Bulletin T.CLIII de l'Académie serbe des sciences et des arts − 2020
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 45

THE AHLFORS-SCHWARZ LEMMA, CURVATURE, DISTANCE AND DISTORTION

MIODRAG MATELJEVIĆ

(Presented at the 5th Meeting, held on June 16, 2021)

A b s t r a c t. In this paper, we consider versions of the Ahlfors-Schwarz lemma related to ultrahyperbolic metric, quasiconformal harmonic and holomorphic functions (including several dimensions versions) and harmonic mappings. Our exposition includes some recently obtained results.

AMS Mathematics Subject Classification (2020): 30F45, 32G15

Key Words: Hyperbolic metric, ultrahyperbolic metric, harmonic quasiregular maps, negatively curved metrics.

1. *Introduction*

The paper contains new results, but also a background that is of a review character, but the author believes that there are also novelties in the parts that describe related results. It can be considered as continuation of our paper [61].

In section 2., we focus on ultrahyperbolic and pseudohermitian metrics, the Ahlfors-Schwarz lemmas (see Chapter 1 of Ahlfors lovely book [2]) and to the estimate opposite to the Ahlfors-Schwarz lemma proved by the author.

In particular, we outline proof of Theorem 2.8, which is announced in the previous author work: If ρ is a complete C^2 density on the unit disk $\mathbb{D}$ whose curvature is bounded from below by a constant $-k^2, k > 0$, then $Hyp_{\mathbb{D}} \leq k\rho$ on $\mathbb{D}$, where $Hyp_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}$. It turns out that our proof is relies on the generalized maximum principle of Cheng and Yau which is used for example in proof Lemma 1.2 [79] by Tam and Wan.

The planar Schwarz lemma for holomorphic and harmonic functions and the Schwarz type inequalities for harmonic functions in multidimensional ball are considered in Section 3..

In the literature the Schwarz lemma for harmonic maps from $\mathbb{U}$ into itself which fix the origin, Theorem 3.2 below, is often attributed to E. Heinz [27]. Later, in 1977, H. W. Hethcote [28] improved the planar results by removing the assumption $f(0) = 0$. B. Burgeth [14] (1992) (see also the papers by H.A. Schwarz and E.J.P.G. Schmidt cited there) considered the Schwarz lemma for harmonic and hyperbolic-harmonic functions in higher dimensions and in [15] (1994) (see also the R.B. Burckel book cited there) the Schwarz type inequalities for harmonic functions in the ball. In Axler et al. book [7] and Khavinson [34] also versions of the Schwarz lemma for harmonic maps which fix origin are given. Although the Burgeth method covers the case of dimension two and applies it to mappings that do not fix the coordinate origin, researchers usually overlook his work. We have tried to draw attention to his work in the author works with M.Svetlik, A. Khalfallah, M. Mhamdi and B. Purtić [64, 36, 62, 67] and explain that it covers the case of dimension two, and in this section we will also sketch the Burgeth method. M. Mateljević and M. Sveltik [64](2020) basically using the strip method and principle of subordination proved a Schwarz lemma for real harmonic functions defined on the planar unit disk with values in $(-1, 1)$. Note that their results apply to mappings that do not fix the coordinate origin. Burgeth method is presented in subsection 3..

The paper [32] by Kalaj-Vuorinen turned our attention to the subject and later we realized that forms of the Schwarz lemma given in F. Colonna [20] and Khavinson [34] are tightly related to [32].

At the end of this section we further discuss the subject including Liu conjecture and announce new results concerning gradient estimate for functions represented by the Poisson type kernels.

In section 4. we consider versions of the Ahlfors-Schwarz lemma related to negative curvatures and strip codomains. New results basically relied on the strip method, which can be extended to pluriharmonic functions in several variables, simplify and improve M. Marković results in [47].

Section 5. is devoted to the Ahlfors-Schwarz lemma for harmonic quasiregular maps. We discuss briefly some properties of harmonic maps and the author result with M. Knežević [41].

Concerning new results we prove that a harmonic quasi-isometric map between pinched Hadamard surfaces is bi-Lip, Theorem 5.5. The proof is based on Y. Benoist-D. Hulin result that a harmonic quasi-isometric map between pinched Hadamard surfaces is quasiconformal (previously proved by V. Marković in the case of hyperbolic disks; see Section 5. for more details).

In section 6. a short review of [6] is given. As an application of the Schwarz lemma we outline a proof that the Bers space is characterized by means of maximal φ-disks.

For the convenience of the reader we have tried to present the basic definitions and properties of the subject we are considering. We are aware that our presentation is incomplete, among other things, due to the goals of the journal to usually publish the original results of a limited number of pages.

Also, due to the limited time for preparing the text, some parts of the text were not ironed. Since there is a huge literature some relevant results are not cited and we apologize to their authors.

2. *The Ahlfors-Schwarz lemma*

Hyperbolic distance and Schwarz lemma. By Δ, $\mathbb{D}$ or $\mathbb{U}$ we denote the unit disk. Let B be the disk with center at z_0 and radius r. Using the conformal automorphisms $\phi_a(z) = (z-a)/(1-\overline{a}z)$, $a \in \Delta$, of Δ, one can define pseudo-hyperbolic distance on Δ by

$$\delta(a,b) = |\phi_a(b)|, \quad a,b \in \Delta\,.$$

Next, using the conformal map $A(\zeta) = (\zeta - z_0)/r$ from B onto Δ, one can define pseudo-hyperbolic distance on B by

$$\delta_B(z,w) = \delta(A(z), A(w))$$

and the *hyperbolic distance* on B by

$$\lambda(z,w) = \log \frac{1+\delta_B(z,w)}{1-\delta_B(z,w)}$$

for $z,w \in B$.

