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TWO PEROV TYPE GENERALIZED GRAPH CONTRACTIONS

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Dedicated to Professor Aleksandar Ivić (1949–2020)

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A b s t r a c t. In [*Približ. Metod. Rešen. Differencial'. Uravnen.* **2** (1964), 115 − 134] *A. I. Perov generalized the Banach contraction principle by employing matrices instead of contraction constants. In this paper, we introduce and study two kind of Perov type contractive mappings. Fixed point results of such mappings are obtained in the framework of cone b-metric spaces endowed with a graph and associated with a generalized c-distance. Our results and methods are new. Some corollaries and examples are presented to support the main result proved herein. These results unify, extend and generalize various comparable results in the literature. plications to quasi-singular integrals.*

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1. *Introduction*

Fixed point theory is a powerful and important tool in the study of various nonlinear phenomena. The interplay between the notion of a nearness among abstract objects of a set and fixed point theory is very strong and fruitful. This gives rise to an interesting branch of nonlinear functional analysis called metric fixed point theory. This theory is studied in the framework of a set equipped with some notion of a distance along with appropriate mappings satisfying certain contraction conditions and has many applications in economics, computer science and other related disciplines.

The concept of a b-metric is one of the important measure of nearness defined by Bakhtin [3] and Boriceanu [6]. The reader interested in fixed point results in setup of b-metric spaces is referred to [2, 7, 16, 21].

Huang and Zhang [23] defined the concept of cone metric space by replacing the range of a distance function with an ordered normed space equipped with an order structure induced by a cone and proved some fixed point results for contraction type mappings on such spaces [40]. After that, the concept of b-metric space was extended to cone b-metric space or cone metric type space ([13, 26]).

Kada et al. [28] introduced the concept of w-distance on metric spaces and solved non-convex minimization problems. Cho et al. [8] defined the notion of a c-distance which is the cone version of a w-distance.

Recently, Hussain et al. [25] defined the concept of wt-distance on b-metric spaces and proved some fixed point theorems under a wt-distance in partially ordered b-metric spaces (also, see [19, 29]). Bao et al. [4] defined generalized c-distance in cone b-metric spaces and obtained some fixed point results in ordered cone b-metric spaces.

On the other hand, Perov [33] generalized the Banach contraction principle by employing matrices instead of contraction constants. The extension of Perov theorem on a cone metric space was presented in [14, 12].

Recently, Abbas et al. [1] obtained common fixed points results of multivalued Perov type contractions on cone metric spaces with a directed graph.

The aim of this paper is to establish the existence and uniqueness of fixed points for Perov type mappings defined on cone b-metric spaces endowed with a graph and associated with a generalized c-distance. Our results generalize and extend various results in the existing literature. It is worth mentioning that we have employed the weaker version of continuity of the mapping called orbitally G-continuity.

2. *Preliminaries*

Let E be a real Banach space. A subset P of E is called a cone if and only if:
(i) P is nonempty, closed and $P \neq \{\theta\}$ (where θ is the zero element of E);
(ii) $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ implies that $ax + by \in P$;
(iii) $P \cap (-P) = \{\theta\}$.
Partial ordering on E is defined with help of a cone P as follows:

$x \preceq y$ if and only if $y - x \in P$.

We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$ and $x \ll y$ stands for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. Unless or otherwise stated, it is assumed that E is a Banach space, P is a cone in E with $\operatorname{int} P \neq \emptyset$ and $\preceq$ is partial ordering on E induced by P.

A cone P is normal or semi monotone if

$$\inf\{\| x + y \|: x, y \in P \text{ and } \| x \| = \| y \| = 1\} > 0$$

or equivalently, if there is a number $K > 0$ such that for all $x, y \in P$, $\theta \preceq x \preceq y$ implies that $\|x\| \leq K \|y\|$. The least positive number satisfying the above inequality is called a normal constant of P. If $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, then $x \preceq y$ means that $x_i \leq y_i$, $i = 1, \ldots, n$. In this case, the set $P = \{x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}$ is a normal cone with $K = 1$.

Lemma 2.1. *Let $u, c \in E$ and $\{x_n\}$ a sequence in E. Then we have the following properties:*

(p_1) *If $u \preceq \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = \theta$;*

(p_2) *If $c \in \operatorname{int} P$, $\theta \preceq x_n$ and $x_n \to \theta$, then there exists n_0 such that for all $n > n_0$ we have $x_n \ll c$.*

Definition 2.1 ([13, 26])**.** *Let X be a nonempty set and $s \geq 1$ a given real number. A mapping $d : X \times X \to E$ is said to be a cone b-metric on X if for any $x, y, z \in X$, the following conditions hold:*

(d_1) *$\theta \preceq d(x, y)$ and $d(x, y) = \theta$ if and only if $x = y$;*

(d_2) *$d(x, y) = d(y, x)$;*

(d_3) *$d(x, z) \preceq s(d(x, y) + d(y, z))$.*

The pair (X, d) is called a cone b-metric space.

Obviously, for $s = 1$, the cone b-metric space is a cone metric space. Moreover, if X is any nonempty set, $E = \mathbb{R}$ and $P = [0, \infty)$, then cone b-metric on X is a b-metric on X.

Definition 2.2 ([4])**.** Let (X, d) be a cone b-metric space and $s \geq 1$ a given real number. A mapping $q : X \times X \to E$ is said to be a generalized c-distance on X if for any $x, y, z \in X$, the following properties are satisfied:

(q_1) $\theta \preceq q(x, y)$,

(q_2) $q(x,z) \preceq s[q(x,y)+q(y,z)]$,

(q_3) If for all $n \geq 1$, $q(x,y_n) \preceq u$ for some $u = u_x$, then $q(x,y) \preceq su$, where $\{y_n\}$ is a sequence in X which converges to $y \in X$;

(q_4) for any $c \in \operatorname{int} P$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply that $d(x,y) \ll c$.

