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MISCELLANEOUS FORMULAE FOR THE CERTAIN CLASS OF COMBINATORIAL SUMS AND SPECIAL NUMBERS

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A b s t r a c t. In a recent paper [*Montes Taurus J. Pure Appl. Math.* **3** (1) (2021), 38–61] *we defined the class of combinatorial sums*

$$y(n,\lambda)=\sum_{j=0}^{n}\frac{(-1)^{n}}{(j+1)\lambda^{j+1}\left(\lambda-1\right)^{n+1-j}}.$$

The purpose of this paper is to give some integral formulas, identities and combinatorial sums using the numbers $y(n,\lambda)$. The obtained results are related to the Bernoulli numbers and their interpolation functions, as well as the Pell numbers, the Harmonic numbers, the alternating Harmonic numbers, the Daehee numbers, and the Catalan-Qi numbers. Moreover, we give answers to some open problems involving the numbers $y(n,\lambda)$.

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1. *Introduction*

Throughout this paper, we use the following notations, formulas, and definitions.

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural numbers, integers, real and complex numbers, respectively. Also, $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Supposing that for $z \in \mathbb{C}$ with $z = x+\mathrm{i}y$ $(x, y \in \mathbb{R})$; $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$ and also $\log z$ denotes the principal branch of the many-valued function $\mathrm{Im}(\log z)$ with the imaginary part of $\log z$ constrained by

$$-\pi < \mathrm{Im}(\log z) \leq \pi,$$

and $\log \mathrm{e} = 1$.

$$0^n = \begin{cases} 1, & n = 0, \\ 0, & n \in \mathbb{N} \end{cases}$$

(cf. [1]–[33]).

The main motivation for this article is to seek some partial answers to the solutions of the open problems given in [30] for the following combinatorial numbers

$$y(n,\lambda) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)\lambda^{j+1}(\lambda-1)^{n+1-j}}. \tag{1.1}$$

In [30], we defined the following numbers

$$y_{8,k}(\lambda; a\ ) = \frac{\partial^k}{\partial t^k}\left\{\frac{\log\left(\lambda+a^t\right)}{a^t+\lambda-1}\right\}\Bigg|_{t=0},$$

assuming that $|a^t/\lambda| < 1$ $(\lambda \neq 0, 1)$ and $a \geq 1$. By using the above formula, we also defined interpolation function of the numbers $y_{8,k}(\lambda; a)$ as follows

$$\begin{aligned} \mathcal{Z}_1(s; a, \lambda) = {} & \frac{\log\lambda}{(\log a)^s}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s(\lambda-1)^{n+1}} \\ & + \frac{1}{(\log a)^s}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\frac{(-1)^n}{(j+1)\lambda^{j+1}(\lambda-1)^{n+1-j}}\right)\frac{1}{(n+1)^s}, \end{aligned} \tag{1.2}$$

where $\lambda \in \mathbb{C}\setminus\{0,1\}$ $(|1/(\lambda-1)| < 1;\ \mathrm{Re}(s) > 1)$, $a \geq 1$, and $s \in \mathbb{C}$ (see, for details, [30]). It is seen that the numbers $y(n,\lambda)$, which is special finite combinatorial sum, springs up in the second series on the right side of (1.2). That is

$$\begin{aligned} \mathcal{Z}_1(s; a, \lambda) \;=\; & \frac{\log\lambda}{(\lambda-1)(\log a)^s}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s(\lambda-1)^n} \\ & + \frac{1}{(\log a)^s}\sum_{n=0}^{\infty}\frac{y(n,\lambda)}{(n+1)^s}. \end{aligned}$$

Combining the above function with the following well-known the polylogarithm function, defined by a power series in z, which is also known as a Dirichlet series in s associated with the Hurwitz–Lerch zeta function

$$\mathrm{Li}_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s},$$

(cf. [8, 5, 33]), we also obtain

$$\mathcal{Z}_1(s;a,\lambda)=\frac{1}{(\log a)^s}\left(\frac{(\log\lambda)\,\mathrm{Li}_s\left(\frac{1}{1-\lambda}\right)}{\lambda-1}+\sum_{n=0}^{\infty}\frac{y(n,\lambda)}{(n+1)^s}\right), \tag{1.3}$$

where $\lambda\in\mathbb{C}\setminus\{0,1\}$ ($|1/(\lambda-1)|<1$; $\mathrm{Re}(s)<1$), $a\geq 1$, and $s\in\mathbb{C}$.