In particular, hyperbolic distance on the unit disk Δ is

$$\lambda(z,\omega) = \log \frac{1+|\frac{z-\omega}{1-z\bar{\omega}}|}{1-|\frac{z-\omega}{1-z\bar{\omega}}|}\,.$$

The classic Schwarz lemma states: If $f : \Delta \to \Delta$ is an analytic function, and if $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality $|f(z)| = |z|$ with $z \neq 0$ or $|f'(0)| = 1$ can occur only for $f(z) = \mathrm{e}^{\mathrm{i}\alpha}\, z,\ \alpha$ is a real constant.

It was noted by Pick that result can be expressed in invariant form. We refer the following result as the Schwarz-Pick lemma.

Theorem 2.1 (Schwarz-Pick lemma)**.** *Let F be an analytic function from a disk B to another disk U. Then F does not increase the corresponding hyperbolic (pseudo-hyperbolic) distances.*

2.1. Curvature

A Riemannian metric given by the fundamental form

$$\mathrm{d}s^2 = \rho^2(\,\mathrm{d}x^2 + vy^2)$$

or $vs = \rho|\,\mathrm{d}z|$, $\rho > 0$, is conformal with euclidian metric.

If $\rho > 0$ is a C^2 function on Δ, the Gaussian curvature of a Riemannian metric defined by $\mathrm{d}s = \rho|\,\mathrm{d}z|$, $z = x + \mathrm{i}y$, (we call ρ metric density or simply density) on Δ is expressed by the formula

$$K = K_\rho = -\rho^{-2}\Delta \log \rho.$$

Also we write $K(\rho)$ instead of K_ρ.

If we use notation

$$\mathrm{d}s^2 = \sigma(\,\mathrm{d}x^2 + \,\mathrm{d}y^2)$$

for metric, then the Gaussian curvature is

$$K = K_\sigma = -\frac{1}{2\,\sigma}\Delta \log \sigma\,.$$

If there is possibility of confusion with the above expression K_ρ we can write here $K_{*\sigma}$ instead of K_σ. Recall that a pseudohermitian metric (density) on Δ is a non-negative upper semicontinuous function ρ such the set $\rho^{-1}(0)$ is discrete in Δ.

If u is an upper semicontinuous function, the *lower generalized Laplacian* of u is defined by ([1], see also [24])

$$\Delta_L\, u(\omega) = 4 \liminf_{r\to 0} \frac{1}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} u(\omega + r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t - u(\omega)\right).$$

When u is a C^2 function , then the lower generalized Laplasian of u reduces to the usual Laplacian

$$\Delta u = u_{xx} + u_{yy}\,.$$

The Gaussian curvature of a pseudohermitian metric density ρ on Δ is defined by the formula

$$K = K_\rho = -\rho^{-2}\Delta_L \log \rho\,.$$

For all $a > 0$ define a family of functions λ_a

$$\lambda_a(z) = \frac{2}{a(1-|z|^2)}\,.$$

Also, it is convenient to write λ instead of λ_1.

By standard calculation

$$\overline{\partial}\partial \log \lambda = -\overline{\partial}\partial \log(1 - z\overline{z}) = \overline{\partial}\frac{\overline{z}}{1 - z\overline{z}} = \frac{1}{(1-z\overline{z})^2}.$$

Thus $\Delta\lambda = -\lambda^2$ and therefore $K(\lambda) = -1$. Hence the Gaussian curvature of density λ_a is $K(\lambda_a) = -a^2$. This family of Hermitian metrics on Δ is of interest because it allows an ordering of all pseudohermitian metrics on Δ in the sense of Lemma 2.1 and Theorem 2.2 below, see [1].

If M is a surface with the metric of the form $\rho(z)|\,\mathrm{d}z|^2$ in local z-coordinate, then $\Delta_M = \Delta_0/\rho$, where $\Delta_0 = 4D_{z\overline{z}}$ is Euclidean Laplacian, and $K_M = -\frac{1}{2}\Delta_M \log\rho$. We also write Δ_ρ instead of Δ_M.

2.2. The Ahlfors lemma 1 and ultrahyperbolic metrics

Lemma 2.1. *If ρ satisfies $K(\rho) \le -1$ everywhere in $\mathbb{D}$, then $\rho(z) \le \lambda(z)$ for an $z \in \mathbb{D}$.*

PROOF. We assume first that ρ has a continuous and strictly positive extension to the closed disk. From $\Delta \log\lambda = \lambda^2$, $\Delta\log\rho \ge \rho^2$ we have $\Delta(\log\lambda - \log\rho) \le \lambda^2 - \rho^2$.

The function $\log\lambda - \log\rho$ tends to $+\infty$ when $|z| \to 1$. It therefore has a minimum in the unit disk.

At the point of minimum $\Delta(\log\lambda - \log\rho) \ge 0$ and hence $\lambda^2 - \rho^2 \ge 0$ at that point, proving that $\lambda \ge \rho$ everywhere. The general case follows easily, one uses $\rho_r(z) = r\rho(rz)$ where $r < 1$ and lets r converge to 1.

