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BOUNDARY VALUES, INTEGRAL TRANSFORMS, AND GROWTH OF VECTOR VALUED HARDY FUNCTIONS

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A b s t r a c t. Banach space valued Hardy functions $H^p,\ 0 < p \leq \infty$, are defined with the functions having domain in tubes $T^C = \mathbb{R}^n + \mathrm{i}C \subset \mathbb{C}^n$; H^2 functions with values in Hilbert space are characterized as Fourier-Laplace transforms of functions which satisfy a certain norm growth property. These H^2 functions are shown to equal a Cauchy integral when the base C of the tube T^C is specialized. For certain Banach spaces and certain bases C of the tube T^C, all H^p functions, $1 \leq p \leq \infty$, are shown to equal the Poisson integral of L^p functions, have boundary values in L^p norm on the distinguished boundary $\mathbb{R}^n + \mathrm{i}\{\overline{0}\}$ of the tube T^C, and have pointwise growth properties. For H^2 functions with values in Hilbert space we show the existence of L^2 boundary values on the topological boundary $\mathbb{R}^n + \mathrm{i}\,\partial C$ of the tube T^C.

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1. *Introduction*

This article can be considered as a review one since the results are the reformulation of the assertions related to scalar cases proved mainly in [12]. For the reader's convenience we give almost all proofs although they are based on ideas already seen in the quoted book. The accumulated knowledge related to vector valued functions can be used for several proofs, but we decide to follow the classical approach in order to show its usefulness.

Let C be a regular cone as defined in Section 2, and let $\mathbf{f}(z), z \in T^C = \mathbb{R}^n + \mathrm{i}C$, be an analytic function with values in a Banach space $\mathcal{X}$. We showed in [5] and [6] that if $\mathbf{f}(z)$ has a distributional boundary value which is an element $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{X}), 1 \leq p \leq \infty$, then $\mathbf{f}(z)$ is an element of the Hardy space $H^p(T^C, \mathcal{X})$. Additionally, in [6] we showed that for $\mathcal{X}$ being a certain type of Banach space, any function $\mathbf{f}(z) \in H^p(T^C, \mathcal{X}), 1 \leq p \leq \infty$, is the Poisson integral of some function $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{X})$.

These results lead us to ask for additional properties of Hardy $H^p(T^C, \mathcal{X})$ analytic functions with values in a Banach space $\mathcal{X}$ and for a more general base C of the tube T^C.

Bochner ([2], [3]) originally defined the Hardy space $H^2(T^C)$ of scalar valued analytic functions on tubes $T^C = \mathbb{R}^n + \mathrm{i}C$ where C is an open convex cone in $\mathbb{R}^n$ and obtained Fourier-Laplace and Cauchy integral representations of the $H^2(T^C)$ functions in terms of $L^2(\mathbb{R}^n)$ functions. Stein, Weiss, and Weiss [11] and Stein and Weiss [12] defined $H^p(T^C)$ functions for $0 < p \leq \infty$ and obtained boundary value and integral representations. Korányi ([9], [10]) has obtained important results concerning $H^p(T^C)$ spaces. We desire to extend scalar valued results to vector valued $H^p(T^C, \mathcal{X})$ spaces with the base C being as general as possible and to obtain new results as appropriate. We obtain boundary value properties of $H^p(T^C, \mathcal{X})$ spaces both on the distinguished boundary $\mathbb{R}^n + \mathrm{i}\{\overline{0}\}$ and on the topological boundary $\mathbb{R}^n + \mathrm{i}\,\partial C$ of the tube T^C where ∂C denotes the boundary of C. We obtain Fourier-Laplace and Cauchy integral representations in addition to the Poisson integral representation. We prove pointwise growth estimates for $H^p(T^C, \mathcal{X})$ spaces.

Before proceeding we note that generalizations of $H^p(T^C)$ functions have been obtained in the scalar valued case. In [4, Chapter 5] we defined scalar valued generalizations of $H^p(T^C)$ functions using sequences which are involved in defining ultradifferentiable functions and ultradistributions and obtained integral representations and boundary value results of these generalizations. Other scalar valued generalizations of $H^p(T^C)$ functions are noted in [4, Chapter 5] as well, and the bibliography of [4] contains a listing of many associated and relevant papers.

2. *Notation and definitions*

All notation and definitions needed in this paper are the same as described or referred in [5] and [6]. We mention and refer to several of the most frequently used notation and definitions here.

We write $\mathcal{X}$, $\mathcal{H}$ and $\mathcal{N}$ to denote a Banach space, a Hilbert space and the norm of the specified Banach or Hilbert space; Θ will denote the zero vector in $\mathcal{X}$ and $\mathcal{H}$. A set $C \subset \mathbb{R}^n$ is a cone with vertex at $\overline{0} = (0, 0, ..., 0)$ in $\mathbb{R}^n$ if $y \in C$ implies $\lambda y \in C$ for all $\lambda > 0$. The intersection of a cone C with the unit sphere $|y| = 1$ is the projection of C and is denoted $\mathrm{pr}(C)$. A cone C' such that $\mathrm{pr}(\overline{C'}) \subset \mathrm{pr}(C)$ is a compact subcone of C, $C' \Subset C$. The dual cone C^* of C is defined as $C^* = \{t \in \mathbb{R}^n : \langle t, y\rangle \geq 0 \text{ for all } y \in C\}$. An open convex cone which does not contain any entire straight line is called a regular cone. Let $v = (v_1, v_2, ..., v_n)$ be any of the 2^n n-tuples whose entries are 0 or 1. The 2^n n-rants $C_v = \{y \in \mathbb{R}^n : (-1)^{v_j} y_j > 0,\ j = 1, 2, ..., n\}$ are examples of regular cones.

The $L^p(\mathbb{R}^n, \mathcal{X})$ functions, $1 \leq p \leq \infty$, with values in $\mathcal{X}$ and their norm $|\mathbf{h}|_p$ are noted in [5, Section 2]. The Fourier transform on $L^1(\mathbb{R}^n)$ or $L^1(\mathbb{R}^n, \mathcal{X})$ is given in [5, Section 2]. All Fourier (inverse Fourier) transforms on scalar or vector valued functions will be denoted $\widehat{\phi} = \mathcal{F}[\phi(t); x]$ $(\mathcal{F}^{-1}[\phi(t); x])$. As stated in [1], the Plancherel theory is not true for vector valued functions except when $\mathcal{X} = \mathcal{H}$, a Hilbert space. The Plancherel theory is complete in the $L^2(\mathbb{R}^n, \mathcal{H})$ setting in that the inverse Fourier transform is the inverse mapping of the Fourier transform with $\mathcal{F}^{-1}\mathcal{F} = I = \mathcal{F}\mathcal{F}^{-1}$ with I being the identity mapping.

