Bulletin T. CLIV de l'Académie serbe des sciences et des arts − 2021
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 46

FIFTY YEARS OF KUREPA'S $!n$ HYPOTHESIS

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Dedicated to Aleksandar Ivić (1949–2020)

(Presented at the 3nd Meeting, held on April 23, 2021)

A b s t r a c t. *Djuro Kurepa formulated in 1971 the so called left factorial hypothesis. This hypothesis is still open, despite much efforts to solve it. Here we give some historical notes and review the current status of the hypothesis. Using probabilistic model we also estimated sums of Kurepa reminders and discussed in details finite Kurepa trees.*

AMS Mathematics Subject Classification (2020): 11-03,11B50, 11K65, 11L20, 05C60
Key Words: Left factorial hypothesis, finite Kurepa tree, tree embedding

1. *Introduction*

Djuro Kurepa defined in 1964 [17] the left factorial function as

$$K_n \equiv\ !n = \sum_{i=0}^{n-1} i!. \tag{1.1}$$

In 1971 [18] he asserted a number theoretical conjecture related to $!n$ with simple formulation which is stated as follows: Assuming $n \geq 2$, prove that the only common divisor of $n!$ and $\sum_{i=0}^{n-1} i!$ is 2. Prime divisors p of $n!$ must satisfy $p \leq n$, so we have immediately the following equivalent reformulation of the hypothesis, also known to Kurepa, which we abbreviate as KHp:

KHp $!p \neq 0 \pmod{p}$ for all odd primes p.

This second formulation is more convenient to work with and most of the papers on this topic refer to it. So we shall adopt the same convention here, i.e by the Kurepa hypothesis we shall assume the statement KHp.

This problem had caught the attention of many mathematicians. For example, it is listed as a Problem B44 in R. Guys book [10], a Problem 37 in Konick-Mercier book [23] and in Sandor – Crstici book [36]. There were number-theoretical attempts to prove it and on the other side many computational efforts to find a counterexample to KHp, a prime p such that $!p = 0 \pmod{p}$. However, up to this moment KHp is still an open problem.

We assume the following notation. The set of natural numbers (nonnegative integers) is denoted by $\mathbb{N}$, $\mathbb{N}^+$ denotes positive integers, $\mathbb{Z}$ is the ring of integers, while $\mathbb{Z}_n$ denotes the ring of integers modulo n. The greatest common divisor of integers a and b is denoted by (a, b). If m and $n > 1$ are integers, the remainder obtained from division of m by n is denoted by $\rho_n(m)$. Observe that ρ_n is an epimorphism from the ring of integers $\mathbb{Z}$ to the ring $\mathbb{Z}_n$. If p is a prime, $\mathrm{GF}(p)$ denotes the Galois field with p elements.

2. *Problem and computational attempts*

Djuro Kurepa is mostly known for his works in set theory, in particular in infinitary combinatorics and on infinite trees. In some of these papers he defined factorial functions in connection with these infinitary objects and studied their properties. So Kurepa surely was inspired with his earlier works [15] and [16] for the introduction of the left factorial function. We shall see that $!n$ is related to domains of some fast branching trees.

Immediately after Kurepa introduced $!n$ in [18], several papers studying properties of this function were published. Kurepa himself published several papers and extended the left factorial function to complex numbers $\mathbb{C}$:

$$!(z+1) = \Gamma(z+1) + !z, \quad z \in \mathbb{C}. \tag{2.1}$$

Notation $K(z) = !z$ is also used and Milovanović in [31] gave an expansion of $K(z)$ what enabled computation of $K(x)$ for any real number x. The same author also defined an interesting sequence of functions which generalize $K(z)$. The first two terms of this sequence are $\Gamma(z)$ and $K(z)$. Mijajlović and Malešević proved in [30] that $\Gamma(z)$ and $K(z)$ are transcendental differential functions, i.e they are not solutions of algebraic differential equations. They also asked if the Milovanović's sequence is an infinite part of a transcendental differential base. Malešević considered in [24], [25] and [26] various inequalities on $K(z)$, while Slavić gave in [40] an interesting integral representation of $K(z)$.

In a short period Carlitz in [6], Stanković in [41] and Šami in [38] proved many identities on $!n$, some of them modulo n, while Slavić tested KHp up to $n = 1000$, see [40].