We sketch another proof of the lemma based on the variation of the above argument. Since $\Delta(\log\rho - \log\lambda) \ge \rho^2 - \lambda^2$, now the function $\eta = \log\rho - \log\lambda$ has a maximum in the unit disk at a point z_0. At the point of maximum $\Delta\eta(z_0) \le 0$ and hence $0 \ge \rho^2 - \lambda^2$.

It is natural to ask if the above lemma has a generalization to the nonconformal case. Hence we have the following question:

Question 1. *Let S be a surface which is parametrized by the unit disk and with first fundamental form* $\mathrm{d}s^2 := E\,\mathrm{d}x^2 + 2F\,\mathrm{d}x\,\mathrm{d}y + G\,\mathrm{d}y^2$. *If $K_S \le -1$ and d_S is the distance on S whether $d_S(z_1, z_2) \le Hyp(z_1, z_2)$, $z_1, z_2 \in \mathbb{D}$?*

We outline an idea to consider this question. We can use conformal representation of surfaces Jost [29] (Chapter 3, Theorem 3.1.1) to write $\mathrm{d}s$ in the form $\mathrm{d}s_* = \rho|\,\mathrm{d}w|$. More precisely there is mapping $z = g(w)$ of the unit disk onto itself which is an isometry between $(S,\,\mathrm{d}s)$ and $(S,\,\mathrm{d}s_*)$, i.e., $d_\rho(z_1, z_2) = d_S(z_1, z_2)$. Then we apply Lemma 2.1.

The following theorem follows from the above lemma.

Theorem 2.2. *Let ρ be a pseudohermitian metric density on Δ such that*

$$K_\rho(z) \le -a^2$$

for some $a > 0$. Then $\rho \le \lambda_a$.

This kind of estimate is similar to Ahlfors-Schwarz lemma.
Ahlfors lemma can be found in Ahlfors [2].

The hyperbolic metric density of a disk $|z| < R$ is given by

$$\lambda_{*R}(z) = \frac{2R}{R^2 - |z|^2}\,.$$

Definition 2.1. A metric ρ is said to be ultrahyperbolic in a region Ω if it has the following properties:

(a) ρ is upper semicontinuous; and

(b) at every $z_0 \in \Omega$ with $\rho(z_0) > 0$ there exits a supporting metric ρ_0, defined and class C^2 in a neighborhood V of z_0, such that $\rho_0 \le \rho$ and $K_{\rho_0} \le -1$ in V, while $\rho_0(z_0) = \rho(z_0)$.

Remark 2.1. It seems that hypotheses (a) and (b) imply that ρ is continuous. Here we outline a simple argument. For given $\varepsilon > 0$ by hypothesis (a) there is $\delta_1 > 0$ such that (i) $\rho(z) - \rho(z_0) < \varepsilon$ for $|z - z_0| < \delta_1$. By hypothesis (b) $\rho_0(z) - \rho_0(z_0) \le \rho(z) - \rho(z_0)$. Since ρ_0 is continuous there is $\delta_2 > 0$ such that $-\varepsilon < \rho_0(z) - \rho_0(z_0)$ and therefore (ii) $-\varepsilon < \rho(z) - \rho(z_0)$. It follows from (i) and (ii) that ρ is continuous.

A metric ρ is said to be a-ultrahyperbolic, $a > 0$, in a region Ω if it has the following properties:

(a) ρ is upper semicontinuous; and

(b) at every $z_0 \in \Omega$ there exits a supporting metric ρ_0, defined and class C^2 in a neighborhood V of z_0, such that $\rho_0 \le \rho$ and $K_{\rho_0} \le -a^2$ in V, while $\rho_0(z_0) = \rho(z_0)$.

Note that ρ is ultrahyperbolic iff ρ/a is a-ultrahyperbolic.

The notion of an ultrahyperbolic metric makes sense, and the theorem remains valid if Ω is replaced by a Riemann surface.

2.3. The Poincaré metric of a domain

If ρ is ultrahyperbolic in $|z| < R$, then $\rho \leq \lambda_{*R}$. In particular, if ρ is ultrahyperbolic in the whole plane, then $\rho = 0$. Hence there is no ultrahyperbolic metric in the whole plane. The same is true of the punctured plane $C^* = \{z : z \neq 0\}$. Indeed, if ρ were ultrahyperbolic metric in the whole plane, then $\rho(\mathrm{e}^z)\ |\ \mathrm{e}^z\ |$ would be ultrahyperbolic in the whole plane. These are only cases in which ultrahyperobolic metric fails to exist.

Theorem 2.3. *In a plane region Ω whose complement has at least two points, there exists a unique maximal ultrahyperbolic metric, and this metric has constant curvature -1.*

Such domains we call hyperbolic domains.

The maximal metric is called the *Poincaré metric* of Ω, and we denote it by λ_Ω or hyp_Ω. It is maximal in the sense that every ultrahyperbolic metric ρ satisfies $\rho \leq \lambda_\Omega$ throughout Ω.

The proof of Theorem 2.3 can be based on nonelementary result that there is complete holomorphic covering $\pi : \mathbb{U} \to \Omega$ (see section 4. for more details). Using holomorphic covering $\pi : \mathbb{U} \to S$, one can define the pseudo-hyperbolic and the hyperbolic metric on Ω. In particular, if $S = G$ is hyperbolic planar domain we can use

(I-1) $\mathrm{hyp}_\Omega(\pi z)|\pi'(z)| = \mathrm{hyp}_{\mathrm{d}}(z)$ and

(I-2) If G and D are hyperbolic domains and f is a conformal mapping of D onto G, then $\mathrm{hyp}_G(fz)|f'(z)| = \mathrm{hyp}_D(z)$.