If (X,d) is a b-metric space, $E = \mathbb{R}$ and $P = [0,\infty)$. Then, wt-distance [25] on a b-metric space X is a generalized c-distance. But the converse does not hold.

Furthermore, if $s = 1$, the generalized c-distance is a c-distance defined in [8]. Also, if in the above definition, we take $s = 1$, $E = \mathbb{R}$ and $P = [0,\infty)$, then we obtain the definition of w-distance [28].

Note that, if q is a generalized c-distance, then $q(x,y) = \theta$ is not necessarily equivalent to $x = y$. Moreover, $q(x,y) = q(y,x)$ does not necessarily hold for all $x,y \in X$.

Let us point out that the definitions of convergent sequence, Caushy sequence and completnes of the space are the same in b- cone metric space as in cone metric space (see e.g., [1, 4, 14]).

Lemma 2.2. *Let (X,d) be a cone b-metric space and q a generalized c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, $\{u_n\}$ and $\{v_n\}$ two sequences in P converging to θ. For any $x,y,z \in X$, the following conditions hold:*

(qp_1) *if for all $n \in \mathbb{N}$, $q(x_n,y) \preceq u_n$ and $q(x_n,z) \preceq v_n$, then $y = z$. In particular, if $q(x,y) = \theta$ and $q(x,z) = \theta$, then $y = z$;*

(qp_2) *if for all $n \in \mathbb{N}$, $q(x_n,y_n) \preceq u_n$ and $q(x_n,z) \preceq v_n$, then $\{y_n\}$ converges to z;*

(qp_3) *if for $m,n \in \mathbb{N}$, with $m > n$, we have $q(x_n,x_m) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X;*

(qp_4) *if for all $n \in \mathbb{N}$, $q(y,x_n) \preceq u_n$ then $\{x_n\}$ is a Cauchy sequence in X.*

PROOF. Following arguments similar to those given in [8], the Lemma follows.

On the other hand, the interplay between the order among abstract objects of underlying mathematical structure and fixed point theory is very strong and fruitful. This gives rise to an interesting branch of nonlinear functional analysis called order oriented fixed point theory. This theory is studied in the framework of a partially ordered sets along with appropriate mappings satisfying certain order conditions and has many applications in economics, computer science and other related disciplines.

Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [36], and then by Nieto and Lopez [31].

Jachymski [27] introduced a new approach in metric fixed point theory by replacing order structure with a graph structure on a metric space. In this way, the results obtained in ordered metric spaces are generalized. Employing the notion of orbits, Nicolae et al. [30] obtained some fixed point results for a new type of contraction mappings and for G-asymptotic contraction mapping in metric spaces endowed with a graph. Bojor [5] defined the notion of G-Reich type mappings and obtained a fixed point theorem for such mappings in metric spaces endowed with a graph. Fallahi and Aghaniansyz [18] introduced G-quasi-contractions using directed graphs in metric spaces with a graph and showed that this contraction generalizes a large number of contractions. Cholamjiak [9] proved fixed point theorems for a Banach type contractive mapping on a complete Topological vector-cone metric spaces endowed with a graph. Also, Hussain et al. [24] proved new fixed point results for graphic weak ψ-contractive mappings. Recently, in 2018, Fallahi, Petrusel and Rad [20] have studied the existence of the fixed points for pointwise Chatterjea type mappings with respect to a c-distance in cone metric spaces endowed with a graph.

The following definitions and notations will be needed in the sequel.

Let (X, d) be a cone b-metric space and Δ denotes the diagonal of $X \times X$. Let G be a directed graph such that set $V(G)$ of its vertices is X and $E(G)$ be the set of edges of a graph G which contains all loops; that is, $(x, x) \in \Delta \subset E(G)$ for all $x \in X$. Assume further that graph G has no parallel edges.

Thus one can identify the graph G with the ordered pair $(V(G), E(G))$. If $x, y \in X$, then a finite sequence $\{x_i\}_{i=0}^k$ consisting of $k+1$ vertices is called a path in G from x to y whenever $x_0 = x$, $x_k = y$ and (x_{i-1}, x_i) is an edge of G for $i = 1, \dots, k$.

The graph G is called connected if there exists a path in G between any two vertices of G. The symbols G^{-1} and $\widetilde{G}$ denote the graph which is obtained from G by reversing the directions of its edges and an undirected graph obtained from G by ignoring the directions of the edges, respectively. In other words, $V(G^{-1}) = V(\widetilde{G}) = X$, $E(G^{-1}) = \big\{(x, y) : (y, x) \in E(G)\big\}$ and $E(\widetilde{G}) = E(G) \cup E(G^{-1})$.

We denote by $\mathrm{Fix}(T)$ the set of all fixed points of a self mapping T on X and X_T the set of all points $x \in X$ such that (x, Tx) is an edge of a graph G, that is,

$$X_T = \{x \in X : \ (x, Tx) \in E(G)\}.$$

Following is the analogue of the concept of Picard operators [34] in cone b-metric spaces.

Definition 2.3. Let (X, d) be a cone b-metric space. A mapping $T : X \to X$ is called a Picard operator if T has a unique fixed point x_* in X and $T^n x \to x_*$ for any $x \in X$.

Consistent with Jachymski [27, Definition 2.4], we introduce the concept of orbitally G-continuous for self mapping T on a cone b-metric space (see also [10]).

Definition 2.4. A mapping $T : X \to X$ is called orbitally G-continuous on X if for any $x, y \in X$ and a sequence $\{b_n\}$ of positive integers with $(T^{b_n}x, T^{b_{n+1}}x) \in E(G)$ for all $n \geq 1$, such that $T^{b_n}x \to y$ implies $T(T^{b_n}x) \to Ty$.

Note that a continuous mapping on a cone b-metric space is orbitally G-continuous for all graphs G but the converse is not true in general.