Let C be an open subset of $\mathbb{R}^n$ and $\mathcal{X}$ be a Banach space. The Hardy space $H^p(T^C, \mathcal{X})$, $0 < p < \infty$, consists of those analytic functions $\mathbf{f}(z)$ on the tube $T^C = \mathbb{R}^n + \mathrm{i}C \subset \mathbb{C}^n$ with values in the Banach space $\mathcal{X}$ such that for some $M > 0$, and every $y \in C$,

$$\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}(x + \mathrm{i}y)))^p \,\mathrm{d}x \ \leq\ M;$$

the usual modification is made for the case $p = \infty$.

Let C be a regular cone in $\mathbb{R}^n$ and C^* be the corresponding dual cone of C. The Cauchy kernel corresponding to T^C is

$$K(z - t) = \int_{C^*} \mathrm{e}^{2\pi \mathrm{i}\langle z - t, \eta\rangle} \,\mathrm{d}\eta,\ t \in \mathbb{R}^n,\ z \in T^C.$$

The Poisson kernel corresponding to T^C is

$$Q(z; t) = \frac{K(z - t)\overline{K(z - t)}}{K(2\mathrm{i}y)} = \frac{|K(z - t)|^2}{K(2\mathrm{i}y)},\ t \in \mathbb{R}^n,\ z \in T^C.$$

Referring to [4, Chapters 1 and 4] for details, we know for $z \in T^C$ that $K(z-\cdot) \in \mathcal{D}(*, L^p) \subset \mathcal{D}_{L^p},\ 1 < p \le \infty$; and $Q(z;\cdot) \in \mathcal{D}(*, L^p) \subset \mathcal{D}_{L^p},\ 1 \le p \le \infty$, where $*$ is Beurling (M_p) or Roumieu $\{M_p\}$. These ultradifferentiable functions are contained in the Schwartz space $\mathcal{D}_{L^p} = \mathcal{D}(L^p, \mathbb{R}^n)$. Because of the combined properties of the Cauchy and Poisson kernels from [4] and [7], we know that the Cauchy and Poisson integrals

$$\int_{\mathbb{R}^n} \mathbf{h}(t) K(z-t)\, \mathrm{d}t \ \text{ and } \ \int_{\mathbb{R}^n} \mathbf{h}(t) Q(z;t)\, \mathrm{d}t,\ z \in T^C,$$

are well defined for $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{X}),\ 1 \le p < \infty$, and $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{X}),\ 1 \le p \le \infty$, respectively, where $\mathcal{X}$ is a Banach space.

We use [5, Lemma 3.4] several times in this paper. For convenience to the reader we state it here to conclude this section.

Lemma 2.1. *Let $\boldsymbol{f}$ be analytic in $T^C = \mathbb{R}^n + \mathrm{i}C$ with values in a Banach space $\mathcal{X}$, where C is a regular cone in $\mathbb{R}^n$, and have the Poisson integral representation*

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n} \boldsymbol{h}(t) Q(z;t)\, \mathrm{d}t,\ z \in T^C,$$

for $\boldsymbol{h} \in L^p(\mathbb{R}^n, \mathcal{X}),\ 1 \le p \le \infty$. We have $\boldsymbol{f} \in H^p(T^C, \mathcal{X}),\ 1 \le p \le \infty$. For $p = \infty$, $\boldsymbol{f}(x+\mathrm{i}y) \to \boldsymbol{h}(x)$ in the weak-star topology of $L^\infty(\mathbb{R}^n, \mathcal{X})$ as $y \to \overline{0},\ y \in C$; for $1 \le p < \infty$, $\boldsymbol{f}(x+\mathrm{i}y) \to \boldsymbol{h}(x),\ x \in \mathbb{R}^n$, in $L^p(\mathbb{R}^n, \mathcal{X})$ as $y \to \overline{0},\ y \in C$. Let $1 < p \le 2$. Then for every compact subcone $C' \Subset C$, there exists $M(C') > 0$ such that

$$\mathcal{N}(\boldsymbol{f}(x+\mathrm{i}y)) \le M(C') |\boldsymbol{h}|_p |y|^{-n/p},\ z = x+\mathrm{i}y \in T^{C'},$$

while for every $y \in C$ there exists $M_y > 0$ such that

$$\mathcal{N}(\boldsymbol{f}(x+\mathrm{i}y)) \le M_y |\boldsymbol{h}|_p |y|^{-n/p},\ z = x+\mathrm{i}y \in T^C.$$

Let $2 < p < \infty$. For all $C' \Subset C$ and $r > 0$ there exists $M(C', r) > 0$ such that

$$\begin{aligned} &\mathcal{N}(\boldsymbol{f}(x+\mathrm{i}y)) \le M(C', r) |\boldsymbol{h}|_p, \\ &z = x+\mathrm{i}y \in T(C', r) = \left\{ z = x+\mathrm{i}y : x \in \mathbb{R}^n, y \in (C' \setminus (C' \cap N(\overline{0}, r))) \right\}, \end{aligned}$$

while for every $y \in C$ there is $M_y > 0$ such that

$$\mathcal{N}(\boldsymbol{f}(x+\mathrm{i}y)) \le M_y |\boldsymbol{h}|_p,\ z = x+\mathrm{i}y \in T^C.$$

3. *Fourier-Laplace and Cauchy integral representations*

In this section we consider the important case of $p = 2$. Our analysis builds upon that of Stein and Weiss [12, III.2 and III.3] in some instances.

Let $\mathcal{X}$ be a Banach space, C be an open connected subset of $\mathbb{R}^n$ and $\mathbf{g}$ be a measurable function on $\mathbb{R}^n$ with values in $\mathcal{X}$ such that for some $M > 0$

$$\sup_{y\in C}\int_{\mathbb{R}^n} \mathrm{e}^{-4\pi\langle y,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^2\,\mathrm{d}t \le M < \infty \tag{3.1}$$

The estimate (3.1) is the characterizing property of functions $\mathbf{g}$ whose Fourier-Laplace transforms are elements of $H^2(T^C, \mathcal{X})$.