Let $C_{a,b}$ be the complement of two point set $\{a, b\}$. By Theorem 2.4 below there is a complete holomorphic covering $C_{a,b}$ by the unit disk and therefore there is the Poincaré metric on $C_{a,b}$ which we denote by $\lambda_{a,b}$. Before statement of Theorem 2.4 we introduce a group Γ.

Let Γ be the group generated by σ and τ, where

$$\sigma(z) = \frac{z}{2z+1}, \quad \tau(z) = z + 2. \tag{2.1}$$

Let Q be the set of all z which satisfy $y > 0, -1 \leq x < 1, |2z+1| \geq 1, |2z-1| \geq 1$. Q is a *fundamental domain* of Γ.

Theorem 2.4. (i) *There exists a function $\phi = P_{0,1}$ analytic on the upper half plane $\mathbb{H}^+$, which is invariant under Γ and the range $\Omega_* = \mathbb{C}_{0,1}$ of ϕ is the region consisting of all complex number different from* 0 *and* 1.

(ii) *ϕ has the real axis as its natural boundary.*

PROOF. We construct such map, the so-called *modular function*, as follows. Let $\mathrm{E} = \{0 < x < 1, y > 0, |2z-1| > 1\}$. Then $\partial\mathrm{E} = l_1 \cup l_2 \cup l_3$, where $l_1 = \{iy : y > 0\}$, $l_2 = \{z : |2z-1| = 1, y > 0\}$ and $l_3 = \{1 + iy : y > 0\}$. In view of the Riemann mapping theorem, there is ϕ mapping E to $\mathbb{H}$, l_1, l_2, l_3 to $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$ respectively, and fixing $0, 1, \infty$. Let E^* denote the reflection of E through $\{|2z-1| = 1\}$. By the Schwarz reflection principle we can extend ϕ to a conformal mapping of $\mathrm{E} \cup \mathrm{E}^* \cup l_2$ to $\mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$. By continuing to reflect we can extend ϕ to all of $V_0 = \{0 < x < 1, y > 0\}$ taking its values in $\mathbb{C} \setminus \{0, 1\}$. Define $L_n = \{x = n\}$, where n is integer. We can first reflect across L_0 and extend ψ to all of $V_1 = \{-1 < x < 1, y > 0\}$ and then reflect across L_1 and then reflect across a convenient subsequence of L_n one can extend ϕ to all of $\mathbb{H}$. From the construction the extended ϕ is a covering map of $\mathbb{H}$ over Ω_*.

One can prefer to construct directly covering of $\mathbb{C}_{0,1}$ with d, cf. [72]; let D_1 be an ideal triangle with vertices $z_1 = 1$, $z_2 = \mathrm{e}^{\mathrm{i}2\pi/3}$ and $z_3 = \mathrm{e}^{-\mathrm{i}2\pi/3}$, and $l_1 = z_1z_2$, $l_2 = z_2z_3$ and $l_3 = z_3z_1$. In view of the Riemann mapping theorem, there is λ mapping D_1 to $\mathbb{H}$, l_1, l_2, l_3 to $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$ respectively, and fixing $0, 1, \infty$; further proceed by the reflection as above.

Example 2.1. If $a \in \mathbb{D}$, use the mapping $\varphi = \varphi_{-a} \circ \exp$ from $\{z : \operatorname{Re} z < 0\}$ to $\mathbb{D} \setminus \{a\}$ construct hyperbolic metric on $\mathbb{D} \setminus \{a\}$.

2.4. The Ahlfors lemma 2

The next theorem generalize Lemma 2.1.

Theorem 2.5 (Ahlfors Lemma 1)**.** *Suppose ρ is an ultrahyperbolic metric on Δ. Then $\rho \le \lambda$.*

PROOF. Since $\log\lambda - \log\rho$ is lower semicontinuous the existence of a minimum is assured at a point z_0 with $\rho(z_0) \neq 0$. The function $\log\lambda - \log\rho$ has also local minimum at z_0 and we apply the same reasoning as before.

The version presented in [22] has a slightly modified definition of supporting metric. This modification and formulation is due to Earle. This version has been used (see [22]) to prove that $Teichmüller$ distance is less than equal to $Kobayashi's$ on $Teichmüller$ space.

Ahlfors [2] proved a stronger version of Schwarz's lemma and Ahlfors lemma 1.

Theorem 2.6 (Ahlfors lemma)**.** *Let f be an analytic mapping of Δ into a region on which there is given ultrahyperbolic metric ρ. Then $\rho[f(z)]\,|f'(z)| \le \lambda$.*

The proof consists of observation that $\rho[f(z)]\,|f'(z)|$ is an ultrahyperbolic metric on Δ. Observe that the zeros of $f'(z)$ are singularities of this metric.

Note that if f is the identity map on Δ we get Theorem 2.5 (Ahlfors lemma 1) from Theorem 2.6.

Ahlfors [2], used Theorem 2.6 to prove Bloch and the Picard theorems. Ultrahypebolic metrics (without the name) were introduced by Ahlfors. They found many applications in the theory of several complex variables.

To understand the notion of ultrahyperbolic metric, we advise the interested reader to construct an example of ultrahyperbolic metric which is not hyperbolic; see Example 2.3 below.