For more than 15 years, since 1974 till the beginning of 1990's there were no publications on $!n$. Mijajlović published the first paper [29] at that time where a computational method for testing KHp was presented. There were two major novelties. The first one was the introduction of recurrent formulas in $\mathrm{GF}(p)$ on which the computation was based. Strange enough, even if the modular arithmetic was used already in the 1970's for proving identities involving $!n$, that was the first application of arithmetic in $\mathbb{Z}_n$ in verification of KHp. The second one was the parallel software for testing KHp, while the computation was carried out on parallel computers based on transputer CPU's T800. The left factorial hypothesis was verified for primes $p \leq 311009$, extending a Wagstaff result $n \leq 50000$, for which we could find only a secondarily reference [10]. After that and following similar ideas several extensions appeared, e.g. Gogić [9] for primes $p \leq 2^{20}$, Malešević [26] for $p \leq 2^{22}$, Živković [43], [44] for 2^{24}, Gallot [8] for $p \leq 2^{26}$, Jobling [13] for $p \leq 2^{27}$ and using modern technologies for parallel computing on GPU's Ilijašević [11] tested KHp up to $p \leq 2^{31}$. Andrejić and Tatarević extended in [1] the verification to $p \leq 2^{34}$. This computation took 240 days. Finally, in the recent paper [2] Andrejić, Bostan and Tatarević published that they verified KHp up to the amazing $p \leq 2^{40}$. In this computation they used a new algorithm based on matrix multiplication and fast algorithms for multiplication of large integers. The method is also independently described and studied in details, but without formal program implementation, in the Rajkumar's master thesis [34].

In attempts to solve KHp new and interesting identities were found and other open problems were solved. Probably the most notable achievement is the Živković's solution [43] of the neighboring problem B43 in [10] which asks if there are infinitely many primes of the form $A_n = \sum_{i=0}^{n}(-1)^{n-i}i!$. Živković gave a negative answer as he has shown that $p = 3612703$ divides A_n for all $n \geq p$. In some cases particular formulas, or their variations, were reinvented by several authors. For example, in [29] it was proved that in the Galois field $\mathrm{GF}[p]$ the following formula is valid:

$$!p = \sum_{k=0}^{p-1} \frac{(-1)^{k+1}}{k!}. \tag{2.2}$$

Using Wilson theorem one can then immediately infer that for a prime p:

$$!p = \left[\frac{(p-1)!}{\mathrm{e}}\right] + 1 \pmod{p}. \tag{2.3}$$

Another proof of (2.3) can be found in [12] using analytic continuation $K(z)$ of $!n$. These formulas were rediscovered later several times, for example in [1]. In connection with this I have to mention the following interesting fact. Aleksandar Ivić used to keep a neat math diary of all his mathematical inventions. In one occasion, in the middle of nineties, he had shown me there his unpublished note dating the beginning of 1970's which contained the mentioned analytic proof of (2.3).

We have to mention also that there are inappropriate names assignments related to Kurepa's left factorial function. The most remarkable example is that $!n$ is also called Smarandache-Kurepa function at the rather reputed WolframMathWorld portal [45]. The first mention of this term we found in Ashbacher somewhat ghostly (I could not find it) reference [3], published 33 years after Kurepa introduced the left factorial function [17] and posed his problem and after several tens of papers on this matter were published. We do not know any contribution of F. Smarandache to this topic, except the mention of his name in the left factorial problem.

It should be mentioned that there were several announcements, even published papers with solution of KHp. However, these alerts were never confirmed, while solutions were wrong. Kurepa himself announced the solution (he ringed me up one early morning in the spring of 1992 to tell me this), but he never published the solution. Richard Guy in a letter to me in 1991 mentioned that R. Bond from G. Britain might solved the left factorial hypothesis, but this proof did not appear either up to now. Finally, we mention Barsky – Benzaghou paper [4] with the erroneous proof. It passed almost ten years before the authors declared the mistake [5].

