$$= \left(10^{8k^2+2(r-1)} (\log q) p^{4k^2+2r-1}\right)^{\frac{1}{4k^2+2(r-1)}},$$

$$N^{4k^2+2(r-1)} > 10^{8k^2+2(r-1)} p^{4k^2+2r-1} \log q = \frac{(10p)^{8k^2+2(r-1)}}{p^{4k^2-1}} \log q,$$

$$N^{8k^2+2(r-1)} > (10p)^{8k^2+2(r-1)} \frac{N^{4k^2}}{p^{4k^2-1}} \log q, \quad \frac{N^{4k^2}}{p^{4k^2-1}} \log q < \left(\frac{N}{10p}\right)^{8k^2+2(r-1)}$$

and, finally,

$$\left(\frac{N^{4k^2} \log q}{p^{4k^2-1}}\right)^{\frac{1}{8k^2+2(r-1)}} < \frac{N}{10p}.$$

This means that the condition $X < N/(10p)$ is satisfied. Next, the following inequalities are obvious

$$X \geqslant \left[\left(\frac{p^{4k^2} \log q}{p^{4k^2-1}}\right)^{\frac{1}{8k^2+2(r-1)}}\right] = \left[\left(p \log q\right)^{\frac{1}{8k^2+2(r-1)}}\right] > 1.$$

Therefore,

$$|S_q(N)| \leqslant c_3 N \left(\frac{q \log q}{N^{r-1}}\right)^{\frac{1}{4k^2+r-1}},$$

where $c_j = c_j(k,r) > 0$. Setting $k = \frac{1}{2}\, r(r-1) + 1$, we arrive at the assertion of Theorem 1.1. □

PROOF OF THEOREM 1.2. We express the variable n, $N < n \leqslant N_1$, as follows: $n = px + y$, where y runs through the reduced residual system modulus p and x runs through the interval

$$\frac{N-y}{p} < x \leqslant \frac{N_1 - y}{p},$$

for fixed y. Thus we get

$$
\begin{aligned}
S_q(N) &= \sum_{N<n\leqslant N_1} e_q\Big(\frac{a}{n}\Big) = \sum_{y=1}^{p-1} \sum_{(N-y)/p<x\leqslant(N_1-y)/p} e_q\Big(\frac{a}{y+px}\Big) \\
&= \sum_{y=1}^{p-1}\Big(\sum_{N/p<x\leqslant N_1/p} e_q\Big(\frac{a\overline{y}}{1+px\overline{y}}\Big) + 2\theta_1\Big) \\
&= \sum_{y=1}^{p-1} \sum_{N/p<x\leqslant N_1/p} \exp\Big(\frac{2\pi\mathrm{i}}{p^3}\,(a\overline{y}-pa\overline{y}^2x+p^2a\overline{y}^3x^2)\Big) + 2\theta_2 p \\
&= \sum_{y=1}^{p-1} e_q(a\overline{y})S(y) + 2\theta_2 p,
\end{aligned}
$$

where

$$
S(y) = \sum_{N/p<x\leqslant N_1/p} \exp\Big(2\pi\mathrm{i}\Big(\frac{a\overline{y}^3}{p}\,x^2 - \frac{a\overline{y}^2}{p^2}\,x\Big)\Big).
$$

Let $a\overline{y}^3 \equiv A \pmod p$ and $-a\overline{y}^2 \equiv Bp + C \pmod{p^2}$ for some A, B and C satisfying the conditions $(AC, p) = 1$, $|C| < p/2$. Then

$$
S(y) = \sum_{N/p<x\leqslant N_1/p} \exp\Big(\frac{2\pi\mathrm{i}}{p}(Ax^2+Bx)\Big) f(x), \quad f(x) = \mathrm{e}^{2\pi\mathrm{i}(Cx/p^2)}.
$$

Setting

$$
G(v) = \sum_{N/p<x\leqslant v} \mathrm{e}^{\frac{2\pi\mathrm{i}}{p}(Ax^2+Bx)}
$$

and noting that $N_1/p \leqslant 2N/p < p$, by Lemma 2.3 we have $|G(v)| \leqslant c_1\sqrt{p}$ for any v, $N/p < v \leqslant N_1/p$. Hence, the application of Abel summation formula to $S(y)$ together with the inequality

$$
|f'(x)| = \frac{2\pi|C|}{p^2} < \frac{\pi}{p}
$$

yield

$$
S(y) = G\Big(\frac{N_1}{p}\Big) f\Big(\frac{N_1}{p}\Big) - \int_{N/p}^{N_1/p} G(v)f'(v)\,\mathrm{d}v,
$$

$$
|S(y)| \leqslant c_1\sqrt{p}\Big(1 + \int_{N/p}^{N_1/p} \frac{\pi}{p}\,\mathrm{d}v\Big) \leqslant c_2\sqrt{p}.
$$