Let $M_{n\times n}(\mathbb{R}^+)$ be the set of all $n \times n$ matrices with non negative elements. It is well known that if A is any square matrix of order n, then $A(P) \subset P$ if and only if $A \in M_{n,n}(\mathbb{R}^+)$.

A matrix $A \in M_{n,n}(\mathbb{R}^+)$ is said to be convergent to zero if $A^n \longrightarrow \Theta$ as $n \longrightarrow \infty$, where Θ is the null matrix of size n.

Regarding this class of matrices we have the following classical result in matrix analysis (see [1], [35], [37] and [39]).

Theorem 2.1. *Let $A \in M_{n,n}(\mathbb{R}^+)$. The following statements are equivalent:*

1. $A^n \to \Theta$, *as* $n \to \infty$*;*

2. *the eigenvalues of A lies in the open unit disc, that is, $|\lambda| < 1$, for all $\lambda \in \mathbb{C}$ with* $\det(A - \lambda I_n) = 0$*;*

3. *the matrix $I_n - A$ is non-singular and*

$$(I_n - A)^{-1} = I_n + A + A^2 + \cdots + A^m + \cdots ;$$

4. *the matrix $(I_n - A)$ is non-singular and $(I_n - A)^{-1}$ has nonnegative elements;*

5. *the Av and v^tA converges to zero for each $v \in \mathbb{R}^+$.*

Perov [33] obtained the following generalization of a Banach contraction principle.

Theorem 2.2. *Let (X, d) be a complete generalized metric space, $T : X \to X$ and $A \in M_{n,n}(\mathbb{R}^+)$ a matrix convergent to zero. If for any $x, y \in X$, we have*

$$d(T(x), T(y)) \leq A(d(x, y)).$$

Then the following statements hold:

1. *T has a unique fixed point $x^* \in X$;*

2. *The Picard iterative sequence $x_n = T^n(x_0)$, $n \in \mathbb{N}$ converges to x^* for all $x_0 \in X$;*

3. *$d(x_n, x^*) \leq A^n(I_n - A)^{-1}(d(x_0, x_1))$, $n \in \mathbb{N}$;*

4. *if $S : X \to X$ satisfies the condition $d(T(x), T(x)) \leq c$ for all $x \in X$ and some $c \in \mathbb{R}^n$, then for the sequence $y_n = S^n(x_0)$, $n \in \mathbb{N}$, the following inequality*

$$d(y_n, x^*) \leq (I_n - A)^{-1}(c) + A^n(I_n - A)^{-1}(d(x_0, x_1))$$

is valid for all $n \in \mathbb{N}$.

The role of vector valued norm is important in the study of semi linear operator systems. For details, we refer to [32] and [35].

We write $\mathcal{B}(E)$ for the set of all bounded linear operators on E and $L(E)$ for the set of all linear operators on E. Note that $\mathcal{B}(E)$ is a Banach algebra, and if $A \in \mathcal{B}(E)$, then

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \inf_n \|A^n\|^{\frac{1}{n}}$$

is the spectral radius of A. We write $\mathcal{B}(E)^{-1}$ for the set of all invertible elements in $\mathcal{B}(E)$. Let us remark that if $r(A) < 1$, then

1. Series $\sum\limits_{n=0}^{\infty} A^n$ is absolutely convergent;

2. $I - A$ is invertible in $\mathcal{B}(E)$.

$$\sum_{n=0}^{\infty} A^n = (I - A)^{-1}.$$

If $A, B \in \mathcal{B}(E)$ and $AB = BA$ then $r(AB) \leq r(A)r(B)$.

If $A \in \mathcal{B}(E)$ and $A^{-1} \in \mathcal{B}(E)$ exists, then $r(A^{-1}) = 1/r(A)$.

Furthermore, if $\| A \| < 1$, then $I - A$ is invertible and

$$\left\|(I - A)^{-1}\right\| \leq \frac{1}{1 - \|A\|}.$$

The above is known as Geometric series theorem. Note that $r(A) \leq \|A\|$.

Remark 2.1 ([15]). Let X be a cone metric space, $P \subseteq E$ cone in E and $A : E \to E$ a linear operator. The following conditions are equivalent:

1. A is increasing, that is, $x \preceq y$ implies that $A(x) \preceq A(y)$;

2. A is positive, that is, $A(P) \subset P$.

Remark 2.2. Let $P \subseteq E$ be a cone in E and $A : E \to E$ a bounded linear operator with $r(A) < 1$ and $A(P) \subset P$. If for any u in P, we have

$$u \preceq A(u), \tag{1.1}$$

then $u = 0$.

PROOF. From equation (1.1), we have

$$u \preceq (I - A)^{-1}(0) = 0$$

implies $u = 0$.

Now we give the following definition:

Definition 2.5 (PEROV TYPE GRAPH CONTRACTIONS)**.** Let (X, d) be a complete cone b-metric space associated with the generalized c-distance q and endowed with the graph G and $s \geq 1$ be a given real number. A map $T : X \to X$ is said to be a

1. PEROV TYPE-I GRAPH CONTRACTION if
   (a) T preserves the edges of G; that is, $(x, y) \in E(G)$ implies that $(Tx, Ty) \in E(G)$ for all $x, y \in X$; and
   (b) there exists a linear bounded operator $A : E \to E$ with $r(sA) < 1$ and $A(P) \subset P$ such that
   $$q(Tx, Ty) \preceq A\left(U(x, y)\right),$$
   hold, where
   $$U(x, y) \in \{q(x, y), q(x, Tx)), q(y, Ty)\}$$
   for all $x, y \in X$ with $(x, y) \in E(G)$.
2. PEROV TYPE-II GRAPH CONTRACTION if
   (a) T preserves the edges of G; that is, $(x, y) \in E(G)$ implies that $(Tx, Ty) \in E(G)$ for all $x, y \in X$; and

(b) there exists linear bounded operators $A_1, A_2, A_3 : E \to E$ such that $A_k(P) \subset P$ for $k = 1, 2, 3$, exists $(I - A_3)^{-1}$, $(I - A_3)^{-1} \in \mathcal{B}(E)$, $r[s(I - A_3)^{-1}(A_1 + A_2)] < 1$ and

$$q(Tx, Ty) \preceq A_1(q(x, y)) + A_2(q(x, Tx)) + A_3(q(y, Ty))$$

hold for all $x, y \in X$ with $(x, y) \in E(G)$.