3. *Probabilistic method*

A number of algorithms for testing KHp is reduced to the search of first odd prime p such that $r_p \equiv \rho_p(K_p) \neq 0$. Several author had heuristic assumption that r_p behaves as a random variable with uniform distribution, taking any integer value in $[0, p-1]$ with the probability $1/p$. Under this assumption, many question can be answered including if KHp is true. Namely, assuming the probabilistic nature of r_p, the probability of r_p not being 0 is $1 - 1/p$. So, also assuming that they are probabilistically independent, the probability to appear 0 in the sequence r_p, $p \in \mathbb{P}$, $\mathbb{P}$ is the set of odd primes, would be

$$1 - \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)\left(1 - \frac{1}{11}\right)\cdots = 1, \tag{3.1}$$

yielding the violation of KHp. Edwin Clark [7] first mentioned such a probabilistic model already in 1994 at sci.math forum. Later, Živković [44] and Andrejić and Tatrević [1] discussed, assuming this premisse, in depth the distribution of odd primes p such that $r_p = 0$. For example, if p_0 violates KHp, then the infinite product $\prod(1 - 1/p)$ over primes where $p > p_0$ is still divergent i.e., tends to 0, hence according to (3.1) there is another $p_1 > p_0$ which violates KHp. By this argument it follows that there are infinitely many counterexamples to KHp. On the other hand one can easily see that the probability of being $r_p = 0$ for $p \in [2^n, 2^{n+1}]$ is equal to $1/(n+1)$, reducing to chance for real finding such a prime p if not already found.

There are computational supports for adopting probabilistic models. Živković used chi square test and the result was concordat. We add several more arguments in favor of probabilistic model. The first one is "white noise" that makes a visual presentation of reminders r_p for the last 60000 primes $p < 8\,000\,000$. Under the

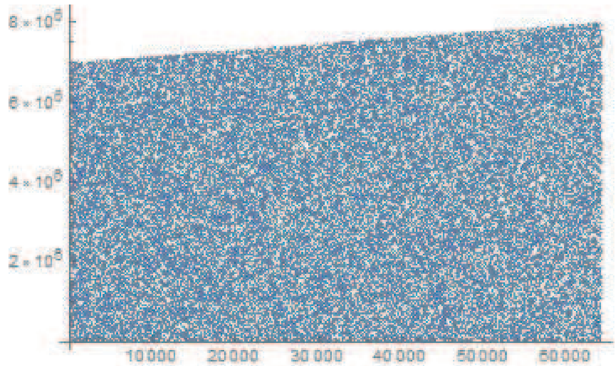

Figure 1: "White noise" produced by r_p

assumption that r_p is a random variable taking values in $[0, p-1]$, the expected value of r_p is $v_p = (p-1)/2$. Hence the average value for v_p for the first n odd primes would be $w_n = (\sum_p v_p)/n$. On the other hand, the effective mean of computed r_p is $w'_n = (\sum_p r_p)/n$. For primes bellow $8\,000\,000$ we found $w' = 1.92815 \times 10^6$, while $w = 1.92717 \times 10^6$ giving the relative difference $(w' - w)/w' = 0.0005$.

However, we note that it is easy to construct similar examples that have seemingly uniform probabilistic distribution on $[0, m-1]$ but the "random" variable s_p never takes the value 0. For that we can take $m = 2q$ where q is a moderately large prime and the sequence $s_p = \rho_m(p)$, p is a prime.

Using the probabilistic model, we can also resolve some other questions. Even if it seems obvious, there is no proof that there are infinitely many primes p such that $r_p \neq 0$. As the probability of r_p being $\neq 0$ is $1 - 1/p$ it follows that the for almost all p, $r_p \neq 0$. Obviously it would be interesting to give a proper proof of the above statement. If the primality condition for p is omitted, we write then r_n. Kurepa had shown in [18] that in this case there are infinitely many n such that $r_n \neq 0$. For that we can take for example $n = 3k$. Really, if $r_{3k} = 0$ for some k, then $k > 1$ and $3k|!(3k)$, and so 3 divides $!3 + 4! + 5! + \cdots + (3k-1)!$, i.e. 3 divides $!3 = 4$, a contradiction.