Therefore,

$$|S_q(N)| \leqslant \sum_{y=1}^{p-1} c_2\sqrt{p} + 2p \leqslant cp\sqrt{p} = c\sqrt{q}.$$

Theorem 1.2 is proved. □

4. *M. A. Korolev: Short memoirs about professor Aleksandar Ivić*

I have heard the name of professor Ivić in the end of 1990s from my advisor professor Anatoly Alekseevich Karatsuba (1937-2008) when he spoke about consecutive zeros γ_n, γ_{n+1} of Hardy's function $Z(t)$. He mentioned the famous result of professor Ivić asserting that $\gamma_{n+1} - \gamma_n \ll \gamma_n^{\alpha+\varepsilon}$, where $\alpha = 0.1559458\ldots$ is the constant coming from the method of exponential pairs.

Figure 1: Professor A. Ivić at "27th Journées Arithmétiques" (Vilnius, Lithuania, 2011)

In May, 2005, A. A. Karatsuba told to us, his students, about one problem posed and partially solved by A. Ivić. I mean the problem concerning the true order of the integral of Hardy's function, that is the function

$$F(T) = \int_0^T Z(t)\,\mathrm{d}t.$$

Professor Ivić proved that $F(T) \ll T^{1/4+\varepsilon}$ for any fixed ε and asked whether is it possible to remove ε. After some efforts, I proved the estimate $F(T) = O(T^{1/4})$. After this result was published (in 2007), professor Karatsuba told me to send this paper to professor Ivić. Unfortunately, I did not keep his answer, but I remember that it was very benevolent. Moreover, professor Ivić sent this paper to professor Matti Jutila because he studied this problem the same time. Later, professor Ivić mentioned our results in his last book "The Theory of Hardy's Z-Function" (2013). He kindly sent me the file of this book a long before its publication.

Figure 2: Professor A. Ivić and M. A. Korolev at "27th Journées Arithmétiques" (Vilnius, Lithuania, 2011)

After millennium, digital technologies began to interfere persistently to our life. In 2010, I had to submit my paper to `arXiv` for the first time. To do this, one need some approval from the person who had already at least one submission to `arXiv`. I do not know why I decided to ask professor Ivić for such an approval. I wrote him an e-mail, and after a day received an answer. He wrote: "Yes, but first I have to look the paper". After checking the text, he made this approval, and thus my first submission was successful.

As usual, the mathematicians who live in different countries meet face-to-face only at the conferences (I do not take into account the format of scientific communication after the pandemic). So, all my meetings with professor Ivić were quite sporadic.

In 2011, I took part in "27th Journées Arithmétiques" in Vilnius University (Vilnius, Lithuania). It was the first experience of foreign conferences for me. I knew that professor Ivić will visit this conference, too, and I took with me the hard copy of his book "The Riemann Zeta-Function. Theory and Applications" to get the autograph. It happend that our talks stay one by one in the schedule of the first day (June, 27). Moreover, professor Ivić was a chairman of our section. During a coffee-break, I came to him and introduced myself. Professor kindly listened to my request, took the book and wrote in Russian: "*Дорогому М. Королеву. С уважением, Александар Ивич. Вильнюс, 27.06.2011.*"[3] I do not remember, but it seems that we did not speak a lot during this conference.

In the beginning of 2012, professor Ivić helped me a lot with the publication of my paper concerning the so-called Gram's law in the theory of Riemann zeta-function. This paper had a quite difficult fate, but finally it was published in "Publications de linstitut Mathématique (Nouvelle série)" due to professor. This year, I met with him in August, at "Elementare und Analytische Zahlentheorie" conference (ELAZ) organized by professor Jörn Steuding in Frankenakademie Schloss Schney (Lichtenfels, Germany). I remember only that we spoke with professor about the famous Russian mathematician A.I. Vinogradov during a coffee-break. Next time we met, I think, at "28th Journées Arithmétiques" at Joseph Fourier University (Grenoble, France), but the details dropped from my memory.