Lemma 3.1. *Let* $\mathbf{g}$ *be a measurable function on* $\mathbb{R}^n$ *with values in the Banach space* $\mathcal{X}$ *such that* (3.1) *holds for* C *being an open connected subset of* $\mathbb{R}^n$. *For any point* $y_o \in C$ *there is a ball* $N(y_o, s) = \{y \in C : |y - y_o| < s\} \subset C$ *such that* $\mathrm{e}^{-2\pi\langle y,t\rangle}\mathcal{N}(\boldsymbol{g}(t))$ *is bounded on* $\mathbb{R}^n$ *by an integrable scalar valued function of* $t \in \mathbb{R}^n$ *independent of* $y \in N(y_o, s)$.

PROOF. Let S be a compact subset of C such that $y_o \in N(y_o, r) \subset S \subset C$. Using (3.1) we have for $y \in N(y_o, r)$ and suitable $M > 0$

$$\begin{aligned} M &\ge \int_{\mathbb{R}^n} \mathrm{e}^{-4\pi\langle y,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^2\,\mathrm{d}t \\ &= \int_{\mathbb{R}^n} \mathrm{e}^{-4\pi\langle y-y_o,t\rangle}\mathrm{e}^{-4\pi\langle y_o,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^2\,\mathrm{d}t. \end{aligned} \tag{3.2}$$

Decompose $\mathbb{R}^n$ into a union of a finite number of non-overlapping cones $C_1, C_2, \ldots, C_k$, having vertices at $\overline{0}$ and such that whenever two points y_1 and y_2 belong to one of these cones the angle between the rays from $\overline{0}$ to y_1 and from $\overline{0}$ to y_2 is less that $\pi/4$ radians. There is a $\delta > 0$ such that $0 < \delta < r$ and $\{y : |y - y_o| = \delta\} \subset N(y_o, r)$. Let $\varepsilon = 4\pi\delta/2^{1/2}$, and choose y_j such that $(y_o - y_j) \in C_j$ and $|y_j - y_o| = \delta$ for each $j = 1, 2, \ldots, k$. For $t \in C_j$ let θ_j denote the angle between the rays from $\overline{0}$ to $(y_o - y_j)$ and from $\overline{0}$ to t; we have

$$\langle y_o - y_j, t\rangle = |y_o - y_j||t|\cos(\theta_j) \ge |y_o - y_j||t|/2^{1/2}$$

and hence

$$\begin{aligned} \varepsilon|t| &= 4\pi\frac{\delta|t|}{2^{1/2}} = 4\pi\frac{|y_o - y_j||t|}{2^{1/2}} \\ &\le 4\pi|y_o - y_j||t|\cos(\theta_j) = -4\pi\langle y_j - y_o, t\rangle. \end{aligned}$$

Thus, for each $j = 1, 2, \dots, k$, by (3.2)

$$\begin{aligned}\int_{\mathbb{R}^n} &\mathrm{e}^{-4\pi\langle y_o,t\rangle}\mathrm{e}^{\varepsilon|t|}(\mathcal{N}(\mathbf{g}(t)))^2\,\mathrm{d}t \\ &= \sum_{j=1}^{k}\int_{C_j}\mathrm{e}^{-4\pi\langle y_o,t\rangle}\mathrm{e}^{\varepsilon|t|}(\mathcal{N}(\mathbf{g}(t)))^2\,\mathrm{d}t \\ &\le \sum_{j=1}^{k}\int_{C_j}\mathrm{e}^{-4\pi\langle y_o,t\rangle}\mathrm{e}^{-4\pi\langle y_j-y_o,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^2\,\mathrm{d}t \\ &\le k\int_{\mathbb{R}^n}\mathrm{e}^{-4\pi\langle y_o,t\rangle}\mathrm{e}^{-4\pi\langle y_j-y_o,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^2\,\mathrm{d}t \le kM < \infty. \quad (3.3)\end{aligned}$$

This implies

$$\int_{\mathbb{R}^n}\mathrm{e}^{-2\pi\langle y_o,t\rangle}\mathrm{e}^{\varepsilon|t|/4}\mathcal{N}(\mathbf{g}(t))\,\mathrm{d}t \le |\mathrm{e}^{-2\pi\langle y_o,t\rangle}\mathrm{e}^{\varepsilon|t|/2}\mathbf{g}(t)|_2||\mathrm{e}^{-\varepsilon|t|/4}||_{L^2(\mathbb{R}^n)} < \infty.$$

Now let $y \in \{y \in C : |y - y_o| < \varepsilon/8\pi\}$; we have for such y and $t \in \mathbb{R}^n$,

$$\begin{aligned}\mathrm{e}^{-2\pi\langle y,t\rangle}\mathcal{N}(\mathbf{g}(t)) &= \mathrm{e}^{-2\pi\langle y_o,t\rangle}\mathrm{e}^{2\pi\langle y_o-y,t\rangle}\mathcal{N}(\mathbf{g}(t)) \\ &\le \mathrm{e}^{-2\pi\langle y_o,t\rangle}\mathrm{e}^{2\pi(\varepsilon/8\pi)|t|}\mathcal{N}(\mathbf{g}(t)) \\ &= \mathrm{e}^{-2\pi\langle y_o,t\rangle}\mathrm{e}^{\varepsilon|t|/4}\mathcal{N}(\mathbf{g}(t))\end{aligned}$$

where, by the assumption, the right side is an integrable function independent of $y \in \{y \in C : |y - y_o| < \varepsilon/8\pi\}$. We thus let $s = \varepsilon/8\pi$ in the conclusion of this lemma. □

Up to the end of this section **g** denotes a measurable function and $\mathcal{H}$ is a Hilbert space.

Theorem 3.1. *Let C be an open connected subset in $\mathbb{R}^n$ and* **g** *satisfy* (3.1). *We have*

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n}\boldsymbol{g}(t)\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\,\mathrm{d}t,\ z \in T^C,$$

is analytic in T^C, and there exists $M > 0$ such that

$$|\boldsymbol{f}(x+\mathrm{i}y)|_2 = |\mathrm{e}^{-2\pi\langle y,t\rangle}\boldsymbol{g}(t)|_2 \le M^{1/2} < \infty,\ y \in C;$$

that is, $\boldsymbol{f}(z) \in H^2(T^C, \mathcal{H})$.