In [12], page 9, the following hypothesis related to the above numerical example is posed:

Does there exist a constant $C > 0$ such that

$$\text{(Q)} \qquad \sum_{p \leq x} r_p = \frac{Cx^2}{\ln(x)}, \quad (x \to \infty)? \tag{3.2}$$

We give a proof of the statement (Q) assuming probabilistic model for r_q. Namely we prove

Theorem 3.1 (assuming probabilistic model)**.** *We have*

$$\sum_{p \leq x} r_p = \frac{1}{4} \frac{x^2}{\ln(x)} + O\left(\frac{x^2}{\ln(x)^2}\right). \tag{3.3}$$

PROOF. If v_p is a random variable with the uniform distribution taking the values in the interval $[0, p-1]$, by the probabilistic model we may take $r_p = E(v_p) = (p-1)/2$ where E is the expectation operator. Using the additivity of the operator E, for $v = \sum_{p \leq x} v_p$ we have

$$E(v) = \sum_{p \leq x} r_p = \sum_{p \leq x} \frac{p-1}{2} = \frac{1}{2} \sum_{p \leq x} p - \frac{1}{2} x. \tag{3.4}$$

As, for derivation see for example [12],

$$\sum_{p \leq x} p = \frac{x^2}{2\ln(x)} + O\left(\frac{x^2}{\ln(x)^2}\right), \tag{3.5}$$

substituting $\sum_{p \leq x} r_p$ in (3.4) by the righthand side of (3.5) and observing that the term $\frac{1}{2}x$ is absorbed by $O\left(\frac{x^2}{\ln(x)^2}\right)$, we obtain (3.3). □

In fact we found that $C = 1/4$ in the statement (Q). In the above numerical example, $x = 8\,000\,000$, we computed

$$a \equiv \sum_{p \leq x} r_p = 1.04077 \times 10^{12}, \quad b \equiv \frac{x^2}{4\ln(x)} = 1.00661 \times 10^{12},$$

giving the relative difference $(a-b)/a = 0.034$, well concordant with the statement (Q).

4. *Finite Kurepa's tree*

First we remind some terminology on trees. A partially ordered set $\mathbf{S} = (S, \leq, 0)$ with the least element 0 is called a tree with the root 0 if for every $a \in S$ the set $[0, a]_S = \{s \in S \colon s \leq a\}$ is a well ordered chain. Here we shall consider only finite trees, so the condition "well-ordered" is automatically fulfilled. Depending on the context we shall identify $\mathbf{S}$ with it's domain S. Rank function $r \colon S \to \mathbb{N}$ is defined recursively taking $r(0) = 0$ and for $a > 0$, $r(a) = \max\{r(x) \colon x < a\} + 1$. The set $L_i(S) = \{a \in S \colon r(a) = i\}$, $i \in \mathbb{N}$, is called the i-th level of S. The height of $\mathbf{S}$ is $\operatorname{ht}\mathbf{S} = \max\{r(a) \colon a \in S\}$. A tree S is called smooth if all maximal elements of S belong to the same level L. We see that for a smooth tree L is the top level of S and that for $a \in L$, $r(a) = \operatorname{ht}(S)$.

Kurepa defined in [20] the following tree $(T_n, <, v)$. Let $I_0 = \emptyset$ and for $n > 0$, $I_n = \{0, 1, \ldots, n-1\}$. Then $T_0 = \{v\}$, where v is the empty function, i.e., $v \equiv \emptyset$. If $n > 0$ then: For every $n \in \mathbb{N}$ let T_n be the tree formed of the empty sequence v and of the all sequences

$$a = (a_1, a_2, \ldots, a_j) \quad \text{where} \quad 1 \leq i \leq j \leq n, \quad a_i \in I_i. \tag{4.1}$$

The tree T_n is ordered by the relation $<$ (Kurepa used the sign $\dashv$) where $a < a'$ means that the sequence a is an initial part of the sequence a'. In particular, $v < a$ for every $a \in T_n$. We see that T_n is a smooth tree. The maximal branch $v002$ of the tree T_3 corresponds to the sequence $f_0 \subseteq f_1 \subseteq f_2 \subseteq f_3$ of functions, where $f_0 = v \equiv \emptyset$, $f_1 = \{(0,0)\}$, $f_2 = \{(0,0), (1,0)\}$ and $f_3 = \{(0,0), (1,0), (2,2)\}$.