2.5. Elementary examples

By $\mathbb{D}'$ and $C_{a,b}$ we denote the punctured disk $\mathbb{D}\setminus\{0\}$ and the complement of the set $\{a,b\}$ and its Poincaré metric densities by σ^0 and $\lambda_{a,b}$ respectively. We also by H^+ and P^- denote the upper half plane $\{z : \operatorname{Im} z > 0\}$ and the left half plane $\{z : \operatorname{Re} z < 0\}$ respectively.

If $\phi : G_* \to G$ is conformal and ρ a metric density on G , then by
(1): $\rho_*(w) = \rho(\phi(w))|\phi'(w)|$,
we can define a metric density on G^* (pull back of ρ). Using notation $z = \phi(w)$, we can rewrite (1) as $\rho_*(w) = \rho(z)|z'(w)|$. In particular, if $\rho = \mathrm{hyp}_G$ then $\rho^* = \mathrm{hyp}_{G^*}$.

Example 2.2. 1. Check that $\lambda_{H^+}(z) = 1/y$ and $\lambda_{P^-}(z) = -1/x$.
2. Prove that

$$\lambda_{0,1}(z) \le \sigma^0(z) = \left(|z| \log \frac{1}{|z|}\right)^{-1} \tag{2.2}$$

for $z \in \mathbb{D}'$.

We outline a proof of (2.2).

Proof. Note that mapping E defined by $w \to \mathrm{e}^w$ maps P^- onto $\mathbb{D}'$ and locally has inverse e^{-1} which we write simply as $w = w(z)$. If $z = \mathrm{e}^w$, then (2): $Rew = \log|z|$, and $z' = \mathrm{e}^w$, $|z'(w)| = |z|$ and therefore (3): $|w'(z)| = 1/|z|$. By (1) $\sigma^0(z) = \lambda_{P^-}(w)|w'(z)| = \frac{1}{|\operatorname{Re} w|}|w'(z)|$ and then by (2) and (3)

$$\sigma^0(z) = \frac{1}{|\log|z||}\frac{1}{|z|} = \left(|z| \log \frac{1}{|z|}\right)^{-1}.$$

Hence we obtain (2.2).

Example 2.3. (i) For $0 < a < 1$, $K(a\lambda_{*s}) \leq -1$ and if $s < a/b$, then $a\lambda_{*s}(0) = 2a/s > 2b = b\lambda(0)$. Choose $b < a < 1$ and $s = a/b > 1$. Then in addition $K(b\lambda) < -1$ and $a\lambda_{*s}(0) = 2b$. Since $\lambda = \infty$ for $|z| = 1$ equation $b\lambda - a\,\lambda_{*s} = 0$ in z has a solution $r_1 \in (0,1)$. Set $\rho = a\lambda_{*s}$ for $|z| \leq r_1$ and $\rho = b\lambda$ for $r_1 < |z| < 1$. For every fixed $y \in (-1,1)$, $\rho(x,y)$ as function of x has minimum for $x = 1/2$.

Is ρ an ultrahyperbolic metric on $\mathbb{U}$? The answer is yes.

(ii) Let ρ is defined on $D = \mathbb{U} \cap B(1,1)$ by $\rho = \lambda$ for $\operatorname{Re} z \geq 1/2$ and $\rho(z) = \lambda(1-z)$ for $\operatorname{Re} z \leq 1/2$.

Is ρ an ultrahyperbolic metric on D ? The answer is yes.

(iii) Let $G_1 = \{|z| < 1, |z| < |z-1|\}$, $G_2 = \{|z-1| < 1, |z| > |z-1|\}$, $G = G_1 \cup G_2$ and G_3 the complement of G.

Let us construct hyperbolic density on $G_1 \setminus \{0\}$ which has continuous extension to bG_1. Set $l = [1,\infty)$, $C_l = C \setminus l$ and let $\zeta(z)$ maps conformally on the unit disk. On $\mathbb{D}'$ we consider hyperbolic density σ^1 which is restriction of Poincare density σ of $B(0,\mathrm{e}^4)'$ (the punctured disk $0 < |\zeta| < \mathrm{e}^4$) and then we transform σ to hyperbolic density ρ_l on C_l. Here we have explicit formulas

$$\sigma^1(\zeta) = \frac{1}{|\zeta|}(4 - \log|\zeta|)^{-1}$$

and $\rho_l(z) = \sigma^1(\zeta)|\zeta'(z)|$. Let ρ on G_1 is the restriction of ρ_l, and extend ρ on G_2 and G_3 by symmetry $\rho(1-z) = \rho(z)$ and $\rho(1/z) = |z|^2\rho(z)$. The extended metric is clearly continuous. Show that ρ is ultrahyperbolic on G and therefore $\lambda_{0,1} \geq \rho$ on G_1.

(iv) Hence $\log\lambda_{0,1} = -\log|z| - \log\log|1/z| + O(1)$ for $z \to 0$ (see[2] for details).

The mapping $\zeta(z)$ is given explicitly by

$$\zeta(z) = A(\sqrt{1-z}),$$

where $A(w) = (w-1)/(w+1)$ with $\operatorname{Re}\sqrt{1-z} > 0$.

2.6. An inequality opposite to the Ahlfors-Schwarz lemma

Mateljević [52] proved an estimate opposite to the Ahlfors-Schwarz lemma.

A metric $H|dz|$ is said to be superhyperbolic in a region Ω if it has the following properties :

(a) H is continuous (more general, lower semicontinuous) on Ω.

(b) at every z_0 there exists a supporting metric (from above) H_0, defined and class C^2 in a neighborhood V of z_0, such that $H_0 \geq H$ and $K_{H_0} \geq -1$ in V, while $H_0(z_0) = H(z_0)$.