PROOF. By (3.1) and Lemma 3.1, $\mathbf{f}(z)$ is analytic in T^C and $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H}),\ y \in C$. Hence $\mathbf{f}(z) = \mathcal{F}[e^{-2\pi\langle y,t\rangle}\mathbf{g}(t); x],\ y \in C$, in $L^2(\mathbb{R}^n, \mathcal{H})$; and with suitable $M > 0$,

$$|\mathbf{f}(x + iy)|_2 = |e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)|_2 \le M^{1/2} < \infty,\ y \in C.$$

Thus, $\mathbf{f}(z) \in H^2(T^C, \mathcal{H})$. □

Before proving Theorem 3.2, the converse of Theorem 3.1, we need a lemma.

Lemma 3.2. *Let C be an open connected subset of $\mathbb{R}^n$. Let $\mathbf{f}(x + iy) \in H^2(T^C, \mathcal{H})$ and $\varepsilon > 0$. Put*

$$\boldsymbol{g}_{\varepsilon,y}(t) = \int_{\mathbb{R}^n} e^{-\varepsilon\sum_{j=1}^n z_j^2}\mathbf{f}(x + iy)e^{-2\pi i\langle x+iy,t\rangle}\, dx,\ y \in C,\ t \in \mathbb{R}^n,$$

and

$$\boldsymbol{g}_y(t) = \mathcal{F}^{-1}[e^{2\pi\langle y,t\rangle}\boldsymbol{f}(x + iy); t],\ y \in C,\ t \in \mathbb{R}^n,$$

in $L^2(\mathbb{R}^n, \mathcal{H})$. Then, for every $t \in \mathbb{R}^n$,

$$\lim_{\varepsilon\to 0+} |\boldsymbol{g}_{\varepsilon,y}(t) - \boldsymbol{g}_y(t)|_2 = 0,\ \textit{as well as}\ \boldsymbol{g}_{\varepsilon,y}(t) \to \boldsymbol{g}_y(t),\ \varepsilon \to 0+,\ y \in C. \quad (3.4)$$

PROOF. First note that, as functions of $x \in \mathbb{R}^n$, $e^{2\pi\langle y,t\rangle}e^{-\varepsilon\sum_{j=1}^n z_j^2}\mathbf{f}(x + iy)$ is in $L^1(\mathbb{R}^n, \mathcal{H}) \cap L^2(\mathbb{R}^n, \mathcal{H})$ and $e^{2\pi\langle y,t\rangle}\mathbf{f}(x + iy) \in L^2(\mathbb{R}^n, \mathcal{H}),\ y \in C,\ t \in \mathbb{R}^n$. In this proof we may assume $0 < \varepsilon \le 1$ since we are letting $\varepsilon \to 0+$ here. We have

$$\begin{aligned} |\mathbf{g}_{\varepsilon,y}(t) - \mathbf{g}_y(t)|_2 &= |\mathcal{F}^{-1}[e^{2\pi\langle y,t\rangle}(e^{-\varepsilon\sum_{j=1}^n z_j^2} - 1)\mathbf{f}(x + iy); t]|_2 \\ &= |e^{2\pi\langle y,t\rangle}(e^{-\varepsilon\sum_{j=1}^n z_j^2} - 1)\mathbf{f}(x + iy)|_2. \end{aligned} \quad (3.5)$$

Since for every $y \in C$, $|e^{-\varepsilon\sum_{j=1}^n z_j^2} - 1|$ converges to zero and it is bounded as a function of x, by (3.5), the Lebesgue theorem and the Hölder inequality, we have (3.4). □

Theorem 3.2. *Let C be an open connected subset of $\mathbb{R}^n$, and $\boldsymbol{f}(z) \in H^2(T^C, \mathcal{H})$. Assume that $\boldsymbol{f}$ is bounded for $x = \mathrm{Re}(z) \in \mathbb{R}^n$ and $y = \mathrm{Im}(z)$ in any compact subset of C. There exists a measurable function $\boldsymbol{g} \in L^2(\mathbb{R}^n, \mathcal{H})$ satisfying (3.1) such that for $z \in T^C$,*

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n} \boldsymbol{g}(t)e^{2\pi i\langle z,t\rangle}\, dt \quad \textit{and} \quad \sup_{y\in C} |\boldsymbol{f}(x + iy)|_2 = \sup_{y\in C} |e^{-2\pi\langle y,t\rangle}\boldsymbol{g}(t)|_2. \quad (3.6)$$

PROOF. Let $\varepsilon > 0$. Put

$$\mathbf{f}_\varepsilon(z) = \mathrm{e}^{-\varepsilon\sum_{j=1}^n z_j^2}\mathbf{f}(z),\ z \in T^C,$$

which is an element of $L^1(\mathbb{R}^n, \mathcal{H}) \cap L^2(\mathbb{R}^n, \mathcal{H})$ as a function of $x \in \mathbb{R}^n$ for $y \in C$, and

$$\mathcal{N}\left(\int_{\mathbb{R}^n} \mathbf{f}_\varepsilon(x + \mathrm{i}y)\,\mathrm{d}x\right) \leq ||\mathrm{e}^{-\varepsilon\sum_{j=1}^n z_j^2}||_{L^2}|\mathbf{f}(x + \mathrm{i}y)|_2,\ y \in C.$$

Put $\mathbf{g}_{\varepsilon,y}(t) = \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon(x + \mathrm{i}y)\mathrm{e}^{-2\pi\mathrm{i}\langle x+\mathrm{i}y,t\rangle}\,\mathrm{d}x,\ t \in \mathbb{R}^n,\ y \in C$. Let S be any compact subset of C, and let $y \in S$. We have

$$|\mathrm{e}^{-\varepsilon\sum_{j=1}^n z_j^2}| \leq \mathrm{e}^{\varepsilon na^2}\mathrm{e}^{-\varepsilon|x|^2},\ x \in \mathbb{R}^n,\ y \in S,$$

where $a = \max_{y\in S}\{|y_1|, |y_2|, \ldots, |y_n|\}$. For $y \in S \subset C$ and $t \in \mathbb{R}^n$