Kurepa also observed that that the height of T_n is $\operatorname{ht}(T_n) = n$ and that every element a at the level L_k, $k < n$, splits into $k+1$ elements, i.e., has $k+1$ immediate successors. Hence, each level L_k, $k \leq n$, has $k!$ elements. Since levels are mutually disjoint sets, there is the following connection between $!n$ and the cardinality of T_n:

Theorem 4.1 (Dj. Kurepa, 1974)**.** *If* $n \geq 1$ *then* $|T_{n-1}| = !n$.

From this theorem immediately follows the following combinatorial reformulation of the left factorial hypothesis:

KHp2 *For no prime* p *there is a partition of* T_{p-1} *into* p *subsets of the same size.*

Using the branching property of elements of T_n it is easy to find a simple recursive definition of this notion taking $T_0 = v$ and for $n \geq 1$:

$$T_n = T_{n-1} \cup \big\{ f \colon I_n \to I_n \mid \text{there is } g \in T_{n-1} \text{ so that } g \subseteq f \big\}. \tag{4.2}$$

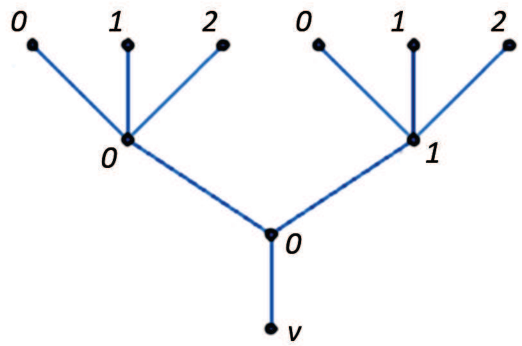

Figure 2: Finite Kurepa tree T_3

Following this definition, we constructed more examples of the trees T_n using the programming package Wolfram Mathematica (see Figure 3).

The Kurepa tree T_n was discussed in the late 1990's and in this century, e.g. [33], but without reference to Kurepa's definition from 1974 and it's connection to $!n$. Even more, in [37], A003422, the notion of the tree T_n is attributed to Arkadiusz Wesolowski (2012) and Jon Perry (2013).

Now we consider some combinatorial properties of T_n. Let B_n be full binary tree. Let us remind that B_n is a tree of the height $\mathrm{ht}(B_n) = n$ in which every element $b \in B_n$ of rank $r(b) < n$ splits into exactly two elements. We say that a tree S is embedded into a tree S' if there is an order preserving $1-1$ map $\tau : S \to S'$. which maps the root od S into the root of S'. In this case we write $S \hookrightarrow S'$. For example,

$$T_0 \hookrightarrow B_0, \quad T_1 \hookrightarrow B_1, \quad T_2 \hookrightarrow B_2, \quad T_3 \hookrightarrow B_4, \tag{4.3}$$

but $T_3 \not\hookrightarrow B_3$. The next proposition speaks about reasonable small m relative to n such that T_n is embeddable into B_m.

Theorem 4.2. *Let $n \in \mathbb{N}$, $n \geq 6$ and $m = [n \log_2(n)]$. Then $T_n \hookrightarrow B_m$.*

For the simplicity of the notation, let us introduce $Q_m = T_{m-1}$, $m > 0$. We see that Q_{m+1} is obtained from Q_m by branching each element $a \in Q_m$ of the highest rank, i.e., $r(a) = m - 1$, into m new elements. First we prove the following statement:

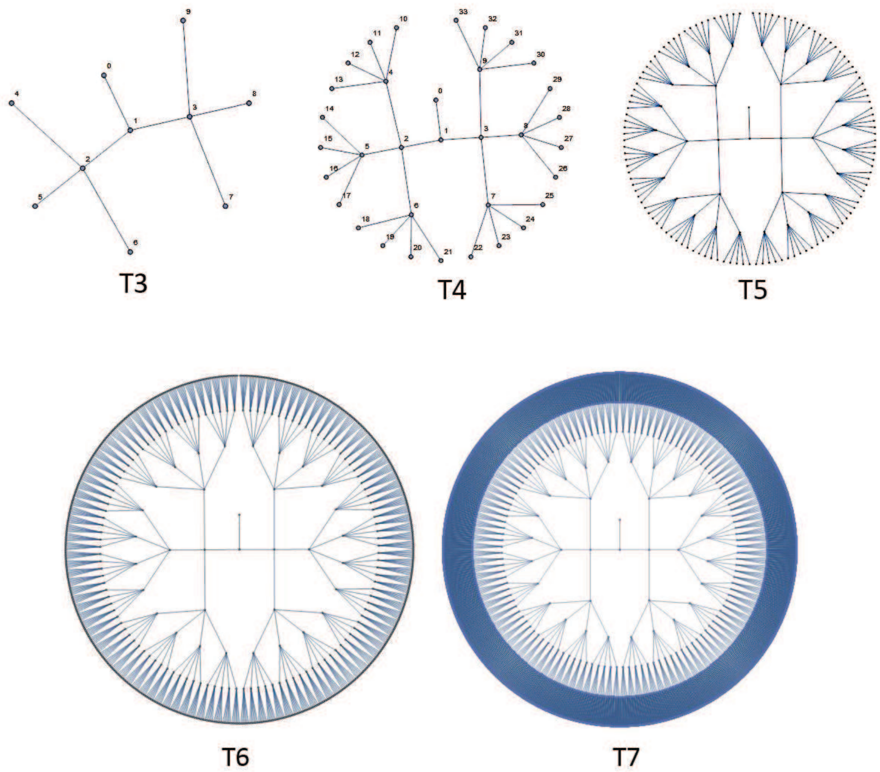

Figure 3: Finite Kurepa trees T_n for $n = 3, 4, 5, 6$, and 7

Lemma 4.1. *Suppose $Q_m \hookrightarrow B_n$ and assume $2^{k-1} < m \le 2^k$. Then*

$$Q_{m+1} \hookrightarrow B_{n+k}.$$

PROOF. Suppose $\tau\colon Q_m \hookrightarrow B_n$, let for $a \in Q_m$, $a' = \tau(a)$ and $L = L_{m-1}(Q_m)$ the set of elements which lay on top of Q_m. For each $a \in$ Ł there is $b_a \in B_n$, such that $r(b_a) = n$ and $b_a \ge a'$. We see that all elements b_a are at the same, the last level of B_n. To each b_a let us attach a copy B^a of the tree B_k on top of the element a, i.e., a being a root of B^a. Then we obtain a binary tree

$$S = B_n \cup \bigcup_{a \in L} B^a \tag{4.4}$$

with inherited orderings from B_n and trees B^a. The tree S is obviously embedded into B_{n+k}. As $m \le 2^k$ and B^a as a copy of B_k has 2^k elements at the top level

W, we can choose mutually incomparable elements $b_1^a, b_2^a, \ldots, b_m^a \in W$. Let us take $S^a = \{b_1^a, b_2^a, \ldots, b_m^a\}$ and

$$Q' = \tau(Q_m) \cup \bigcup_{a \in L} S^a. \tag{4.5}$$

Then Q' is a subtree of B_{n+k} in which for every $a \in L$, we remind that L is the top level of Q_m, a' splits into m elements $b_1^a, b_2^a, \ldots, b_m^a$. Hence, the tree Q' is isomorphic to Q_{m+1} and $Q' \subset B_{n+k}$ i.e $T_m \hookrightarrow B_{n+k}$. This ends the proof of the lemma and we continue with the proof of the theorem. □

PROOF OF THEOREM 4.2. Suppose $T_{m-1} \hookrightarrow B_n$ and $2^{k-1} < m \le 2^k$. Then $\log_2(m) \le k < \log_2(m)+1$ and so $[\log_2(m)] \le k \le [\log_2(m)]+1$. The embedding relation is transitive, hence by Lemma

$$T_m \hookrightarrow B_{n+[\log_2(m)]+1}. \tag{4.6}$$

As $T_2 \hookrightarrow B_2$, iterating (4.6) we obrain

$$T_3 \hookrightarrow B_{2+[\log_2(3)]+1}, \quad T_4 \hookrightarrow B_{2+[\log_2(3)]+1+[\log_2(4)]+1}, \quad \ldots\ ,$$

$$T_m \hookrightarrow B_{2+\sum_{i=3}^m([\log_2(i)]+1)}.$$

As

$$\sum_{i=3}^m \big[\log_2(i)\big] \le \left[\sum_{i=3}^m \log_2(i)\right],$$

we have

$$T_m \hookrightarrow B_{m+[\log_2(m!)]}. \tag{4.7}$$

It is easy to prove by induction:

$$\text{If} \quad n \ge 6 \quad \text{then} \quad 2^n \cdot n! < n^n.$$

Hence $n + \log_2(n!) < n\log(n)$, and so by (4.7)

$$\text{If} \quad n \ge 6 \quad \text{then} \quad T_n \hookrightarrow B_{[n\log_2(n)]}.$$ □