A metric $H|\,\mathrm{d}z|$ is said to be b-superhyperbolic in a region Ω if it has the following properties:

(a) H is continuous (more general, lower semicontinuous) on Ω.

(b) there is $b > 0$ such that at every z_0 there exists a supporting metric (from above) H_0, defined and class C^2 in a neighborhood V of z_0, such that $H_0 \geq H$ and $K_{H_0} \geq -b^2$ in V, while $H_0(z_0) = H(z_0)$.

Theorem 2.7 ([52]). *Suppose H is a superhyperbolic metric on Δ for which*
(c) *$H(z)$ tends to $+\infty$ when $|z|$ tends to 1_-.*
Then $\lambda \leq H$.

PROOF. Let $\rho_r(z) = 2r(1-|rz|^2)^{-1}$, where $r \in (0,1)$, and let

$$\Psi_r(z) = \log|H(z)| - \log\rho_r(z)\,.$$

By the hypothesis of theorem Ψ_r has a minimum on $\mathbb{D}$ at a point z_0. Let H_0 be supporting metric density from above to H at z_0 in a neighborhood V and

$$\tau_r(z) = \log|H_0(z)| - \log\rho_r(z)\,.$$

τ_r has a minimum on V at z_0 and so

$$0 \leq \Delta\tau_r(z_0) = \Delta\log|H_0(z_0)| - \Delta\log\rho_r(z_0)\,. \tag{1}$$

By the hypothesis, we have

$$K_{H_0}(z_0) = -H_0(z_0)^{-2}(\Delta\log H_0)(z_0) \geq -1\,,$$

that is

$$(\Delta\log H_0)(z_0) \leq H_0(z_0)^2\,,$$

and

$$(\Delta\log\rho_r)(z_0) = (\rho_r(z_0))^2\,.$$

Hence by (1),

$$0 \leq \Delta\tau_r(z_0) = \Delta\log|H_0(z_0)| - \Delta\log\rho_r(z_0) \leq H_0^2(z_0) - (\rho_r(z_0))^2 \tag{2}$$

and therefore $\rho_r(z_0) \leq H_0(z_0)$. Since $H_0(z_0) = H(z_0)$ it follows that Ψ_r has non-negative minimum at z_0 and hence we conclude that $\rho_r \leq H$ for every $z \in \mathbb{D}$. If r tends to 1_-, we find $\rho \leq H$ on $\mathbb{D}$.

We need a special case of known lemma of Cheng and Yau (the generalized maximum principle of Cheng and Yau).

Lemma 2.2. *Let (S, g) be a complete surface with the Gaussian curvature bounded below by some constant. If there two positive constant $a, b > 0$ such that*

$$\Delta_g u \geq a\mathrm{e}^{bu} - c$$

near the boundary of M, then u is bounded from above.

Remark 2.2. 1. Let H be complete density on $\mathbb{D}$ with curvature $K_H \geq -1$. Since $K_H = -\Delta_{H^2} \log H$, by the hypothesis, we have $\Delta_{H^2} u \geq -1$, where $u = -\log H$ and we can not directly apply Lemma 2.2.

Note here that in subsection 2. (below) we state the generalized maximum principle of Cheng and Yau as Lemma 2.4 which is much more general result than Lemma 2.2 (see below). Since the function $f = -1$ does not satisfy (2.3) in this lemma we also can not directly apply it here.

2. But if we consider $\eta = \log\lambda - \log H$, then

$$\Delta_{H^2}\eta \geq \frac{\lambda^2}{H^2} - 1$$

and hence since $\frac{\lambda}{H} = \mathrm{e}^{\eta}$, we obtain $\Delta_{H^2}\eta \geq \mathrm{e}^{2\eta} - 1$. By Lemma 2.2, we conclude that there is a real number c such that $\eta \leq c$ and therefore $\log\lambda - c \leq \log H$, i.e., $H \geq \mathrm{e}^{-c}\lambda$. Hence H tends $+\infty$ if $|z| \to 1_{-0}$.

3. There is interesting phenomenon in our approach. Namely we can not prove directly that $u_0 = -\log H$ is bounded above but if we add term λ which tends to $+\infty$, we can apply the generalized maximum principle. It seems that we need to investigate this phenomenon.

By item (3) of the remark we have:

Claim 1. *if H is a complete density on $\mathbb{D}$ with curvature $K_H \geq -1$, then H tends to $+\infty$ as $|z|$ tends to 1.*

Repeat

$$K(z_0) = -H_0(z_0)^{-2}(\Delta \log H_0)(z_0) \geq -1\,,$$

that is

$$(\Delta \log H_0)(z_0) \leq H_0(z_0)^2\,.$$

By applying a method developed by Yau in [87] (or by generalized maximum principle of Cheng and Yau [18]), it follows that this result holds if we suppose instead of (c) that

(d) H is a complete metric on Δ.

Theorem 2.8. *If ρ is a complete C^2 density on the unit disk $\mathbb{D}$ whose curvature bounded from below by a constant $-k^2$, $k > 0$, then $hyp_{\mathbb{D}} \leq k\rho$ on $\mathbb{D}$.*

We postpone for moment the proof of this result.[1]

During the consideration of this theorem, we had a problem to justify application of the maximum principle and we realized that we can not apply directly Theorem 2.7. Then we found that in proof Lemma 1.2 [79] Tam-Wan justified application of the generalized maximum principle.