The function $\mathcal{Z}_1(s;a,\lambda)$ interpolates the numbers $y_{8,n}(\lambda;a)$ when $s=-n$ with $n\in\mathbb{N}$. That is

$$\mathcal{Z}_1(-n;a,\lambda)=y_{8,n}(\lambda;a)$$

where the numbers $y_{8,n}(\lambda;a)$ are given by the following generating functions:

$$\frac{\log\left(\lambda+a^t\right)}{a^t+\lambda-1}=\sum_{n=0}^{\infty}y_{8,n}(\lambda;a)\frac{t^n}{n!}$$

(cf. [30, Thm. 4.10, p. 52]).

The open problems about the numbers $y(n,\lambda)$ raise at reference [30] are presented as follows:

1. *One of the first questions that comes to mind what is generating function for the numbers $y(n,2)$ and the numbers $y(n,\lambda)$.*

2. *Some of the other questions are what are the special families of numbers the numbers $y(n,2)$ are related to.*

3. *What are the combinatorial applications of the numbers $y(n,2)$?*

4. *Can we find a special arithmetic function representing this family of numbers?*

Let us add one more question to the above open problems:

Considering the second series on the right side of equation (1.3) *as the Dirichlet series, what are the arithmetic functions with which the numbers $y(n,\lambda)$ are related?*

We have been working on the solutions of the above problems recently (cf. [30] and [31]). The main purpose of this paper is investigate the other partial solutions of these problems.

The following some results, including partial solutions of some of the above open problems, have been obtained:

For $n \in \mathbb{N}_0$, in [31], we gave many novel identities for the numbers $y(n,\lambda)$. Some of them are given as follows:

$$y(n,\lambda) = \frac{(-1)^n}{(\lambda-1)^{n+2}} \int_0^{\frac{\lambda-1}{\lambda}} \frac{1-x^{n+1}}{1-x}dx, \tag{1.4}$$

$$y\left(n,\frac{1}{2}\right) = \frac{1}{2^{n+2}} \int_0^1 \frac{1+(-1)^n x^{n+1}}{1+x}dx, \tag{1.5}$$

$$y(n,\lambda) = \frac{(-1)^n}{(\lambda-1)^{n+2}} \int_0^{\frac{\lambda-1}{\lambda}} \frac{1-x^{n+1}}{1-x}dx, \tag{1.6}$$

and

$$\int_0^1 \frac{1-x^{\left[\frac{n}{2}\right]}}{1-x}dx - \int_0^1 \frac{1-x^n}{1-x}dx = \frac{y\left(n,\frac{1}{2}\right)}{2^{n+2}} + \frac{(-1)^n}{n+1}. \tag{1.7}$$

Putting $\lambda = 2$ and $\lambda = -1$ in (1.1), we [30] gave the following combinatorial sums:

$$y(n,2) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)2^{j+1}} \tag{1.8}$$

and

$$y(n,-1) = \frac{1}{2(n+1)} \sum_{j=0}^{n} \frac{1}{\binom{n}{j}}. \tag{1.9}$$

(cf. [30]).

For some special values of the variable λ, relations between the numbers $y(n,\lambda)$ numbers and the following special sums involving binomial coefficients can also be investigated:

$$\sum_{j=0}^{n} \frac{1}{\binom{n}{j}}, \quad \sum_{j=0}^{n} \frac{(-1)^j}{\binom{n}{j}}, \quad \sum_{j=0}^{n} \binom{n}{j} j^k, \quad \sum_{k=0}^{n} \binom{n}{k}^m$$

and so on. There are many different methods in order to give the values of the above special combinatorial sums (cf. [6, 4, 13, 14, 20, 23, 24, 25, 26, 27, 28, 29, 30]).