By (4.3) it follows that the theorem is valid also for $n = 0, 1, 2, 3$. It would be interesting to check if it is also true for $n = 4, 5$, i.e., if $T_4 \hookrightarrow B_8$ and $T_5 \hookrightarrow B_{11}$. We also note that there are better bounds for m in Theorem 4.2 if one uses sharper bounds for $n!$, e.g. variants of Stirling's formula, for example (Robins [35]):

$$n! < \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \mathrm{e}^{\frac{1}{12n}} \quad \text{for} \quad n \in N^+. \tag{4.8}$$

REFERENCES

[1] V. Andrejić, M. Tatarević, *Searching for a counterexample to Kurepa's conjecture*, Math. Comp. **85**, (2016), No. 302, 3061–3068.

[2] V. Andrejić, A. Bostan, M. Tatarević, *Improved algorithms for left factorial residues*, Inform. Process. Lett. **167** (2021), 106078, 3 pp.

[3] C. Ashbacher, *Some Properties of the Smarandache-Kurepa and Smarandache-Wagstaff functions*, Math. Informatics Quart. **7** (1997), 114–116.

[4] D. Barsky, B. Benzaghou, *Nombres de Bell et somme de factorielles*, J. Théor. Nombres Bordx. **16** (2004), 1–17.

[5] D. Barsky, B. Benzaghou, *Erratum á l'article nombres de Bell et somme de factorielles*, J. Théor. Nombres Bordx. **23** (2011), 527–527.

[6] L. Carlitz, *A note on the left factorial function*, Math. Balkanica **5**, (1975) 37–42.

[7] E. Clark, *Edwin Clark*, University of South Florida, 1994 (`http://shell.cas.usf.edu/˜wclark/`).

[8] Y. Gallot, *Is the number of primes* $\frac{1}{2}\sum_{i=0}^{n-1} i!$ *finite*, `http://yves.gallot.pagesperso-orange`, 2000.

[9] G. Gogić, *Parallel algorithms in arithmetic*, Master Thesis, Belgrade University, 1991.

[10] R. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag NY, 2004.

[11] I. Ilijašević, *Verification of Kurepa's left factorial conjecture for primes up to* 2^{31}, IPSI Trans. on Internet Research (former "The IPSI BgD Transactions on Internet Research"), **11**, No. 1, (2015), 7–12.

[12] A. Ivić, Ž. Mijajlović, *On Kurepa problems in number theory*, Publ. Inst. Math. (Beograd) **57** (71), (1995), 19–28.

[13] P. Jobling, *A couple of searches*, `https://groups.yahoo.com/neo/groups/primeform/conversations/topics/5095` (2004).

[14] B.C. Kellner, *Some remarks on Kurepa's left factorial*, arXiv:math/0410477v1 [mathNT], 2004.

[15] Dj. Kurepa, *O faktorijelima konačnih i beskonačnih brojeva*, Rad JAZU, 296, (1953) 105–112.

[16] Dj. Kurepa, *Über die Faktoriellen der endlichen und unendlichen Zahlen*, Ac. Sci. Yougoslave, Classe Math. **4** (1954), 51–64.

[17] Dj. Kurepa, *Factorials of cardinal numbers and trees*, Glasnik Mat. Fiz. Astr. **19** (1-2) (1964), 7–21.

[18] Dj. Kurepa, *On the left factorial function !N*, Math. Balkanica **1** (1971), 147–153.

[19] Dj. Kurepa, *Left factorial function in complex domain*, Math. Balkanica **3** (1973), 297–307.

[20] Dj. Kurepa, *On some new left factorial propositions*, Math. Balkanica **4** (1974), 383–386.