In 2015, I met professor at "29th Journées Arithmétiques" at Institute of Mathematics (Debrecen, Hungary). This time, I took with me my wife and my small daughter. We stayed at small hotel at Peterfia utca. That year, the tram railways coming along this street were closed for reconstruction. So, I had to go to the University by foot. My wife and daughter accompanied me for a half of the way: there was a beautiful Nagyerdei-park near the University. Near the entrance to this park, there was the hotel "Aquaticum" with a very nice swimming-pool. Professor Ivić stayed at this hotel, and we met him several times in the park. One day, during a break for the dinner, I was sitting with my daughter on the bench near the pond with tortoises in Nagyerdei-park. We were reading a book, when I saw professor going to the af-

[3]"*To dear M. Korolev. Faithfully yours, Aleksandar Ivić.*"

Figure 3: Professor A. Ivić at “27th Journées Arithmétiques” (Vilnius, Lithuania, 2011)

ternoon conference session to the University. He saw me, too. I’ve interrupted the reading and put a book on the bench in order to come to professor and greet him, but professor began to gesture: “Please, do not interrupt!” Nevertheless, I came to him, and we spoke to each other ten words.

In May, 2017, Steklov Mathematical Institute together with Moscow State University (Moscow, Russia) organized the memorial conference dedicated to 80th anniversary of professor A.A. Karastuba. To our pleasure, professor Ivić found a possibility to join us (it is necessary to note that he knew A.A. Karatsuba personally). He stayed in guest rooms at Steklov Institute, and it was quite convenient for him because the most part of the sessions was at the same building. We spoke a lot, but I did not remember the details. During the last session (it was at Moscow University), professor Ivić offered me to write a joint paper concerning the value distribution of the function $S(t)$, that is, the argument of the Riemann zeta-function on the critical line. I doubted, because such a paper was written by professor Kai-Man Tsang in 1985. However, professor Ivić insisted that the corresponding problem for any segment $[a, b]$ is still open. This idea seemed me not very good, but I promised to think about it. After some time, I offered him a way how to organize our paper. We decided to include some new results concerning the “precise” formulas for fractional moments of special Dirichlet polynomials approximating $S(t)$ and then to derive the desired value-distribution result as a corollary of this new theorem.

That year, I had a lot of other duties and I could not start our “project” immediately. I told to professor about it, and he said: “Please, tell me the date when you can start this job. I would prefer not to delay this job for an indefinite future. I have no much time”. Yes, at that time, he knew about his illness. I did not realize these words fully, but I promised to begin writing at 1st of October, 2018. This was our last meeting. After starting this job, we exchanged e-mail letters a lot, and I keep our correspondence. It was a great honour for me to work together with professor Ivić. The paper was successfully published at 2019 in “Journal of Number Theory”.

In 2018–2020, we wrote letters to each other with 2–3 month periodicity. I remember that once I asked professor an advice concerning the choice of the problem for one of my students, and he helped me again. In his letters, he wrote about his illness briefly but with unvarnished truth. At the same time, he hoped to overcome this severe affliction and tried to work actively. I think that his family and all his friends had the same hopes … This year, in the 1st day of Orthodox Christmas, January 7, my friend professor Jörn Steuding wrote me that professor Aleksandar Pavle Ivić had passed away.

“Remember, O Lord, the soul of Thy departed servant Aleksandar and forgive him all transgressions, voluntary and involuntary, granting him the kingdom and a portion of Thine eternal good things, and the delight of Thine endless and blessed life”.

5. *I. S. Rezvyakova: Sharing some words about Professor Aleksandar Ivić*

I’ve personally met Professor Aleksandar Ivić for the first time in April 2005 in St. Petersburg during the conference dedicated to the 90th anniversary of academician Yu. V. Linnik. I was introduced to Professor Aleksandar Ivić by my scientific advisor Professor Anatoly Karatsuba, who told me that Aleksandar speaks 5 languages. At this conference there was also Professor Matti Jutila - one of Aleksandar’s best friends.