We think that the most interesting question is to investigate what the relationship or relationships between the numbers $y(n,\lambda)$ numbers and the numbers $y_6(m,n;\lambda,p)$ defined by means of the following generating function:

$$ {}_pF_{p-1}\left[\begin{array}{c} -n,-n,...,-n \\ 1,1,...,1 \end{array};(-1)^p\,\lambda e^t\right] = n!\sum_{m=0}^{\infty} y_6(m,n;\lambda,p)\frac{t^m}{m!}, $$

where

$$ n!y_6(m,n;\lambda,p) = \sum_{k=0}^{n}\binom{n}{k}^p k^m\lambda^k, $$

and ${}_pF_q$ denotes the well-known generalized hypergeometric function which is defined by

$$ {}_pF_q\left[\begin{array}{c} \alpha_1,...,\alpha_p \\ \beta_1,...,\beta_q \end{array};z\right] = \sum_{m=0}^{\infty}\left(\frac{\prod_{j=1}^{p}(\alpha_j)^{\overline{m}}}{\prod_{j=1}^{q}(\beta_j)^{\overline{m}}}\right)\frac{z^m}{m!}, $$

where the above series converges for all z if $p<q+1$, and for $|z|<1$ if $p=q+1$. Assuming that all parameters have general values, real or complex, except for the β_j, $j=1,2,\ldots,q$, none of which is equal to zero or a negative integer and also

$$ (\lambda)^{\overline{v}} = \prod_{j=0}^{v-1}(\lambda+j), $$

and $(\lambda)^{\overline{0}}=1$ for $\lambda\neq 1$, where $v\in\mathbb{N}$, $\lambda\in\mathbb{C}$. For the generalized hypergeometric function and their applications, it is also recommended to refer to the following resource (cf. [10, 28, 34]; and references therein).

The Bernoulli polynomials of order k, $B_n^{(k)}(x)$, are defined by the following generating function:

$$ \left(\frac{t}{e^t-1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \tag{1.10} $$

where $|t|<2\pi$ (cf. [4, 5, 6, 9, 10, 11, 13, 14, 17, 20, 21], [25]–[33] and references therein).

When $x=0$ in (1.10), we have the Bernoulli numbers of order k:

$$ B_n^{(k)} = B_n^{(k)}(0), $$

and putting $k=1$, we have the Bernoulli numbers, B_n:

$$B_n = B_n^{(1)},$$

(cf. [4, 5, 6, 9, 10, 11, 13, 14, 17, 20, 21, 25, 26, 27, 28, 29, 30, 32, 33]; and references therein).

For $m, n \in \mathbb{N}$, we [31] gave the following the following novel results involving the numbers :

$$y\left(n, \frac{1}{e^t}\right) = \sum_{m=0}^{\infty} \sum_{j=0}^{n} (-1)^{j-1} \frac{B_{m+n+1-j}^{(n+1-j)}(n+2)}{(j+1)\binom{m+n+1-j}{n+1-j}(n+1-j)!} \frac{t^m}{m!} \tag{1.11}$$

and

$$y\left(n, \frac{1}{e^t}\right) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)} \zeta_{n+1-j}(-m, n+2) \frac{t^m}{m!}, \tag{1.12}$$

where $\zeta_d(s,x)$ denotes the Hurwitz zeta functions, for $d \in \mathbb{N}$, which defined by

$$\zeta_d(s,x) = \sum_{v=0}^{\infty} \frac{\binom{v+d-1}{v}}{(x+v)^s} = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} \cdots \sum_{v_d=0}^{\infty} \frac{1}{(x+v_1+v_2+\cdots+v_d)^s}$$

where $\mathrm{Re}(s) > d$, when $d=1$, we have the Hurwitz zeta function

$$\zeta(s,x) = \zeta_1(s,x) = \sum_{v=0}^{\infty} \frac{1}{(x+v)^s},$$

(cf. [33]).

It is clear that

$$\zeta_d(-m,x) = \frac{(-1)^d m! B_{m+d}^{(d)}(x)}{(d+m)!} \tag{1.13}$$

and

$$\zeta(-m,x) = \zeta_1(-m,x) = -\frac{B_{m+1}(x)}{m+1} \tag{1.14}$$

where $m \in \mathbb{N}_0$ (cf. [1, 3, 8, 5, 22, 33]). By combining (1.12) with (1.11), we gave the following result (see [31]):

Theorem 1.1. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n} \frac{1}{j+1} \left((-1)^n \zeta_{n+1-j}(-m, n+2) + \frac{(-1)^j B_{m+n+1-j}^{(n+1-j)}(n+2)}{\binom{m+n+1-j}{n+1-j}(n+1-j)!} \right) = 0. \tag{1.15}$$

The Harmonic numbers H_n are defined by means of the following generating function:

$$F_1(z) = \frac{\log(1-z)}{z-1} = \sum_{n=1}^{\infty} H_n z^n, \tag{1.16}$$

where $|z| < 1$ (cf. [4, 6, 20, 32, 33]).