Remark 2.3. Let us remark that instead of more general condition $r(sA) < 1$ (in (1) (b))we could use the inequality $\|sA\| < 1$. Concerning (2), (b), if

$$s(\|A_1\| + \|A_2\|) + \|A_3\| < 1$$

then exists $(I - A_3)^{-1} \in \mathcal{B}(E)$ and $r[s(I - A_3)^{-1}(A_1 + A_2)] < 1$. To check assumptions for norms are sometimes more convenient for various applications.

A clique in an undirected graph $G = (V, E)$ is a subset of the vertex set $W \subset V$, such that for every two vertices in W, there exists an edge connecting the two. This is equivalent to saying that the subgraph induced by W is complete, that is, for every $x, y \in W(G)$, we have $(x, y) \in E(G)$.

3. *Main results*

We start now with the following main result.

Theorem 3.1. *Let (X, d) be a complete cone b-metric space associated with the generalized c-distance q and endowed with the graph G and $s \geq 1$ be a given real number. If the orbitally G-continuous mapping $T : X \to X$ is a Perov type-I graph contraction, then T has a fixed point if and only if $X_T \neq \emptyset$. Moreover, for any x_* in X with $Tx_* = x_*$, we have $q(x_*, x_*) = \theta$. Also, if the subgraph of G with the vertex set* $\text{Fix}(T)$ *is connected, then the restriction of T to X_T is a Picard operator.*

PROOF. As $\text{Fix}(T) \subseteq X_T$, if T has a fixed point then X_T is nonempty. Let x_0 be a given point in X_T. Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$. Clearly, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Thus

$$q(x_n, x_{n+1}) = q(Tx_{n-1}, Tx_n) \preceq A\left(U\left(x_{n-1}, x_n\right)\right),$$

where

$$\begin{aligned} U\left(x_{n-1}, x_n\right) &\in \{q(x_{n-1}, x_n), q(x_{n-1}, Tx_{n-1}), q(x_n, Tx_n)\} \\ &= \{q(x_{n-1}, x_n), q(x_n, x_{n+1})\}. \end{aligned}$$

Now, if $U(x_{n-1}, x_n) = q(x_n, x_{n+1})$, then $q(x_n, x_{n+1}) \preceq A(q(x_n, x_{n+1}))$, which by Remark 2.2 implies that $q(x_n, x_{n+1}) = 0$. Thus

$$q(x_n, x_{n+1}) \preceq A(q(x_{n-1}, x_n))$$

for all $n \in \mathbb{N}$. Continuing this way, we have

$$q(x_n, x_{n+1}) \preceq A^n(q(x_0, x_1)) \tag{3.1}$$

for all $n \in \mathbb{N}$. Let $m > n$.

Now, It follows from (q_2) and $0 \leq r(sA) < 1$ that

$$\begin{aligned} q(x_n, x_m) &\preceq s[q(x_n, x_{n+1}) + q(x_{n+1}, x_m)] \\ &\preceq sq(x_n, x_{n+1}) + s[sq(x_{n+1}, x_{n+2}) + q(x_{n+2}, x_m)] \\ &\vdots \\ &\preceq sq(x_n, x_{n+1}) + s^2 q(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} q(x_{m-1}, x_m)] \\ &\preceq (sA^n + s^2 A^{n+1} \cdots + s^{m-n} A^{m-1})(q(x_0, x_1)) \\ &\preceq sA^n (I - sA)^{-1} (q(x_0, x_1)). \end{aligned}$$

Let $c \gg 0$. Choose $\delta > 0$ such that $c + N_\delta(\theta) \subseteq P$, where $N_\delta(\theta) = \{x \in E : \|x\| < \delta\}$. Also, choose $N_1 \in \mathbb{N}$ such that $sA^n(I - sA)^{-1}(q(x_0, x_1)) \in N_\delta(\theta)$ for all $n > N_1$. Thus for all $m > n > N_1$,

$$q(x_n, x_m) \preceq sA^n(I - sA)^{-1}(d(x_0, x_1)) \ll c$$

implies $\{x_n\}$ is a Cauchy sequence in X. Next we assume that there exists a point $x_* \in X$ such that $x_n = T^n x_0 \to x_*$ as $n \to \infty$. As $x_0 \in X_T$, $(T^n x_0, T^{n+1} x_0) \in E(G)$ for all $n \geq 0$. From orbital G-continuity of T, we get $T^{n+1} x_0 \to Tx_*$ and hence $Tx_* = x_*$. Now if $Tx_* = x_*$ for any $x_* \in X$. Then, since map T is form Perov type-I graph contraction, we have

$$\begin{aligned} q(x_*, x_*) &= q(Tx_*, Tx_*) \\ &\preceq A(U(x_*, x_*)), \end{aligned}$$

where

$$\begin{aligned} U(x_*, x_*) &\in \{q(x_*, x_*), q(x_*, Tx_*), q(x_*, Tx_*)\} \\ &= \{q(x_*, x_*)\}. \end{aligned}$$

Thus,

$$q(x_*, x_*) \preceq A\left(q\left(x_*, x_*\right)\right)$$

and therefore by Remark 2.2, we have $q(x_*, x_*) = \theta$.

