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THE LEGACY OF ALEKSANDAR IVIĆ

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Dedicated to the Memory of Aleksandar Ivić (1949 – 2020)

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A b s t r a c t. The research and expositive writing of Aleksandar Ivić on analytic number theory is surveyed with an emphasis of his contributions to the theory of Riemann's zeta-function.

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1. *MFO and Troika*

To begin with, let me tell how so-called "Troika" originated. In late seventies, a "Tagung" on elementary and analytic number theory, organized by Hans-Egon Richert, Wolfgang Schwarz and Eduard Wirsing, took place at MFO, that is Mathematisches Forschungsinstitut Oberwolfach in idyllic Schwarzwald. Among the participants there were three relatively young number theorists, Aleksandar Ivić, Yoichi Motohashi and myself, sharing similar mathematical interests. Actually I had met Yoichi at the preceding MFO – conference in 1974, and once Aleksandar now joined us, the troika was finally complete. A hot topic in those days was the large sieve method, and it had occupied us with applications to zero-density estimates in mind. The next overwhelming wave, "Kloostermania", was just to come and to captivate us in due course. We three felt intuitive friendship and so our troika came into existence to exist over the decades to come. In view of this long acquaintance, I take the liberty to call below my respected troika-companions by their first names, Aleksandar and Yoichi.

Though mathematics is basically an individual activity, there are and have been well-known mathematical teams or pairs; the most legendary definitely was the pair Hardy - Littlewood. They agreed about rules for their collaboration. But our troika had no rules, and each of us had ones mathematical profile or style, with common goals however. We worked jointly in all possible pairs, having even one paper with the whole troika as authors. As an example of the spontaneous collaboration, Aleksandar and myself completed a joint paper (on gaps between zeta-zeros on the critical line), from beginning to end, during a single MFO – week.

2. *Zeta-powers in mean*

It is difficult to cover the wealth of Aleksandar's production – over 280 papers, four excellent books, many talks and courses – in a limited survey, so let us focus our eyes to distinguish his most favorite area. It is evidently the Riemann zeta-function, for each of his books is devoted to this mysterious function. Moreover, one of these, namely the Tata-book [4], deals solely with mean value problems. Therefore we are going to consider below mainly Aleksandar's contributions related to certain zeta-averages, more specially to the following:

$$\int_0^T Z^k(t)\,\mathrm{d}t, \quad k = 1, 2, 4, \tag{2.1}$$

where $Z(t)$ is the classical Hardy function. Thus $Z(t)$ is real for real t and $|Z(t)| = |\zeta(1/2 + \mathrm{i}t)|$. Though these may seem to be rather narrow topics, it will turns out that here is, in a nutshell and hidden, a lot of traditional and modern mathematics involved.

3. *The Voronoi and Atkinson formulae*

The celebrated formula due to F. V. Atkinson [1] concerns the mean square of Riemann's zeta-function. For background, let us recall first the famous formula due to G. Voronoi (1904) for the sum function of the ordinary divisor function $d(n)$. It implies a more general sum formula involving terms of the form $d(n)f(n)$, where f is a given function. Aleksandar used Voronoi's summation formula in [2] and [3] to prove the approximate functional equation for $\zeta^2(s)$.

Atkinson's formula gives a rather complicated but surprising precise expression for the integral (2.1) for $k = 2$. This formula, which is nowadays a fundamental result in the zeta-function theory, was curiously "sleeping" about thirty years until D. R. Heath-Brown used it for the proof of his estimate for the twelfth moment of the zeta-function. Aleksander also "rehabilitated" Atkinson's formula including its proof and more subtle developments in his zeta-books [2] – [4].

Atkinson pointed out an analogy of his formula with the Voronoi formula, however without going to details. Indeed, there is a visible similarity, for both formulae involve an exponential sum with values of $d(n)$ as coefficients. Moreover, the first few terms of these sums are close to each other, except that there is an extra factor $(-1)^n$ in the terms of Atkinson's formula. However, it turned out that these difficulties can be overcome, as I observed around 1980. First, the sum in Atkinson's formula can be truncated by a suitable smoothing to make it more tractable. Second, the annoying factors $(-1)^n$ can be inserted into the Voronoi formula if the divisor function $d(n)$ is replaced by the modified function

$$\frac{1}{2}(-1)^n d(n). \tag{3.1}$$

The Voronoi-Atkinson analogy now means that there is a correlation between two error terms: one for the mean square of the zeta-function and the other for the sum formula for the divisor function (3.1). In a series of interesting papers, Aleksandar considered various aspects of this analogy. The correlation is "good" in the sense that the deviation between those two error terms turns out to be significantly smaller, in mean, than both error terms as such. Hypothetically we might say that the correlation is "optimally good" if it is as precise as possible. This would require that the deviation is small throughout, not only in mean.Then the consequence would be quite striking: the estimate $\zeta(\frac{1}{2} + \mathrm{i}t) \ll t^{1/12+\varepsilon}$ would follow! Thus the classical exponent $1/6$ in the Lindelöf problem would be halved. A highlight in this context is Aleksandar's paper [9], where the mean square of the above mentioned deviation is refined to an asymptotic formula with an error term; previously only an upper estimate was known. It follows that the correlation cannot be "too good". A comprehensive and up-dated survey on zeta-values on the critical line is [10], which is one Aleksandar's last papers and thus in some sense his legacy.