$$\begin{aligned}\int_S \mathcal{N}(\mathrm{e}^{-\varepsilon\sum_{j=1}^n z_j^2}\mathbf{f}(x + \mathrm{i}y)\mathrm{e}^{-2\pi\mathrm{i}\langle x+\mathrm{i}y,t\rangle})\,\mathrm{d}y &\leq A_S\mathrm{e}^{\varepsilon na^2}\mathrm{e}^{-\varepsilon|x|^2}\int_S \mathrm{e}^{2\pi\langle y,t\rangle}\,\mathrm{d}y \\ &\leq A_S\mathrm{e}^{\varepsilon na^2}\mathrm{e}^{-\varepsilon|x|^2}\int_S \mathrm{e}^{2\pi|y||t|}\,\mathrm{d}y \qquad (3.7)\end{aligned}$$

where A_S is a bound on $\mathcal{N}(\mathbf{f}(x + \mathrm{i}y))$ for $x \in \mathbb{R}^n$ and $y \in S$. The right side of (3.7) approaches 0 as $|x| \to \infty$. Thus an application of the Cauchy-Poincare theorem yields $\mathbf{g}_{\varepsilon,y}$ is independent of $y \in S$ for any $\varepsilon > 0$ and hence independent of $y \in C$ for any $\varepsilon > 0$ since S is an arbitrary compact subset of C. Henceforth we refer to $\mathbf{g}_{\varepsilon,y},\ y \in C$, as $\mathbf{g}_\varepsilon$. We now proceed to obtain the desired function $\mathbf{g}(t)$ with values in $\mathcal{H}$ and conclude the proof. We have $\mathbf{f}(x + \mathrm{i}y) \in H^2(T^C, \mathcal{H})$; hence $\mathrm{e}^{2\pi\langle y,t\rangle}\mathbf{f}(x + \mathrm{i}y) \in L^2(\mathbb{R}^n, \mathcal{H}),\ y \in C,\ t \in \mathbb{R}^n$. Put

$$\mathbf{g}_y(t) = \mathrm{e}^{2\pi\langle y,t\rangle}\mathcal{F}^{-1}[\mathbf{f}(x + \mathrm{i}y); t],\ y \in C, \qquad (3.8)$$

in $L^2(\mathbb{R}^n, \mathcal{H})$. Let y_1 and y_2 both be points in C. Since $\mathbf{g}_\varepsilon = \mathbf{g}_{\varepsilon,y}$ is independent of $y \in C$, for any $\varepsilon > 0$ we have

$$|\mathbf{g}_{y_1}(t) - \mathbf{g}_{y_2}(t)|_2 \leq |\mathbf{g}_{y_1}(t) - \mathbf{g}_{\varepsilon,y_1}(t)|_2 + |\mathbf{g}_{y_2}(t) - \mathbf{g}_{\varepsilon,y_2}(t)|_2. \qquad (3.9)$$

Letting $\varepsilon \to 0+$ in (3.9) and using Lemma 3.2 we have that the right side of (3.9) approaches 0 while the left side of (3.9) is independent of $\varepsilon > 0$. Thus $\mathbf{g}_{y_1}(t) = \mathbf{g}_{y_2}(t)$ a.e., $t \in \mathbb{R}^n$, and $\mathbf{g}_y$ defined in (3.8) is independent of $y \in C$. As a result we write $\mathbf{g}(t)$ in the continuation of this proof instead of $\mathbf{g}_y(t)$, and $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H})$.

From (3.8) we have

$$|\mathrm{e}^{-2\pi\langle y,t\rangle}\mathbf{g}(t)|_2 = |\mathbf{f}(x + \mathrm{i}y)|_2 \leq M,\ y \in C,$$

for the constant M being independent of $y \in C$ from which (3.1) and the second part of (3.6) follow. Since (3.1) is obtained for **g** because of (3.8), we have by the proofs of Lemma 3.1 and Theorem 3.1 that

$$\int_{\mathbb{R}^n} \mathbf{g}(t)\mathrm{e}^{2\pi i\langle z,t\rangle}\,\mathrm{d}t = \mathcal{F}[\mathrm{e}^{-2\pi\langle y,t\rangle}\mathbf{g}(t);x],\ z = x + \mathrm{i}y \in T^C,$$

is analytic in T^C with the Fourier transform being the $L^1(\mathbb{R}^n,\mathcal{H})$ transform. Thus from (3.8)

$$\mathbf{f}(z) = \mathcal{F}[\mathrm{e}^{-2\pi\langle y,t\rangle}\mathbf{g}(t);x] = \int_{\mathbb{R}^n} \mathbf{g}(t)\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\,\mathrm{d}t,\ z \in T^C,$$

with the Fourier transform being in both the $L^1(\mathbb{R}^n,\mathcal{H})$ and $L^2(\mathbb{R}^n,\mathcal{H})$ sense, and the first part of (3.6) is obtained. The proof of Theorem 3.2 is completed. □

In the scalar valued case Stein and Weiss have shown [12, p. 93, Corollary 2.4] that the restriction of the base C of the tube T^C to be an open convex subset of $\mathbb{R}^n$, as opposed to the more general open connected subset of $\mathbb{R}^n$, suffices to obtain the same results as in [12, p. 93, Theorem 2.3], which are similar to Theorems 3.1 and 3.2, in the scalar valued case.

We now restrict the base C in Theorem 3.2 to be an open convex cone in $\mathbb{R}^n$ and obtain additional information on the constructed function **g** and on the Fourier-Laplace integral representation of the assumed function $\mathbf{f}(z),\ z \in T^C$.

Theorem 3.3. *Let C be an open convex cone in $\mathbb{R}^n$, and $\boldsymbol{f}(z) \in H^2(T^C,\mathcal{H})$ satisfy the boundedness hypothesis of Theorem* 3.2. *There exists a measurable function $\boldsymbol{g} \in L^2(\mathbb{R}^n,\mathcal{H})$ satisfying* (3.1) *with* $\operatorname{supp}(\mathbf{g}) \subseteq C^*$ *a.e. for which*

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n} \boldsymbol{g}(t)\mathrm{e}^{2\pi i\langle z,t\rangle}\,\mathrm{d}t = \int_{C^*} \boldsymbol{g}(t)\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\,\mathrm{d}t,\ z \in T^C, \tag{3.10}$$

$$\sup_{y\in C}|\boldsymbol{f}(x+\mathrm{i}y)|_2 = \sup_{y\in C}|\mathrm{e}^{-2\pi\langle y,t\rangle}\boldsymbol{g}(t)|_2 = \left(\int_{C^*}(\mathcal{N}(\boldsymbol{g}(t)))^2\,\mathrm{d}t\right)^{1/2} = |\boldsymbol{g}|_2. \tag{3.11}$$