Here we will consider again their lemma.

Lemma 2.3. *Suppose $u_1, u_2 \in C^2(\mathbb{D})\, \mathrm{e}^{u_2}|\,\mathrm{d}z|$ is a complete metric on $\mathbb{D}$, $\Delta_0 u_1 \geq \mathrm{e}^{2u_1}$ and $\Delta_0 u_2 \leq \mathrm{e}^{2u_2}$. Then* (1) $u_1 \leq u_2$.

Recall that Δ_0 denotes the euclidean Laplacian.

PROOF. Set $\eta = u_1 - u_2$. Then $\Delta_0\eta \geq \mathrm{e}^{2u_1} - \mathrm{e}^{2u_2}$ and therefore (2) $\Delta_{\rho_2}\eta = \mathrm{e}^{-2u_2}\Delta_0\eta \geq \mathrm{e}^{2\eta} - 1$. Hence, by applying generalized maximum principle there is c such that $\eta < c$. Next let $\eta^* = \sup\eta < +\infty$. By Omori-Yau maximum principle (see below) there exists a sequence of points $\{z_k\}, k \in \mathbb{N}$, in $\mathbb{D}$ with the properties

$$\text{(i)}\quad \eta(z_k) > \eta^* - \frac{1}{k},\quad \text{(ii)}\quad |\nabla_{g_2}\eta(z_k)| < \frac{1}{k},\quad \text{and}\quad \text{(iii)}\quad \triangle_{g_2}\eta(z_k) < \frac{1}{k}.$$

By (i), (iii) and (2), $\mathrm{e}^{2\eta(z_k)} - 1 \leq 1/k$ and therefore $\mathrm{e}^{2\eta^*} \leq 1$. Hence $\eta^* \leq 0$.

We can avoid using Omori-Yau maximum principle in the above proof. Under hypothesis of the lemma set $\rho_1 = \mathrm{e}^{u_1}$ and $\rho_2 = \mathrm{e}^{u_2}$. Then $K(\rho_1) \leq -1$ and $K(\rho_2) \geq -1$ If set $\eta = \lambda - u_2$, then as above by applying generalized maximum principle there is c such that $\eta < c$ and therefore $\lambda - \rho_2 < c$ and ρ_2 satisfies hypothesis Theorem 2.7 and by application of this theorem $\lambda \leq \rho_2$. Finally by the Ahlfors lemma $\rho_1 \leq \lambda \leq \rho_2$ (1) follows. Thus we have proved

Claim 2. *Under hypothesis of the lemma, $\rho_1 \leq \lambda \leq \rho_2$.*

From this claim Theorem 2.8 follows.

Versions of the Ahlfors-Schwarz lemma and opposite inequalities to the Ahlfors-Schwarz lemma we shortly call the comparison principle for metrics.

Claim 3. *If ρ^+ and ρ are densities on $\mathbb{D}$, ρ^+ is a complete, $K_{\rho^+}(z) \geq -a^2$ and $K_\rho(z) \leq -b^2$, $a, b > 0$. Then $(a/b)\rho^+ \geq \rho$.*

PROOF. By the comparison principle for metric we have, $\rho^+ \geq \lambda_a$, $\rho \leq \lambda_b$ and therefore $a\rho^+ \geq \lambda \geq b\rho$.

[1] A. Seepi asked me in privative communication for the paper in which this result is proved.

We leave the interested reader to consider the following result.

Theorem 2.9. *If ρ and σ are two metrics on Δ, σ complete and $0 > K_\sigma \geq K_\rho$ on Δ, then $\sigma \geq \rho$.*

This theorem remains valid if ρ is ultrahyperbolic metric and σ superhyperbolic metric on Δ. Also, we can get further generalizations if Δ is replaced by a Riemann surface.
The method of sub-solutions and super-solutions have been used in study harmonic maps between surfaces.

2.7. Omori-Yau (O-Y) maximum principle

We first need definition of Ricci tensor.

Definition 2.2. For a manifold (M, g) of dimension n, the description of the Ricci tensor evaluated on a unit length vector $X \in T_pM$,$p \in M$, is given by

$$Ric(X, X) = \sum_{j=2}^{n} k(L_{X,e_j}),$$

where $(X, e_2, \dots, e_n)$ is an orthonormal basis of T_pM and where $k(L_{X,e_j})$ is the sectional curvature of the plane $L_j = L_{X,e_j}$ generated by X and e_j inside T_pM. Since the sectional curvature of a plane L_j is just the Gaussian curvature at p of the surface $S_j = \exp(L_j)$, we find $Ric(X, X) = \sum_{j=2}^{n} K_{S_j}(p)$, where K_{S_j} is the gaussian curvature of S_j. If $n = 2$ this is $Ric(X, X) = K_M(p)$. Since Ric at a point p is entirely determined by its values on unit vectors as above, it is easy to deduce that you get in fact $Ric = K_M g$ as tensors.

Lemma 2.4 (The generalized maximum principle of Cheng and Yau)**.** *Let (M, g) be a complete manifold with the Ricci curvature bounded below by some constant. Suppose that u is a real valued function, and $u \in C^2(M)$ satisfies $\Delta_g u \geq f(u)$, where $f(x)$ is a continuous function which is positive and non-decreasing near ∞ and satisfies*

$$\int_p^\infty \mathrm{d}t \left(\int_q^t f(x)\,\mathrm{d}x \right)^{-1/2} < \infty \tag{2.3}$$

for some constants $p > q$. Then u is bounded from above.