Now, if the subgraph of G with the vertex set $\mathrm{Fix}(T)$ is connected and $x_{**} \in X$ is a fixed point of T. Then there exists a path $\{x_i\}_{i=0}^N$ in G from x_* to x_{**} such that $x_1, \ldots, x_{N-1} \in \mathrm{Fix}(T)$; that is, $x_0 = x_*$, $x_N = x_{**}$ and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, \ldots, N-1$. As $q(x_{i+1}, x_{i+1}) = q(x_i, x_i) = \theta$, we have

$$q(x_i, x_{i+1}) = q(Tx_i, Tx_{i+1}) \preceq A\left(U\left(x_i, x_{i+1}\right)\right),$$

where

$$\begin{aligned} U\left(x_i, x_{i+1}\right) &\in \{q(x_i, x_{i+1}), q(x_i, Tx_i), q(x_{i+1}, Tx_{i+1})\} \\ &= \{q(x_i, x_{i+1}), \theta\}. \end{aligned}$$

Thus, $q(x_i, x_{i+1}) \preceq A\left(q\left(x_i, x_{i+1}\right)\right)$ and it follows from Remark 2.2 that

$$q(x_i, x_{i+1}) = \theta.$$

Since $q(x_i, x_i) = \theta$ and $q(x_i, x_{i+1}) = \theta$, by Definition 2.2 we have $d(x_i, x_{i+1}) = 0$; that is, $x_i = x_{i-1}$. Consequently,

$$x_* = x_0 = x_1 = \cdots = x_{N-1} = x_N = x_{**}$$

and hence the fixed point of T is unique and the restriction of T to X_T is a Picard operator.

Example 3.1. Let $X = [0, 1]$, $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq 0\} \subset E$. Define $d : X \times X \to E$ by $d(x, y) = (|x - y|^p, \alpha|x - y|^p)$, where $\alpha \geq 1$ and $p > 1$. Then (X, d) is a cone b-metric space with $s = 2^{p-1}$. We fix $p = 2$ (so $s = 2$) and let $q : X \times X \to E$ be given by

$$q(x, y) = \left(y^p, \alpha y^p\right).$$

Then q is a generalized c-distance. Define $T : X \to X$ by

$$T(x) = \begin{cases} \dfrac{x^2}{4} & \text{if } x \neq 1, \\ \dfrac{x}{2} & \text{if } x = 1. \end{cases}$$

Clearly, T is not continuous at $x = 1$. Now assume that X is endowed with a graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x, x) : x \in X\}$. Note that for any $x, y \in X$ with $(x, y) \in E(G)$, we have $x = y$. If $x, y \in X$ and $\{b_n\}$ is a sequence of positive integers with $(T^{b_n}x, T^{b_{n+1}}x) \in E(G)$ for all $n \geq 1$ such that $T^{b_n}x \to y$, then $\{T^{b_n}x\}$ is necessarily a constant sequence. Thus, for some y in X, we have $T^{b_n}x = y$ for all $n \geq 1$ and hence $T(T^{b_n}x) \to Ty$. Define a linear bounded operator $A : E \to E$ by

$$A = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ 0 & \frac{3}{7} \end{bmatrix}.$$

Set $\|\mathbf{x}\| = \max\{|x_1|, |x_2|\}$ for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_i \in \mathbb{R}$, $i = 1, 2$. For arbitrary $\mathbf{x} \in E$,

$$\begin{aligned} \|A\mathbf{x}\| &= \max\left\{\left|\frac{1}{7}x_1 + \frac{2}{7}x_2\right|, \left|0x_1 + \frac{3}{7}x_2\right|\right\} \\ &\leq \max\left\{\frac{3}{7}\max\{|x_1|, |x_2|\}, \frac{3}{7}|x_2|\right\} = \frac{3}{7}\|\mathbf{x}\|. \end{aligned}$$

Thus, $\|A\| \leq \frac{3}{7}$. If $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\|\mathbf{x}\| = 1$, then

$$\|A\mathbf{x}\| = \max\left\{\left|\frac{1}{7} + \frac{2}{7}\right|, \left|0 + \frac{3}{7}\right|\right\} = \max\left\{\frac{3}{7}, \frac{3}{7}\right\} = \frac{3}{7}.$$

Hence $\|A\| = 3/7$. Therefore,

$$\|sA\| = \frac{6}{7} < 1 \quad \text{and} \quad A(P) \subset P.$$

Let $(x, y) \in X$ with $(x, x) \in E(G)$. If $x \neq 1$, then

$$\begin{aligned} q(Tx, Tx) &= \begin{bmatrix} \frac{x^{2p}}{4^p} \\ \alpha(\frac{x^{2p}}{4^p}) \end{bmatrix} \\ &\leq \begin{bmatrix} \left(\frac{1}{7} + \frac{2\alpha}{7}\right)x^p \\ \left(\frac{3\alpha}{7}\right)x^p \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ 0 & \frac{3}{7} \end{bmatrix} \begin{bmatrix} x^p \\ \alpha x^p \end{bmatrix} \\ &= A(U(x, x)), \end{aligned}$$

where

$$U\left(x,x\right)=q\left(x,y\right)\in\left\{q\left(x,y\right),q(x,Tx),q(y,Ty)\right\}.$$

If $x=1$, then we have

$$\begin{aligned}q(Tx,Tx)&=\begin{bmatrix}\frac{1}{2^p}\\ \frac{\alpha}{2^p}\end{bmatrix}\\ &\le\begin{bmatrix}\frac{1}{7}+\frac{2\alpha}{7}\\ 0+\frac{3\alpha}{7}\end{bmatrix}\\ &=\begin{bmatrix}\frac{1}{7}&\frac{2}{7}\\ 0&\frac{3}{7}\end{bmatrix}\begin{bmatrix}1\\ \alpha\end{bmatrix}\\ &=A(U(x,x)),\end{aligned}$$

where

$$U\left(x,x\right)=q\left(x,y\right)\in\left\{q\left(x,y\right),q(x,Tx),q(y,Ty)\right\}.$$

Furthermore, $(0,T0)=(0,0)\in E(G)$, so $X_T\neq\emptyset$. Thus, all the conditions of Theorem 3.1 are satisfied. Moreover, $x_*=0$ is a fixed point of T and $q(0,0)=0$.