PROOF. We showed in the proof of Theorem 3.2 that the function $\mathbf{g}_y(t)$ defined in (3.8) is independent of $y \in C$. Further, $\mathbf{g} \in L^2(\mathbb{R}^n,\mathcal{H})$. We now show that $\operatorname{supp}(\mathbf{g}) \subseteq C^*$ a.e. Let t_o be any point in $\mathbb{R}^n \setminus C^* = C_*$ which is an open set in $\mathbb{R}^n$ since C^* is a closed set in $\mathbb{R}^n$. From the definition of C^* there is a point $y_o \in C$ such that $\langle y_o,t_o\rangle < 0$. Since C_* is open there is a neighborhood $N(t_o,r) = \{t \in \mathbb{R}^n : |t-t_o| < r\} \subset C_*$ and a $\delta > 0$ such that $\langle y_o,t\rangle < -\delta < 0$ for $t \in N(t_o,r)$. For

$\lambda > 0$ being any positive real number, $\lambda y_o \in C$ since C is a cone. From (3.8) we have for arbitrary $\lambda > 0$ that

$$\begin{aligned} e^{2\pi\lambda\delta}\left(\int_{N(t_o,r)} (\mathcal{N}(\mathbf{g}(t)))^2 \, dt\right)^{1/2} &= \left(\int_{N(t_o,r)} e^{4\pi\lambda\delta}(\mathcal{N}(\mathbf{g}(t)))^2 \, dt\right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^n} e^{-4\pi\langle\lambda y_o,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^2 \, dt\right)^{1/2} \\ &= |\mathbf{f}(x+i\lambda y_o)|_2 \leq M \end{aligned}$$

for all $\lambda > 0$ with δ being a fixed real number depending only on y_o and being independent of $t \in N(t_o, r)$. Upon letting $\lambda \to \infty$, we conclude that $\mathbf{g}(t) = \Theta$ a.e. for $t \in N(t_o, r)$. Since t_o is any point of $C_* = \mathbb{R}^n \setminus C^*$ we conclude that $\text{supp}(\mathbf{g}) \subseteq C^*$ a.e. Combining this fact with the conclusion (3.6) of Theorem 3.2 we obtain (3.10).

Since $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H})$ we have

$$\int_{\mathbb{R}^n} e^{-4\pi\langle y,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^2 \, dt = \int_{C^*} e^{-4\pi\langle y,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^2 \, dt \leq |\mathbf{g}|_2^2, \ y \in C. \quad (3.12)$$

From (3.8) and (3.12)

$$\sup_{y\in C} |\mathbf{f}(x+iy)|_2 = \sup_{y\in C} |e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)|_2 \leq |\mathbf{g}|_2 < \infty. \quad (3.13)$$

But $\overline{0} \in C^* = \{t \in \mathbb{R}^n : \langle t, y\rangle \geq 0 \text{ for all } y \in C\}$. Hence the inequality in (3.13) is in fact an equality, and (3.11) is obtained. □

The same form of proof of support which we used to obtain $\text{supp}(\mathbf{g}) \subseteq C^*$ a.e. in Theorem 3.3 can be used to show that the functions $\mathbf{g}_\varepsilon(t)$ in the proof of Theorem 3.2 also have support contained in C^* a.e. for $\varepsilon > 0$ sufficiently close to 0 in the case that the base C of the tube T^C in Theorem 3.2 is an open convex cone.

In the following result we show that if this cone is not regular, that is if C is an open convex cone which does contain an entire straight line, then the function $\mathbf{f}(z) \in H^2(T^C, \mathcal{H})$ in these results is $\mathbf{f}(z) = \Theta, \ z \in T^C$.

Corollary 3.1. *Let C be an open convex cone in $\mathbb{R}^n$ which contains an entire straight line. Let $\boldsymbol{f}(z) \in H^2(T^C, \mathcal{H})$ satisfy the bounded hypothesis of Theorem* 3.2. *We have $\boldsymbol{f}(z) = \Theta, \ z \in T^C$.*

PROOF. From Theorem 3.3 there exists a function $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H})$ with $\text{supp}(\mathbf{g}) \subseteq C^*$ a.e. for which

$$\mathbf{f}(z) = \int_{C^*} \mathbf{g}(t) e^{2\pi i\langle z,t\rangle} \, dt, \ z \in T^C.$$

But by Vladimirov [13, Lemma 1, p. 222] the dual cone C^* of C has measure zero. Thus from the above integral representation of $\mathbf{f}(z)$ we have $\mathbf{f}(z) = \Theta, z \in T^C$. □

The Vladimirov result [13, Lemma 1, p. 222] also yields a converse to Corollary 3.1. That is for a function $\mathbf{f}(z) \in H^2(T^C, \mathcal{H})$ which satisfies the hypotheses of Theorem 3.2 for C being an open convex cone in $\mathbb{R}^n$ with $\mathbf{f}(z) \neq \Theta,\ z \in T^C,\ C$ must be a regular cone.

The fact of functions in $H^2(T^C, \mathcal{H})$ not being identically Θ can be extended to tubes T^B where B is an open convex subset of $\mathbb{R}^n$ which does not contain an entire straight line since for such a subset B of $\mathbb{R}^n$ we have from [12, p. 94] the existence of a regular cone C such that $B \subseteq C$.

For regular cones C we use Theorem 3.3 to obtain a Cauchy integral representation of $H^2(T^C, \mathcal{H})$ functions. It is especially appropriate here that C be a regular cone because of the definition of the Cauchy kernel $K(z-t), z \in T^C, t \in \mathbb{R}^n$, as an integral over C^*.