[21] Dj. Kurepa, *Right and left factorials*, Boll. Unione Math. Ital. (4) **9** Suppl. fasc. 2 (1974), 171–189.

[22] Dj. Kurepa, *Selected Papers of Đuro Kurepa*, Edited and with commentaries by A. Ivić, Z. Mamuzić, Ž. Mijajlović and S. Todorčević, Matematički Institut SANU, Belgrade, 1996.

[23] J.M. De Koninck, A. Mercier, *1001 Problems in Classical Number Theory*, American Mathematical Society, Providence, RI, 2007.

[24] B. Malešević, *Some considerations in connection with Kurepa's function*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **14** (2003), 26–36.

[25] B. Malešević, *Some inequalities for Kurepa's function*, JIPAM. J. Inequal. Pure Appl. Math. **5** (2004), no. 4, Article 84, 5 pp.

[26] B. Malešević, *Some inequalities for alternating Kurepa's function*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **16** (2005), 7—76.

[27] B. Malešević, *Some considerations in connection with alternating Kurepa's function*, Integral Transforms Spec. Funct. **19** (2008), no. 9-10, 747–756.

[28] R. Meštrović, *The Kurepa-Vandermonde matrices arising from Kurepa's left factorial hypothesis*, Filomat **29**:10 (2015), 2207–2215.

[29] Ž. Mijajlović, *On some formulas involving* $!n$ *and the verificaton of the* $!n$*-hypothesis by use of computers*, Publ. Math. Inst. (Beograd) (N.S.) **47** (61), 1990, 24–32.

[30] Ž. Mijajlović, B. Malešević *Differentially transcendental functions*, Bullet. Belgian Math. Soc. - Simon Stevin, 15(2), (2008), 193-201.

[31] G. V. Milovanović, *Expansions of the Kurepa function*, Publ. Inst. Math. (Beograd) (N.S.) 57 (71) (1995), 81-90.

[32] G. V. Milovanović, *A sequence of Kurepa's functions*, Scientific Review, No **19-20** (1996), 137–146.

[33] A. Petojević, M. Žižović, *Trees and the Kurepa hypothesis for left factorial*, Filomat **13** (1999), 31–40.

[34] R. Rajkumar, *Searching for a counterexample to Kurepa's conjecture in average polynomial time*, Masters Thesis, School of Mathematics and Statistics, UNSW Sydney, 2019.

[35] H. Robbins, "*A remark on Stirling's formula*", The American Mathematical Monthly **62** (1) (1955), 26–29.

[36] J. Sándor, B. Crstici, *Handbook of Number Theory II*, Kluwer, Dordrecht, 2004.

[37] N. Sloane, founder, *The on-line encyclopedia of integer sequences*, seq. A003422, https://oeis.org.

[38] Z. Šami, *On the M - hypothesis of Dj. Kurepa*, Math. Balkanica **3** (1973), 530–532.

[39] Z. Šami, *A sequence $u_{n,m}$ and Kurepa's hypothesis on left factorial*, Scientific Review, **19-20**, (1996), 105–113.

[40] D. Slavić, *On the left factorial function of the complex argument*, Math. Balkanica **3** (1973),

[41] J. Stanković, Über einige Relationen zwischen Fakultäten und den linken Fakultäten, Math. Balkanica **3** (1973), 488–497.

[42] V. S. Vladimirov, *Left factorials, Bernoulli numbers, and the Kurepa conjecture*, Publ. Inst. Math. Beograd (N.S.), **72** (86), (2002), 11–22.

[43] M. Živković, *The number of primes* $sum_{i=1..n}(-1)^(n-i)*i!$ is finite, Math. Comp. **68** (1999), 403–409.

[44] M. Živković, *Massive computation as a problem solving tool*, Proc. 10th Congress of Yugoslav Mathematicians, pp. 113–128, 2001.

[45] E. Weisstein, *Smarandache-Kurepa Function*, From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Smarandache-KurepaFunction.html

Bibliographical note: Kurepa's papers [17]–[20] are printed in the book [22]. Digital copy of this book is deposited in the Virtual Library of the Faculty of Mathematics in Belgrade, `http://elibrary.matf.bg.ac.rs`.

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