Theorem 3.2. *Let (X,d) be a complete cone b-metric space associated with the generalized c-distance q and endowed with the graph G and $s\geq 1$ be a given real number. If the orbitally G-continuous mapping $T:X\to X$ is a Perov type-II graph contraction, then T has a fixed point if and only if $X_T\neq\emptyset$. Moreover, for any x_* in X with $Tx_*=x_*$, we have $q(x_*,x_*)=\theta$. Also, if the subgraph of G with the vertex set* $\mathrm{Fix}(T)$ *is connected, then the restriction of T to X_T is a Picard operator.*

PROOF. As $\mathrm{Fix}(T)\subseteq X_T$, if T has a fixed point then X_T is nonempty. Let x_0 be a given point in X_T. Define a sequence $\{x_n\}$ in X by $x_n=Tx_{n-1}=T^nx_0$. Clearly, $(x_n,x_{n+1})\in E(G)$ for all $n\in\mathbb{N}$. Thus

$$\begin{aligned}q(x_n,x_{n+1})&=q(Tx_{n-1},Tx_n)\\ &\preceq A_1\left(q(x_{n-1},x_n)\right)+A_2\left(q(x_{n-1},Tx_{n-1})\right)+A_3\left(q(x_n,Tx_n)\right)\\ &\preceq\big(A_1+A_2\big)(q(x_{n-1},x_n))+A_3\left(q(x_n,x_{n+1})\right),\end{aligned}$$

and so

$$(I-A_3)\big(q(x_n,x_{n+1}\big)\preceq\big(A_1+A_2\big)\big(q(x_{n-1},x_n)\big).\tag{3.2}$$

. Hence,

$$q(x_n, x_{n+1}) \preceq (I - A_3)^{-1} (A_1 + A_2) (q(x_{n-1}, x_n))$$

for all $n \in \mathbb{N}$. Thus,

$$q(x_n, x_{n+1}) \preceq A (q(x_{n-1}, x_n))$$

for all $n \in \mathbb{N}$, where $A = (I - A_3)^{-1} (A_1 + A_2)$. Now, as in the proof of Theorem 3.1 we conclude that $\{x_n\}$ is a Cauchy sequence in X. Next we assume that there exists a point $x_* \in X$ such that $x_n = T^n x_0 \to x_*$ as $n \to \infty$. As $x_0 \in X_T$, $(T^n x_0, T^{n+1} x_0) \in E(G)$ for all $n \geq 0$. From orbital G-continuity of T, we get $T^{n+1} x_0 \to T x_*$ and hence $T x_* = x_*$. Now if $T x_* = x_*$ for any $x_* \in X$, since map T is form Perov type-II graph contraction, we have

$$\begin{aligned} q(x_*, x_*) &= q(Tx_*, Tx_*) \\ &\preceq A_1(q(x_*, x_*)) + A_2(q(x_*, Tx_*)) + A_3(q(x_*, Tx_*)) \\ &= (A_1 + A_2 + A_3) (q(x_*, x_*)) . \end{aligned}$$

Hence,

$$(I - A_3) (q(x_*, x_*)) \preceq (A_1 + A_2) (q(x_*, x_*)) . \tag{3.3}$$

and so

$$q(x_*, x_*) \preceq (I - A_3)^{-1} (A_1 + A_2) (q(x_*, x_*)) . \tag{3.4}$$

Now, Remark 2.2 implies $q(x_*, x_*) = \theta$.

Now, if the subgraph of G with the vertex set $\mathrm{Fix}(T)$ is connected and $x_{**} \in X$ is a fixed point of T. Then there exists a path $\{x_i\}_{i=0}^N$ in G from x_* to x_{**} such that $x_1, \ldots, x_{N-1} \in \mathrm{Fix}(T)$; that is, $x_0 = x_*$, $x_N = x_{**}$ and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, \ldots, N-1$. By $q(x_{i+1}, x_{i+1}) = q(x_i, x_i) = \theta$, we have

$$\begin{aligned} q(x_i, x_{i+1}) &= q(Tx_i, Tx_{i+1}) \\ &\preceq A_1 (q(x_i, x_{i+1})) + A_2 (q(x_i, Tx_i)) + A_3 (q(x_{i+1}, Tx_{i+1})) \\ &= A_1 (q(x_i, x_{i+1})) + A_2 (q(x_i, x_i)) + A_3 (q(x_{i+1}, x_{i+1})) \\ &= A_1 (q(x_i, x_{i+1})) . \end{aligned}$$

It follows from Remark 2.2 that $q(x_i, x_{i+1}) = \theta$. Since $q(x_i, x_i) = \theta$ and $q(x_i, x_{i+1}) = \theta$, by Definition 2.2 we have $d(x_i, x_{i+1}) = 0$; that is, $x_i = x_{i-1}$. Consequently,

$$x_* = x_0 = x_1 = \cdots = x_{N-1} = x_N = x_{**}$$

and hence the fixed point of T is unique and the restriction of T to X_T is a Picard operator.