Corollary 3.2. *Let C be a regular cone and $\boldsymbol{f}(z) \in H^2(T^C, \mathcal{H}),\ z \in T^C$, satisfy the boundedness hypothesis of Theorem 3.2. There exists $\boldsymbol{h} \in L^2(\mathbb{R}^n, \mathcal{H})$ such that*

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n} \boldsymbol{h}(t) K(z-t)\,\mathrm{d}t,\ z \in T^C. \tag{3.14}$$

PROOF. From Theorem 3.3 we have a function $\mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H})$ with $\operatorname{supp}(\mathbf{g}) \subseteq C^*$ a.e. such that (3.10) holds. There exists a function $\mathbf{h} \in L^2(\mathbb{R}^n, \mathcal{H})$ such that $\mathbf{h}(t) = \mathcal{F}[\mathbf{g}(u); t]$ in $L^2(\mathbb{R}^n, \mathcal{H})$. From the proof of [7, Lemma 2.1] we know $I_{C^*}(u)\mathrm{e}^{-2\pi\langle y,u\rangle}\mathrm{e}^{2\pi i\langle x,u\rangle} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),\ z = x + \mathrm{i}y \in T^C$, where $I_{C^*}(u)$ is the characteristic function of C^*. For $z = x + \mathrm{i}y \in T^C$ we have

$$\mathbf{f}(z) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}[\mathbf{h}(t); u] I_{C^*}(u)\mathrm{e}^{-2\pi\langle y,u\rangle}\mathrm{e}^{2\pi i\langle x,u\rangle}\,\mathrm{d}u \tag{3.15}$$

$$= \int_{\mathbb{R}^n} \mathbf{h}(t) \int_{C^*} \mathrm{e}^{-2\pi\langle y,u\rangle}\mathrm{e}^{2\pi\mathrm{i}\langle x,u\rangle}\mathrm{e}^{-2\pi\mathrm{i}\langle t,u\rangle}\,\mathrm{d}u\,\mathrm{d}t$$

$$= \int_{\mathbb{R}^n} \mathbf{h}(t) K(z-t)\,\mathrm{d}t. \tag{3.16}$$

□

We can obtain the Poisson integral representation of the function $\mathbf{f}(z)$ of Corollary 3.2 from Corollary 3.2 by the same technique that we used in [5, Theorem 4.2, Eq. (26)] to obtain the Poisson integral representation from the Cauchy integral representation. But we will consider the Poisson integral representation of $H^p(T^C, \mathcal{X})$ functions more generally in the next section.

4. *Poisson integral representation*

Note [6, Section 4] for terminology, definitions, and references here. In [6, Theorem 2] we proved that if $\mathcal{X}$ is a dual Banach space having the Radon-Nikodým property (this holds in particular if $\mathcal{X}$ is reflexive), if C is a regular cone, and if $1 \le p \le \infty$, then each $\mathbf{f}(z) \in H^p(T^C, \mathcal{X})$ has the Poisson integral representation

$$\mathbf{f}(z) = \int_{\mathbb{R}^n} \mathbf{h}(t) Q(z;t)\,\mathrm{d}t,\ z \in T^C, \tag{4.1}$$

for some $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{X})$.

Using [6, Theorem 2] and [5, Lemma 3.4], which is stated as Lemma 2.1 in this paper for convenience, we have:

Theorem 4.1. *Let C be a regular cone in $\mathbb{R}^n$, and let $\mathcal{X}$ be a dual Banach space having the Radon-Nikodým property. Let $\boldsymbol{f}(z) \in H^p(T^C, \mathcal{X}),\ 1 \le p \le \infty$. There exists $\boldsymbol{h} \in L^p(\mathbb{R}^n, \mathcal{X}),\ 1 \le p \le \infty$, such that* (4.1) *holds. For $p = \infty$, $\boldsymbol{f}(x + \mathrm{i}y) \to \boldsymbol{h}(x)$ in the weak-star topology of $L^\infty(\mathbb{R}^n, \mathcal{X})$ as $y \to \overline{0},\ y \in C$; for $1 \le p < \infty$, $\boldsymbol{f}(x + \mathrm{i}y) \to \boldsymbol{h}(x),\ x \in \mathbb{R}^n$, in $L^p(\mathbb{R}^n, \mathcal{X})$ as $y \to \overline{0},\ y \in C$. Let $1 < p \le 2$. For every compact subcone $C' \Subset C$ there exists $M(C')$ such that*

$$\mathcal{N}(\boldsymbol{f}(x + \mathrm{i}y)) \le M(C') |\boldsymbol{h}|_p |y|^{-n/p},\ \ z = x + \mathrm{i}y \in T^{C'},$$

while for every $y \in C$ there exists $M_y > 0$ such that

$$\mathcal{N}(\boldsymbol{f}(x + \mathrm{i}y)) \le M_y |\boldsymbol{h}|_p |y|^{-n/p},\ \ z = x + \mathrm{i}y \in T^C.$$

Let $2 < p < \infty$. Then for every compact subcone $C' \Subset C$ and $r > 0$ there exists $M(C', r) > 0$ such that

$$\mathcal{N}(\boldsymbol{f}(x + \mathrm{i}y)) \le M(C', r) |\boldsymbol{h}|_p,$$
$$z = x + \mathrm{i}y \in T(C', r) = \big\{ z = x + \mathrm{i}y : x \in \mathbb{R}^n,\ y \in (C' \setminus (C' \cap N(\overline{0}, r))) \big\},$$

while for every $y \in C$ there exists $M_y > 0$ such that

$$\mathcal{N}(\boldsymbol{f}(x + \mathrm{i}y)) \le M_y |\boldsymbol{h}|_p,\ \ z = x + \mathrm{i}y \in T^C.$$

5. *Boundary values*

In this section we desire to obtain boundary value results for functions $\mathbf{f}(z) \in H^2(T^C, \mathcal{H})$ on the topological boundary $\mathbb{R}^n + \mathrm{i}\,\partial C$ for C being a regular cone and $\mathcal{H}$ being a Hilbert space; here ∂C represents the boundary of C. Regular cones are important both mathematically and in physical applications. Interesting regular cones are the self-dual regular cones; a cone $C \subset \mathbb{R}^n$ is said to be self-dual if $C^* = \overline{C}$. For example, any of the 2^n n-rants C_v defined in Section 1.1 are self-dual cones as are the future and past light cones described in Vladimirov [13, p. 219]. Our boundary value result on the topological boundary $\mathbb{R}^n + i\partial C$ for functions $\mathbf{f}(z) \in H^2(T^C, \mathcal{H})$ for regular cones C is as follows where we choose the cone C to be regular to avoid the result of Corollary 3.1 and we choose $p = 2$ and the Hilbert space, instead of a more general Banach space, because we use the Fourier transform in our proof.