4. *The fourth moment of the zeta-function*

The fourth moment in question is the case $k = 4$ of the integral (2.1). No analogue for Atkinson's formula is known for the fourth moment. Nevertheless, *two* ideas of Atkinson turned out to be relevant in the modern theory. The *first* idea goes back to Atkinson's paper from 1941, where he showed a connection between the fourth moment and the additive divisor problem. The latter is concerned with sums of $d(n)d(n+f)$, where the "shift" f is a positive integer. Since Kloosterman sums appear in the theory of the additive divisor problem, we have, after all, a connection between the fourth moment and Kloosterman sums. Here is one of the roots of "Kloostermania". The traditional argument made use of estimates of individual Kloosterman sums. A fundamental novelty was the trace formula due to N. V. Kuznetsov around 1980 expressing sums of Kloosterman sums in terms of the spectral theory of automorphic functions. In this way it was possible to show cancelation between different Kloosterman sums, a new feature which gave significant improvements in applications such as the estimation of the error term for the fourth moment.

The *second* idea, going back to Atkinson's proof of his celebrated formula discussed above, is so-called Atkinson-dissection. He considered two zeta-values giving rise to pairs m, n of positive integers, and the dissection of these pairs into three subsets was made by the conditions $m < n$, $m = n$, and $m > n$. Yoichi's idea was to generalize Atkinson's argument to the fourth moment. Thus one has now a product of fourth zeta-values and there are quadruples k, ℓ, m, n to be dissected. The conditions for the dissection are $km < \ell n$, $km = \ell n$, and $km > \ell n$. Eventually, the argument leads to Kloosterman sums and spectral theory. The final result is a nicely explicit formula for the fourth moment and its error term is about of the order $\ll T^{2/3}$.

In their remarkable joint work, Aleksandar and Yoichi [12], [13] used the above mentioned formula to prove mean value and omega estimates for the error term. Analogous results were also obtained, in [14], for the additive divisor problem. Aleksandar gives a detailed exposition of the theory of the fourth moment in his Tata-lectures [4].

One of Aleksandar's highlights is related to the formula for the fourth moment. It involved so-called "central values" $H(1/2)$, where $H(s)$ is an L-function attached to a Maass form. The functional equation for $H(s)$ and the convexity principle give an estimate for $H(1/2)$ which may be called a "trivial" estimate. The problem concerning stronger "subconvexity" estimates is highly non-trivial, for the arithmetic nature of the coefficients of the L-functions is quite mysterious. As a remarkable breakthrough, Aleksandar [7] obtained the first known subconvexity estimate for $H(1/2)$. The proof was based on a spectral cubic mean for the central values and on their non-negativity. Thus an estimate for a sum gave an estimate for a single term.

Here is a personal story to be told. Aleksandar did the above mentioned work around 2000, the Millenium. And what else happened in those days? As many perhaps remember – at least those living in the Balkan area – that a war was then raging in Serbia and Beograd was even bombed. A miserable situation indeed! In 1999 my colleague Tauno Metsänkylä and myself where organizing a conference in memory of our late teacher Professor Kustaa Inkeri. The troika-friends Aleksandar and Yoichi were of course among the invited speakers. What about Aleksandar? Let me quote from the preface of the conference volume: "… the plenary lecture by A. Ivić (Beograd), who due to unfortunate circumstances could not attend the symposium, was presented by M. Jutila". The main theme of Aleksandar's paper [6] published in the conference volume was the role of Mellin transforms in analytic number theory. The troika paper [11] was also devoted to the same topic. In general, integral transforms occurred repeatedly in our works, both as methods and as subjects of study.

The coincidence of the war with the Turku conference was of course painful for the organizers, and definitely the situation was even more painful for Aleksandar. It is amazing how he managed to do his excellent work in those circumstances. Maybe mathematics is a rescue in difficult times. Indeed, there are known examples from the years of World War II.

5. *The primitive of Hardy's function*

G. H. Hardy proved in 1914 that Riemann's zeta-function has infinitely many zeros on the critical line. For this purpose he needed a non-trivial estimate for the primitive of the function $Z(t)$ bearing now his name. Thus we are now concerned with the case $k = 1$ of the integral (2.1). Let $F(T)$ denote the integral in question. An estimate of the form $F(T) = O(T^c)$ for some constant c with $0 < c < 1$ suffices for a proof of Hardy's theorem. For instance, in the zeta-book of Titchmarsh the value $c = 7/8$ is given. But what is the smallest possible value for c? In a pioneering paper, Aleksandar [8] tackled this question obtaining the drastically smaller value $c = 1/4 + \varepsilon$ for any small $\varepsilon > 0$. Moreover, he stated two interesting conjectures:

$$F(T) = O(T^{1/4}), \quad F(T) = \Omega(T^{1/4}).$$

These conjectures were proved by M. A. Korolev and myself independently and by different methods (I used an analogue of Atkinson's formula for $F(T)$). These developments gave rise to further research by Aleksandar and others. In Finland we were pleased to have Aleksandar as one of the lecturers in the Arctic Number Theory School in May 2011 at the University of Helsinki. On the basis of his lectures on Hardy's function he wrote the monograph [5] which is the first book dedicated solely to this function. In addition to classical and modern material, it contains open problems and conjectures remaining a valuable handbook in future.

Aleksandar's Helsinki lectures were just one of the instances to enjoy his gentlemanly company in Finland, most often in Turku. In Finland he made joint work with Tom Meurman and Eero Saksman beside me and he was always ready to put his expertise at disposal in various academic issues. The warmth and humour of his personality will remain in the minds of his many friends in Finland.

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