I would say that Aleksandar was a very open person and was always ready to help his friends. For example, in 1989 at Amalfi conference on number theory he was translating the conference talk of A. A. Karatsuba (who could speak only Russian and German). Professor Aleksandar Ivić was an opponent for numerous thesis defences and was ready to give any help, advice or recommendation letter if one of

Figure 4: Professor A. Ivić among the participants of the conference at Yonsei University (Seoul, South Korea, 2009; fragment)

his colleagues needs. If the rules of the Russian thesis attestation commission would allow an opponent to get a degree not only in Russia (or Soviet Union), then I am sure that Aleksandar Ivić could be an opponent of many thesis defences in Russia too (and this would be of great significance). Similar to Aleksandar's native country Serbia, which is close to both Russia and other Europe, he provided a good link between Russian mathematicians and others. I have met Professor Aleksandar Ivić at several number theoretical conferences: in September 2006 at CIRM (Lumini) at the conference dedicated to 60th birthday of Professor J.-M. Deshouillers, in September 2009 at Yonsei University (Seoul) at the conference organized by Haseo Ki, at Journées Arithmétiques conferences and in Moscow at the conference dedicated to the memory of A. A. Karatsuba. We had few conversations during those conferences and I can say that he was a great teacher not only in mathematics but also in everyday life. Aleksandar can briefly give you some information or advice with an open heart, and you can just realize (at that moment or later) that this resembles your way or philosophy. In the end I would like to say that he was super punctual in replying letters: his answers were very prompt and profound. He followed this principle even in the last months of his life. And I will always remember Aleksandar's artistic Christmas tree with his greetings by email every December for his friends.

Figure 5: Professor A. Ivić among the participants of the "A. A. Karatsuba's 80th Birthday Conference in Number Theory and Applications" (Moscow, Russia, 2017; fragment)

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REFERENCES

[1] A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci. USA **34** (1948), 204–207.

[2] A. Weil, *Basic Number Theory*, 3rd ed. Springer-Verlag, New York Inc., 1974.

[3] H. D. Kloosterman, *On the representation of numbers in the form* $ax^2+by^2+cz^2+dt^2$, Acta Math. **49** (1926), 407–464.

[4] H. Salie, *Uber die Kloostermanschen Summen* $S(u,v;q)$, Math. Z. **34** (1931), 91–109.

[5] A. A. Karatsuba, *Distribution of inverse values in a residue ring modulo a given number*, Russian Acad. Sci. Dokl. Math. **48**:3 (1994), 452–454.

[6] A. A. Karatsuba, *Fractional parts of functions of a special form*, *Izvestiya: Math.* **59**:4 (1995), 721–740.

[7] A. A. Karatsuba, *Analogues of Kloosterman sums*, Izvestiya: Math. **59**:5 (1995), 971–981.

[8] A. A. Karatsuba, *Sums of fractional parts of functions of a special form*, Dokl. Math. **54**:1 (1996), 541.

[9] J. Bourgain, M. Z. Garaev, *Sumsets of reciprocals in prime fields and multilinear Kloosterman sums*, Izv. Math. **78**:4 (2014), 656–707.

[10] M. A. Korolev, *Karatsuba's method for estimating Kloosterman sums*, Sb. Math. **207**:8 (2016), 1142–1158.

[11] A. A. Karatsuba, *New estimates of short Kloosterman sums*, Math. Notes **88**:3 (2010), 347–359.

[12] A.G. Postnikov, *On the sum of characters with respect to a modulus equal to a power of a prime number*, Izv. Akad. Nauk SSSR Ser. Mat. **19**:1 (1955), 11–16.

[13] A.G. Postnikov, *On Dirichlet L-series with the character modulus equal to the power of a prime number*, J. Indian Math. Soc. (N.S.) **20** (1956), 217–226.

[14] M.A. Korolev, *Methods of estimating of incomplete Kloosterman sums*, Chebyshevskii Sb. **17**:4 (2016), 79–109 (in Russian).

[15] S.A. Stepanov, I.E. Shparlinski, *On the estimate of exponential sums with rational and algebraic functions*, In: Authomorphic functions and number theory. Part I. V.A. Bykovsky, N.I. Zavorotny (eds.) Vladivostok, 1989, pp. 5–18 (in Russian).

[16] J. Bourgain, C. Demeter, L. Guth, *Proof of the main conjecture in Vinogradovs mean value theorem for degrees higher than three*, `arXiv:1512.01565v3 [math.NT]` (2016).

[17] J. Bourgain, *On the Vinogradov mean value*, `arXiv:1601.08173 [math.NT]` (2016).

[18] H. Fielder, W. Jurkat, O. Köner, *Asymptotic expansions of finite theta series*, Acta Arith. **32**:2 (1977), 129–146.

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