Theorem 5.1. *Let C be a regular cone in $\mathbb{R}^n$ and $\mathcal{H}$ be a Hilbert space. Let $\boldsymbol{f}(z) \in H^2(T^C, \mathcal{H})$ and satisfy the bounded hypothesis of Theorem* 3.2. *Let $y_o \in \partial C$. There exists a function $\boldsymbol{F}(x + \mathrm{i}y_o) \in L^2(\mathbb{R}^n, \mathcal{H})$ such that*

$$\lim_{y \to y_o} \boldsymbol{f}(x + \mathrm{i}y) = \boldsymbol{F}(x + \mathrm{i}y_o), \;\; y \in C, \;\; \textit{in } L^2(\mathbb{R}^n, \mathcal{H}). \tag{5.1}$$

PROOF. From Theorem 3.3 there is a measurable function $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H})$ satisfying (3.1) with $\text{supp}(\mathbf{g}) \subseteq C^*$ a.e. such that (3.10) and (3.11) hold. Let $y_o \in \partial C$, and let $\{y_m\}, m = 1, 2, \ldots$, be a sequence of points in C which converges to y_o. By Fatou's lemma

$$\int_{\mathbb{R}^n} \mathrm{e}^{-4\pi\langle y_o, t\rangle} (\mathcal{N}(\mathbf{g}(t)))^2 \, \mathrm{d}t \le \limsup_{y_m \to y_o} \int_{\mathbb{R}^n} \mathrm{e}^{-4\pi\langle y_m, t\rangle} (\mathcal{N}(\mathbf{g}(t)))^2 \, \mathrm{d}t \le |\mathbf{g}|_2^2 < \infty.$$

Thus for any $y_o \in \partial C, \; \mathrm{e}^{-2\pi\langle y_o, t\rangle} \mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H})$. We form

$$\mathbf{F}(x + \mathrm{i}y_o) = \mathcal{F}[\mathrm{e}^{-2\pi\langle y_o, t\rangle} \mathbf{g}(t); x], \; y_o \in \partial C, \tag{5.2}$$

and $\mathbf{F}(x + \mathrm{i}y_o) \in L^2(\mathbb{R}^n, \mathcal{H})$. From (3.8), (5.2), and the Plancherel equality we have

$$\begin{aligned} |\mathbf{f}(x + \mathrm{i}y) - \mathbf{F}(x + \mathrm{i}y_o)|_2 &= |\mathcal{F}[(\mathrm{e}^{-2\pi\langle y, t\rangle} - \mathrm{e}^{-2\pi\langle y_o, t\rangle}) \mathbf{g}(t); x]|_2 \\ &= |(\mathrm{e}^{-2\pi\langle y, t\rangle} - \mathrm{e}^{-2\pi\langle y_o, t\rangle}) \mathbf{g}(t)|_2 \end{aligned} \tag{5.3}$$

for $y \in C$ and $y_o \in \partial C$.

By assumption C is a regular cone. Thus $\text{supp}(\mathbf{g}) \subseteq C^*$ a.e. and both $\langle y, t\rangle \ge 0$ and $\langle y_o, t\rangle \ge 0, \; y \in C, \; y_o \in \partial C, \; t \in C^*$. Since supp $(\mathbf{g}) \subseteq C^*$ a.e. we have for all $y \in C$ and almost every $t \in \mathbb{R}^n$

$$(\mathcal{N}((\mathrm{e}^{-2\pi\langle y, t\rangle} - \mathrm{e}^{-2\pi\langle y_o, t\rangle}) \mathbf{g}(t)))^2 \le 4(\mathcal{N}(\mathbf{g}(t)))^2 \tag{5.4}$$

a.e. for $y_o \in \partial C$. Put $G(t) = 4 I_{C^*}(t) (\mathcal{N}(\mathbf{g}(t)))^2, \; t \in \mathbb{R}^n$, where $I_{C^*}(t)$ is the characteristic function of C^*. Since $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H}), G(t)$ is an integrable function of $t \in \mathbb{R}^n$, is independent of $y \in C$, and

$$(\mathcal{N}((\mathrm{e}^{-2\pi\langle y, t\rangle} - \mathrm{e}^{-2\pi\langle y_o, t\rangle}) \mathbf{g}(t)))^2 \le G(t)$$

for almost all $t \in \mathbb{R}^n$. By the Lebesgue dominated convergence theorem and (5.3) we obtain (5.1) in $L^2(\mathbb{R}^n, \mathcal{H})$ as $y \to y_o,\ y \in C,\ y_o \in \partial C$. □

Stein and Weiss have proved a boundary value result for scalar valued $H^2(T^C)$ functions at points y_o on the ∂C where C is an open polyhedron in $\mathbb{R}^n$ [12, Corollary 2.9, p. 97]. They define an open polyhedron in $\mathbb{R}^n$ to be the interior of the convex hull of a finite subset of $\mathbb{R}^n$. Using our analysis in Theorem 5.1 together with the extension of the analysis used to prove [12, Corollary 2.9, p. 97] extended to functions $\mathbf{f}(z) \in H^2(T^C, \mathcal{H})$, we have the following extensions of the results [12, Corollary 2.9, Corollary 2.10, pp. 97–98].

Theorem 5.2. *Let C be an open polyhedron in $\mathbb{R}^n$ and $\mathcal{H}$ be a Hilbert space. Let $\boldsymbol{f}(z) \in H^2(T^C, \mathcal{H})$ and satisfy the bounded hypothesis of Theorem 3.2. Let $y_o \in \partial C$. There exists a function $\boldsymbol{F}(x + \mathrm{i}y_o) \in L^2(\mathbb{R}^n, \mathcal{H})$ such that (5.1) holds in $L^2(\mathbb{R}^n, \mathcal{H})$.*

Theorem 5.3. *Let $\mathcal{H}$ be a Hilbert space. Let B be an open convex subset of $\mathbb{R}^n$ and y_o be a point on its boundary. Let $\boldsymbol{f}(z) \in H^2(T^B, \mathcal{H})$ and satisfy the bounded hypothesis of Theorem 3.2. Let C be an open polyhedron contained in B having y_o as a boundary point. There exists a function $\boldsymbol{F}(x + \mathrm{i}y_o) \in L^2(\mathbb{R}^n, \mathcal{H})$ such that (5.1) holds in $L^2(\mathbb{R}^n, \mathcal{H})$ as $y \to y_o,\ y \in C$.*

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