Example 3.2. Let $X = [0, \infty)$, $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq 0\} \subset E$. Define the mapping $d : X \times X \to E$ by $d(x, y) = \left(|x - y|^2, |x - y|^2\right)$ for all $x, y \in X$. Then (X, d) is a cone b-metric space with constant $s = 2$. Define $q : X \times X \to E$ by

$$q(x, y) = \left(y^2, y^2\right).$$

Then q is a generalized c-distance. Define $T : X \to X$ by

$$T(x) = \begin{cases} \dfrac{x^2}{3} & \text{if } x < 1\,, \\ \dfrac{1}{6} & x = 1, \\ \dfrac{x}{4} & \text{if } x > 1. \end{cases}$$

Clearly, T is not continuous at $x = 1$. Now assume that X is endowed with a graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x, x) : x \in X\}$. Note that for any $x, y \in X$ with $(x, y) \in E(G)$, we have $x = y$. If $x, y \in X$ and $\{b_n\}$ is a sequence of positive integers with $(T^{b_n}x, T^{b_{n+1}}x) \in E(G)$ for all $n \geq 1$ such that $T^{b_n}x \to y$, then $\{T^{b_n}x\}$ is necessarily a constant sequence. Thus, for some y in X, we have $T^{b_n}x = y$ for all $n \geq 1$ and hence $T(T^{b_n}x) \to Ty$. Define linear bounded operators $A_1, A_2, A_3 : E \to E$ by

$$A_1 = \begin{bmatrix} \frac{1}{6} & \frac{1}{8} \\ 0 & \frac{2}{9} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{15} \end{bmatrix}, \quad A_3 = \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{9} \end{bmatrix}.$$

Then clearly $A_k(P) \subset P$ for $k = 1, 2, 3$, with $s(\|A_1\| + \|A_2\|) + \|A_3\| < 1$.

Let $(x, y) \in X$ with $(x, x) \in E(G)$. If $x < 1$, then

$$\begin{aligned} q(Tx, Tx)(t) &= \begin{bmatrix} \frac{x^4}{9} \\ \frac{x^4}{9} \end{bmatrix} \\ &\preceq \begin{bmatrix} \frac{7}{24}x^2 + \frac{5}{1512}x^4 \\ \frac{13}{45}x^2 + \frac{2}{5}x^4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{6} & \frac{1}{8} \\ 0 & \frac{2}{9} \end{bmatrix} \begin{bmatrix} x^2 \\ x^2 \end{bmatrix} + \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{15} \end{bmatrix} \begin{bmatrix} \frac{x^4}{9} \\ \frac{x^4}{9} \end{bmatrix} + \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{9} \end{bmatrix} \begin{bmatrix} \frac{x^4}{9} \\ \frac{x^4}{9} \end{bmatrix} \\ &= A_1(q(x, y)) + A_2(q(x, Tx)) + A_3(q(x, Tx)). \end{aligned}$$

When $x = 1$, we have

$$\begin{aligned} q(Tx,Tx)(t) &= \begin{bmatrix} \frac{1}{64} \\ \frac{1}{64} \end{bmatrix} \\ &\prec \begin{bmatrix} \frac{603}{2016} \\ \frac{188}{1620} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{6} & \frac{1}{8} \\ 0 & \frac{2}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{15} \end{bmatrix} \begin{bmatrix} \frac{1}{36} \\ \frac{1}{36} \end{bmatrix} + \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{9} \end{bmatrix} \begin{bmatrix} \frac{1}{36} \\ \frac{1}{36} \end{bmatrix} \\ &= A_1(q(x,y)) + A_2(q(x,Tx)) + A_3(q(x,Tx)). \end{aligned}$$

In case $x > 1$, then we have

$$\begin{aligned} q(Tx,Tx)(t) &= \begin{bmatrix} \frac{x^2}{16} \\ \frac{x^2}{16} \end{bmatrix} \\ &\prec \begin{bmatrix} \frac{353}{1152}x^2 \\ \frac{45}{203}x^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{6} & \frac{1}{8} \\ 0 & \frac{2}{9} \end{bmatrix} \begin{bmatrix} x^2 \\ x^2 \end{bmatrix} + \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{15} \end{bmatrix} \begin{bmatrix} \frac{x^2}{16} \\ \frac{x^2}{16} \end{bmatrix} + \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{9} \end{bmatrix} \begin{bmatrix} \frac{x^2}{16} \\ \frac{x^2}{16} \end{bmatrix} \\ &= A_1(q(x,y)) + A_2(q(x,Tx)) + A_3(q(x,Tx)). \end{aligned}$$

Furthermore, $(0, T0) = (0, 0) \in E(G)$, so $X_T \neq \emptyset$. Thus, all the conditions of Theorem 3.2 are satisfied and $x_* = 0$ is a fixed point of T has a fixed point and $q(0, 0) = 0$.

If a cone b-metric space X is endowed with the complete graph G_0 whose vertex set coincides with X; that is, $V(G_0) = X$ and $E(G_0) = X \times X$ and we set $G = G_0$ in Theorem 3.1, then the set X_T coincides with the whole set X, where T is a self mapping on X. Thus, we have the following corollary.

Corollary 3.1. *Let (X, d) be a complete cone b-metric space with constant $s \geq 1$ associated with the generalized c-distance q and $T : X \to X$ a orbitally continuous* *mapping. If there exist linear bounded operators* $A_k : E \to E$ *for* $k = 1, 2, 3$ *with* $A_k(P) \subset P$ *for* $k = 1, 2, 3$ *and* $s(\|A_1\| + \|A_2\|) + \|A_3\| < 1$ *such that*

$$q(Tx, Ty) \preceq A_1\left(q(x, y)\right) + A_2\left(q(x, Tx)\right) + A_3\left(q(y, Ty)\right)$$

for all $x, y \in X$*. Then* T *is a Picard operator.*

Suppose that $(X, \sqsubseteq)$ is a partially ordered set (poset). Let G_1 be the graph such $V(G_1) = X$ and $E(G_1) = \{(x, y) \in X \times X : x \sqsubseteq y\}$. Since $\sqsubseteq$ is reflexive, it follows that $E(G_1)$ contain all the loops. If we take $G = G_1$ in Theorem 3.2, then we obtain the following corollary.