The alternating Harmonic numbers, denoted by $\mathcal{H}_n$, are defined by means of the following generating function:

$$F_2(z) = \frac{\log(1+z)}{z-1} = \sum_{n=1}^{\infty} \mathcal{H}_n z^n, \tag{1.17}$$

where $|z| < 1$ (cf. [4, 7, 31, 32]).

In [32], the Harmonic numbers are given by the following integral representation:

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx, \tag{1.18}$$

and

$$H_n = -n \int_0^1 x^{n-1} \log(1-x) dx = -n \int_0^1 (1-x)^{n-1} \log(x) dx, \tag{1.19}$$

where $H_0 = 0$.

In [31], we gave the following formula involving the harmonic numbers and the numbers $y\,(n, 1/2)$:

$$y\left(n, \frac{1}{2}\right) = 2^{n+2} \left(H_{\left[\frac{n}{2}\right]} - H_n + \frac{(-1)^{n+1}}{n+1} \right), \tag{1.20}$$

where $[x]$ denotes the integer part of x.

In [31, Eq. (20)], we showed that the following formula

$$y\left(n, \frac{1}{2}\right) = 2^{n+2} \sum_{j=0}^{n} \frac{(-1)^{j+1}}{j+1} \tag{1.21}$$

is related to the following well-known alternating harmonic numbers

$$\mathcal{H}_n = \sum_{j=1}^{n} \frac{(-1)^j}{j} = H_{\left[\frac{n}{2}\right]} - H_n, \tag{1.22}$$

(cf. [4, 7], [32, Eq. (1.5)], [31, Eq. (20)]).

The Stirling numbers of the first kind, $S_1(n,k)$, are defined by the following generating function:

$$F_s(z) = \frac{(\log(1+z))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k) \frac{z^n}{n!} \tag{1.23}$$

with $S_1(n,k) = 0$ if $k > n$, and $k \in \mathbb{N}_0$ (*cf.* [4, 5, 21, 29, 30, 32, 33]; and references therein).

The Fibonacci-type polynomials in two variables are defined by means of the following generating function:

$$\frac{1}{1 - x^k z - y^m z^{m+n}} = \sum_{j=0}^{\infty} \mathcal{G}_j(x, y; k, m, n) z^j \tag{1.24}$$

where

$$\mathcal{G}_j(x, y; k, m, n) = \sum_{c=0}^{\left[\frac{j}{m+n}\right]} \binom{j - c(m+n-1)}{c} y^{mc} x^{jk - mck - nck}$$

(*cf.* [16]). Relations among the polynomials $\mathcal{G}_j(x, y; k, m, n)$, the Bernoulli polynomials, the Euler polynomials and the Genocchi polynomials were given in [17].

Substituting $x = 2$, $y = 1$ and $k = m = n = 1$ into (1.24), we have

$$\frac{1}{1 - 2z - z^2} = \sum_{j=0}^{\infty} \mathcal{G}_j(2, 1; 1, 1, 1) z^j.$$

After some calculations, we obtain generating function for the Pell numbers P_n:

$$\frac{z}{1 - 2z - z^2} = \sum_{j=0}^{\infty} \mathcal{G}_j(2, 1; 1, 1, 1) z^{j+1} = \sum_{j=0}^{\infty} P_j z^j. \tag{1.25}$$

Therefore for $j \in \mathbb{N}$, we get

$$\mathcal{G}_{j-1}(2, 1; 1, 1, 1) = P_j = \sum_{c=0}^{\left[\frac{j-1}{2}\right]} \binom{j - c - 1}{c} 2^{j-2c-1}. \tag{1.26}$$

By using (1.26), few values of the Pell numbers P_j are given as follows:

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025,$$

$$470832, 1136689, 2744210, 6625109, 15994428, 38613965, 93222358, 225058681,$$
$$543339720, 1311738121, 3166815962, 7645370045, 18457556052, 44560482149,$$

and so on (sequence A000129 in the OEIS).

By using (1.25), it is easy to have the following following recurrence relation for the Pell numbers:

$$P_0 = 0, \quad P_1 = 1$$

and for $n \geq 2$

$$P_n = 2P_{n-1} + P_{n-2}$$

(cf. [12, 17]).

In [9], with the aid of the Volkenborn integral on the set of p-adic integers, Kim defined the Daehee numbers D_n by the following generating function:

$$\frac{\log(1+z)}{z} = \sum_{n=0}^{\infty} D_n \frac{z^n}{n!}, \tag{1.27}$$

where $z \neq 0$ and $|z| < 1$ (cf. [9]).

By using the above equation, the following explicit formula for the Daehee numbers D_n is found:

$$D_n = (-1)^n \frac{n!}{n+1} \tag{1.28}$$

(cf. [9]).

Let us briefly summarize the next sections of the article.

In Section 2, by using special values of the numbers $y\left(n, \frac{1+\sqrt{2}}{2}\right)$, we give some formulas involving the Pell numbers. We also give two formulas involving the Daehee numbers and the Catalan-Qi numbers derived from the numbers $y(n, \lambda)$.

In Section 3, we give some novel integral formulas with respect to λ for the numbers $y(n, \lambda)$.

In Section 4, we give conclusion section.

2. *Formulas involving the Pell numbers, the Daehee numbers and the Catalan-Qi numbers derived from the numbers* $y(n, \lambda)$

In this section, by the aid of the numbers $y(n, \lambda)$, we give some formulas involving the Pell numbers, the Daehee numbers and the Catalan-Qi numbers.

Putting $\lambda = \frac{1+\sqrt{2}}{2}$ into (1.1), we get

$$y\left(n, \frac{1+\sqrt{2}}{2}\right) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)\left(\frac{1+\sqrt{2}}{2}\right)^{j+1}\left(\frac{1+\sqrt{2}}{2}-1\right)^{n+1-j}}. \tag{2.1}$$

After some calculation in the above equation, we get

$$y\left(n,\frac{1+\sqrt{2}}{2}\right)=2^{n+2}\sum_{j=0}^{n}\frac{(-1)^{j-1}}{(j+1)\left(1+\sqrt{2}\right)^{j+1}\left(1-\sqrt{2}\right)^{n+1-j}}. \tag{2.2}$$

Combining equation (2.2) with the following well-known identities involving the Pell numbers:

$$\left(1+\sqrt{2}\right)^{j+1}=h_{j+1}+\sqrt{2}P_n \tag{2.3}$$

and

$$\left(1-\sqrt{2}\right)^{n-j+1}=h_{n-j+1}-\sqrt{2}P_{n-j+1} \tag{2.4}$$

where

$$h_j=\frac{\left(1+\sqrt{2}\right)^j+\left(1-\sqrt{2}\right)^j}{2}$$

and

$$\sqrt{2}P_j=\frac{\left(1+\sqrt{2}\right)^j-\left(1-\sqrt{2}\right)^j}{2}$$

(cf. [11, 12, 35]), after some elementary calculations, we get the following theorem:

Theorem 2.1. *Let $n\in\mathbb{N}_0$. Then we have*

$$y\left(n,\frac{1+\sqrt{2}}{2}\right)=2^{n+2}\sum_{j=0}^{n}\frac{(-1)^{j-1}}{(j+1)\left(h_{j+1}+\sqrt{2}P_{j+1}\right)\left(h_{n-j+1}-\sqrt{2}P_{n-j+1}\right)}.$$

By using (2.2), we also obtain

$$y\left(n,\frac{1+\sqrt{2}}{2}\right)=2^{n+2}\sum_{j=0}^{n}\frac{\left(1-\sqrt{2}\right)^{2j-n}}{j+1}.$$

Combining the above combinatorial sum with (2.4), we arrive at the following corollary:

Corollary 2.1. *Let $n\in\mathbb{N}_0$. Then we have*

$$y\left(n,\frac{1+\sqrt{2}}{2}\right)=2^{n+2}\sum_{j=0}^{n}\frac{1}{(j+1)\left(h_{n-2j}-\sqrt{2}P_{n-2j}\right)},$$

if $n-2j<0$ in the sum, the related term is assumed to be zero.

Here we note that the following comments about other representations of the numbers $y\left(n, \frac{1+\sqrt{2}}{2}\right)$:

Let $x = 1 + \sqrt{2}$. Then we have

$$x^2 = 1 + 2x,\ \ x^3 = 2 + 5x,\ \ x^4 = 5 + 12x,\ \ x^5 = 12 + 29x,\ \ x^6 = 29 + 70x,$$

and so on, how can general formula(s) for x^n be obtained?

Noting that we can see when the sequence above is examined carefully, the coefficients in front of x give the Pell numbers see also [12].