Corollary 3.2. *Let* $(X, d, \sqsubseteq)$ *be a partially ordered complete cone b-metric space with constant* $s \geq 1$ *associated with the generalized c-distance* q *and endowed with the graph* G_1*. Suppose that* $T : X \to X$ *is a nondecreasing orbitally* G_1*-continuous mapping. If there exist linear bounded operators* $A_k : E \to E$ *for* $k = 1, 2, 3$ *with* $A_k(P) \subset P$ *for* $k = 1, 2, 3$ *and* $s(\|A_1\| + \|A_2\|) + \|A_3\| < 1$ *such that*

$$q(Tx, Ty) \preceq A_1\left(q(x, y)\right) + A_2\left(q(x, Tx)\right) + A_3\left(q(y, Ty)\right)$$

for all $x, y \in X$ *with* $x \sqsubseteq y$*. Then* T *has a fixed point in* X *if and only if there exists* $x_0 \in X$ *such that* $x_0 \sqsubseteq Tx_0$*. Moreover, if* $Tx_* = x_*$ *for any* $x_* \in X$*, then* $q(x_*, x_*) = \theta$*. Also, if the subgraph of* G_1 *with the vertex set* $\mathrm{Fix}(T)$ *is connected, then the restriction of* T *to the set of all points in* $x \in X$ *satisfying* $x \sqsubseteq Tx$ *is a Picard operator.*

Let X be a Poset endowed with the graph G_2 given by $V(G_2) = X$ and

$$E(G_2) = \{(x, y) \in X \times X : x \sqsubseteq y \ \vee \ y \sqsubseteq x\}.$$

That is, an ordered pair $(x, y) \in X \times X$ is an edge of G_2 if and only if x and y are comparable elements of $(X, \sqsubseteq)$. If we set $G = G_2$ in Theorem 3.2, then we obtain the following corollary.

Corollary 3.3. *Let* $(X, d, \sqsubseteq)$ *be a partially ordered complete cone b-metric space with* $s \geq 1$ *associated with the generalized c-distance* q *and endowed with the graph* G_2*. Suppose that* $T : X \to X$ *is a nondecreasing orbitally* G_2*-continuous mapping which maps comparable elements of* X *onto comparable elements. If there exist linear bounded operators* $A_k : E \to E$ *for* $k = 1, 2, 3$ *with* $A_k(P) \subset P$ *for* $k = 1, 2, 3$ *and* $s(\|A_1\| + \|A_2\|) + \|A_3\| < 1$ *such that*

$$q(Tx, Ty) \preceq A_1 q(x, y) + A_2 q(x, Tx) + A_3 q(y, Ty)$$

for all $x, y \in X$, where x and y are comparable. Then T has a fixed point in X if and only if there exists $x_0 \in X$ such that x_0 and Tx_0 are comparable. Moreover, $Tx_* = x_*$ for any x_* in X implies that $q(x_*, x_*) = \theta$. Also, if every two elements of $\mathrm{Fix}(T)$ are comparable, then the restriction of T to the set of all $x \in X$ such x and Tx are comparable is a Picard operator.

Let $e \in int\ P$ with $\theta \ll e$ be a fixed.

Recall that two elements $x, y \in X$ are said to be e-closed if $d(x, y) \prec e$. Define the e-graph G_3 by

$$V(G_3) = X \text{ and } E(G_3) = \big\{(x, y) \in X \times X : \ d(x, y) \prec e\big\}.$$

Note that $E(G_3)$ contains all loops. Finally, if we set $G = G_3$ in Theorem 3.2, then we obtain the following result.

Corollary 3.4. *Let (X, d) be a complete cone b-metric space with $s \geq 1$ associated with the generalized c-distance q endowed with the graph G_3. Suppose that $T : X \to X$ is an orbitally G_3-continuous mapping which maps e-close elements of X onto e-close elements. If there exist linear bounded operators $A_k : E \to E$ for $k = 1, 2, 3$ with $A_k(P) \subset P$ for $k = 1, 2, 3$ and $s(\|A_1\| + \|A_2\|) + \|A_3\| < 1$ such that*

$$q(Tx, Ty) \preceq A_1 q(x, y) + A_2 q(x, Tx) + A_3 q(y, Ty)$$

for all $x, y \in X$, where x and y are e-close elements. Then T has a fixed point in X if and only if there exists $x_0 \in X$ such that x_0 and Tx_0 are e-close. Moreover, if $Tx_* = x_*$ for any $x_* \in X$, then $q(x_*, x_*) = \theta$. Also, if every two elements of $\mathrm{Fix}(T)$ are ε-close, then the restriction of T to the set of all $x \in X$ such x and Tx are e-close is a Picard operator.

Remark 3.1. (I) The map T is need not to the continuous in our results and hence the techniques of Theorem 3.1 and Theorem 3.2 generalize, extend and unify all comparable papers on fixed point theorems in cone b-metric spaces associated with a generalized c-distance and cone metric spaces associated with a c-distances for instance, we refer to [8], [4] and [25] (all references contained in them about w-distance and c-distance). Especially, our Theorem 3.2 is a generalization of Theorem 5 (case (d)) from Suzuki [38].

(II) In 2012, Ćirić et al. [11] showed that the method of Du [17] for contraction mappings in cone metric spaces cannot be applied for contraction mappings in cone metric spaces with a associated c-distance. Also, their notes hold for generalized c-distance in cone b-metric spaces. Thus, our results are new and cannot be derived from the version of wt-distance in b-metric spaces.

(III) Recently, in 2017, in an interesting paper Huang, Radenović and Deng [22] generalize a famous result for Banach-type contractive mapping from $r(A) \in [0, 1/s)$ to $r(A) \in [0, 1)$ in cone b-metric space over Banach algebra with coefficient $s > 1$. It is interestig to know if the similar is true in our results: Theorem 3.1 and Theorem 3.2.

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