Thus, we get the following formula:

Assuming that

$$x^k = a + bx,$$

we have the following presumably new result:

$$x^{k+1} = b + (a + 2b)x.$$

Remark 2.1. With the help of this formula, other representations of equation (2.1) can be given in terms of x.

Substituting z by $-2z$ into (1.27), we get

$$\frac{\log(1-2z)}{-2z} = \sum_{n=0}^{\infty} (-2)^n D_n \frac{z^n}{n!}.$$

Combining the above equation with following generating function for the Catalan-Qi function $C(a, b; n)$:

$$\sum_{n=0}^{\infty} C(1, 2; n) z^n = \frac{\log(1-2z)}{-2z},$$

where

$$C(a, b; n) = \left(\frac{a}{b}\right)^n \frac{(a)^{\overline{n}}}{(b)^{\overline{n}}}$$

(*cf.* [19]), we obtain

$$\frac{(-2)^n}{n!} D_n = C(1, 2; n). \tag{2.5}$$

Since

$$\sum_{n=0}^{m} B_n S_1(m, n) = \frac{(-1)^m m!}{m+1} = D_m \tag{2.6}$$

(cf. [2, p. 117], [9], [20, p. 45, Exercise 19 (b)]), we also get

$$C(1,2;n)=\frac{(-2)^n}{n!}\sum_{d=0}^{n}B_dS_1(n,d). \tag{2.7}$$

By combining equation (2.5) and (2.7) with equation (1.1) yields the following result:

Theorem 2.2. *Let $n\in\mathbb{N}_0$. Then we have*

$$y(n,\lambda)=\sum_{j=0}^{n}\frac{(-1)^{j+1}C(1,2;j)}{2^j\lambda^{j+1}(1-\lambda)^{n+1-j}}. \tag{2.8}$$

Remark 2.2. By combining equation (2.5) and (2.7) with equation (1.1), we also the following results which are modification of Theorem 1 in [31]:

$$y(n,\lambda)=-\sum_{j=0}^{n}\frac{D_j}{j!\lambda^{j+1}(1-\lambda)^{n+1-j}}, \tag{2.9}$$

and

$$y(n,\lambda)=\sum_{j=0}^{n}\frac{1}{j!\lambda^{j+1}(1-\lambda)^{n+1-j}}\sum_{d=0}^{j}B_dS_1(j,d). \tag{2.10}$$

3. *Integral formulas for the numbers $y(n,\lambda)$*

In this section, we modify definition of the numbers $y(n,\lambda)$. By using this modification, we give some novel integral formulas for these numbers.

Let $a,b\in\mathbb{R}$ with $a\neq b$. Substituting $\lambda=(x-a)/(b-a)$ into (1.1), we get

$$y\left(n,\frac{x-a}{b-a}\right)=(b-a)^{n+2}\sum_{j=0}^{n}\frac{(-1)^{j-1}}{(j+1)(x-a)^{j+1}(b-x)^{n+1-j}}.$$

Thus we get the following result:

Theorem 3.1. *Let $a,b\in\mathbb{R}$ with $a\neq b$. Then we have*

$$\sum_{j=0}^{n}\frac{(-1)^{j-1}}{(j+1)(x-a)^{j+1}(b-x)^{n+1-j}}=\frac{1}{(b-a)^{n+2}}y\left(n,\frac{x-a}{b-a}\right). \tag{3.1}$$

Substituting $a=0$ and $b=1$ into (3.1), we get

$$\sum_{j=0}^{n}\frac{(-1)^{j-1}}{(j+1)x^{j+1}(1-x)^{n+1-j}}=y(n,x).$$

Integrating both sides of Equation (3.1), we get

$$\frac{1}{(a-b)^{n+2}}\int y\left(n,\frac{x+a}{a-b}\right)\,\mathrm{d}x=\sum_{j=0}^{n}\frac{(-1)^n}{j+1}\int\frac{\mathrm{d}x}{(x+a)^{j+1}\,(b+x)^{n+1-j}}.$$

Combining the above equation with the following well-known formula:

$$\int\frac{\mathrm{d}x}{(x+a)^m\,(b+x)^n}=\frac{(-1)^{m+n-2}}{(b-a)^{m+n-1}}\sum_{\substack{k=0\\k\neq n-1}}^{m+n-2}\frac{(-1)^k\binom{m+n-2}{k}}{n-k-1}\left(\frac{x+a}{x+b}\right)^{n-k-1}$$
$$+\binom{m+n-2}{n-1}\frac{(-1)^{m-1}}{(b-a)^{m+n-1}}\ln\left|\frac{x+a}{x+b}\right|,$$

(cf. [18, p. 34]), and after some elementary calculations, we arrive at the following theorem:

Theorem 3.2. *Let $a,b\in\mathbb{R}$ with $a\neq b$. Then we have*

$$\frac{1}{a-b}\int y\left(n,\frac{x+a}{a-b}\right)\,\mathrm{d}x=\sum_{j=0}^{n}\sum_{\substack{k=0\\k\neq n-j}}^{n}\frac{(-1)^{n+k+1}\binom{n}{k}}{(n-j-k)\,(j+1)}\left(\frac{x+a}{x+b}\right)^{n-j-k}$$
$$+\log\left|\frac{x+a}{x+b}\right|\sum_{j=0}^{n}\frac{(-1)^{j+1}}{j+1}\binom{n}{n-j}. \tag{3.2}$$

Putting $a=1$ and $b=0$ in (3.2), we obtain the following corollary:

Corollary 3.1.

$$\int y\,(n,x+1)\,\mathrm{d}x=\sum_{j=0}^{n}\sum_{\substack{k=0\\k\neq n-j}}^{n}\frac{(-1)^{k+n+1}\binom{n}{k}}{(n-j-k)\,(j+1)}\left(\frac{x+1}{x}\right)^{n-j-k}$$
$$+\log\left|\frac{x+1}{x}\right|\sum_{j=0}^{n}\frac{(-1)^{j+1}}{j+1}\binom{n}{n-j}. \tag{3.3}$$

Using (3.1), after some elementary calculations, we also arrive at the following result:

Theorem 3.3. *Let $a,b\in\mathbb{R}$ with $a\neq b$. Then we have*

$$\sum_{j=0}^{n}\frac{(-1)^j}{(j+1)\,(x-a)^{j+1}\,(b-x)^{n+1-j}}=\frac{1}{(b-x)^{n+2}}\int_{0}^{\frac{b-x}{x-a}}\frac{(-1)^n y^{n+1}+1}{y+1}\,\mathrm{d}y.$$

Theorem 3.4.

$$\begin{aligned} y(n,2) &= (-1)^{n+1}2\int_0^1 \frac{x^{2n}}{(1+x)(1+x^2)^n}\,\mathrm{d}x + (-1)^n\left(1+\frac{1}{2^n}\right)\log 2 \\ &\quad +(-1)^n\frac{H_n}{2^n} + (-1)^{n+1}\pi\sum_{j=0}^{n}\frac{1}{2^{j+n+1}}\binom{2j}{j} \\ &\quad +(-1)^n\sum_{j=0}^{n-1}\frac{1}{2^{j+n}}\binom{2j}{j}\sum_{k=1}^{j}\frac{2^k}{k\binom{2k}{k}}. \end{aligned}$$

PROOF. By combining (1.1) with the following well-known novel integral formula

$$\begin{aligned} \int_0^1 \frac{x^{2n}}{(1+x)(1+x^2)^n}\,\mathrm{d}x &= \left(1+\frac{1}{2^n}\right)\frac{\log 2}{2} + \frac{H_n}{2^{n+1}} - \sum_{j=0}^{n}\frac{1}{j2^{j+1}} \\ &\quad - \pi\sum_{j=0}^{n}\frac{1}{2^{j+n+2}}\binom{2j}{j} + \frac{1}{2^{n+1}}\sum_{j=0}^{n-1}\frac{1}{2^j}\binom{2j}{j}\sum_{k=1}^{j}\frac{2^k}{k\binom{2k}{k}}, \end{aligned}$$

(cf. [36, p. 173]), we arrive at the desired result.

4. *Conclusion*

Many properties of the numbers $y(n,\lambda)$ were given. Some new novel formulas and relations involving the Euler gamma function, the Bernoulli numbers, the Pell numbers, the Stirling numbers, the Daehee numbers, and the Catalan-Qi numbers are given with the help of some special values of the numbers $y(n,\lambda)$. Some integral formulas for the numbers $y(n,\lambda)$ were also given. In this paper, some partial solutions some of the open problems for the numbers $y(n,\lambda)$ given in [30] were presented. In future, we continue to arrange to investigate another novel solutions of the open questions given in Section 1.

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