In particular if there are two positive constant $a, b > 0$ such that $\Delta_g u \geq a\mathrm{e}^{bu} - c$ near the boundary of M, then u is bounded from above.

For example $f(x) = c(\mathrm{e}^{2x} - \mathrm{e}^{-2x} - 1)$, which is positive and non-decreasing as x tends to $+\infty$. The function $f(x)$ satisfies the above inequality. In our situations, we usually take $f(x) = A\mathrm{e}^{ax} - B\mathrm{e}^{-bx} - C$ for some constants a, b, A, B, and $C > 0$ to apply the Lemma.

For an introduction to the generalized Omori-Yau maximum principle and its applications to geometry see for example L. J. Alías slides lecture [4]. Following the terminology introduced by Pigola,Rigoli and Setti (2005), the Omori-Yau (O-Y) maximum principle is said to hold on a Riemannian manifold M if, for any smooth function $u \in C^2(M)$ with $u^* = \sup u < +\infty$ there exists a sequence of points $\{x_k\}, k \in \mathbb{N}$, in M with the properties

$$\text{(i)}\quad u(x_k) > u^* - \frac{1}{k}, \quad \text{(ii)}\quad |\nabla u(x_k)| < \frac{1}{k}, \quad \text{and} \quad \text{(iii)}\quad \triangle u(x_k) < \frac{1}{k}.$$

Equivalently, for any smooth function $u \in C^2(M)$ with $u_* = \inf u > -\infty$ there exists a points $\{x_k\}, k \in \mathbb{N}$, in M with the properties

$$\text{(i)}\quad u(x_k) > u_* + \frac{1}{k}, \quad \text{(ii)}\quad |\nabla u(x_k)| < \frac{1}{k}, \quad \text{and} \quad \text{(iii)}\quad \triangle u(x_k) > \frac{1}{k}.$$

Theorem 2.10 (Omori-Yau maximum principle)**.** *The O-Y maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below.*

The weak Omori-Yau (O-Y) maximum principle is said to hold on a Riemannian manifold M if, for any smooth function $u \in C^2(M)$ with $u^* = \sup u < +\infty$ there exists a points $\{x_k\}, k \in \mathbb{N}$, in M with the properties (i) $u(x_k) > u^* - 1/k$, and (iii) $\triangle u(x_k) < 1/k$.

As proved by Pigola, Rigoli and Setti (2001), the fact that the weak Omori-Yau maximum principle holds on M is equivalent to the stochastic completeness of the manifold. This is equivalent (among other conditions) to the fact that for every $\lambda > 0$, the only non-negative bounded smooth solution u of $\Delta u \geq \lambda u$ on M is the constant $u = 0$. In particular, every parabolic Riemannian manifold is stochastically complete.

3. *The planar Schwarz lemma for holomorphic functions and multidimensional for harmonic maps*

3.1. The Schwarz lemma for holomorphic functions

(i) By $\mathbb{C}$ we denote the complex plane, by $\mathbb{U}$ the unit disk and by $\mathbb{T}$ the unit circle.

By ∂G or bG we denote the boundary of a set G.

If f is a function on a set X and $x \in X$ sometimes we write fx instead of $f(x)$. For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $u \in T_x\mathbb{R}^n$, we denote by $|u| = |u|_e$ the Euclidean norm of u. Let G be an open set in $\mathbb{R}^n$. For a mapping $f : G \to \mathbb{R}^m$ which is differentiable at $x \in G$ by $f'(x)$ (or $\mathrm{d}f_x = (\mathrm{d}f)_x$ we denote the corresponding linear mapping from the tangent space $T_x\mathbb{R}^n$ into the tangent space $T_{fx}\mathbb{R}^m$ and by $|f'(x)|$ (or $||(\mathrm{d}f)_x||$; shortly $|(\mathrm{d}f)_x|$) its norm with respect to the given norms on G and $f(G)$.

(ii) Throughout this paper by $\mathbb{S}(a, b)$ we denote the set $(a, b) \times \mathbb{R}$, $-\infty \leq a < b \leq \infty$, and in particular we write $\mathbb{S}_0$ for $\mathbb{S}(-1, 1)$. Note that $\mathbb{S}(a, b)$ is a strip if $-\infty < a < b < \infty$ and $\mathbb{S}(a, +\infty)$ is a half-plane if a is a real number, and $\mathbb{S}(-\infty, +\infty) = \mathbb{C}$.

(iii) If w is a complex number by $u = \mathrm{Re}\, w$ we denote the corresponding real part, and in a similar way if f is a complex-valued function defined on set G we usually write $f = u + iv$, where u and v are real valued functions defined on G by $u(z) = \mathrm{Re}\, f(z)$ and $v(z) = \mathrm{Im}\, f(z)$, $z \in G$. We write $u = \mathrm{Re}\, f$ and $v = \mathrm{Im}\, f$ and call it the corresponding real and imaginary part of the function f respectively; and by $\nabla f(z) = (f'_x, f'_y)$ we denote the complex gradient of f.

(iv) Let D be a domain in $z = x + \mathrm{i}y$-plane and $\mathrm{d}s$ a Riemannian metric on D which is conformal with Euclidean metric. Then $\mathrm{d}s$ is given by the fundamental form $\mathrm{d}s = \rho|\,\mathrm{d}z|$, $\rho > 0$. In this paper we suppose that ρ is continuous function on the corresponding domain. In some situations it is convenient to call ρ shortly metric density and denote by d_ρ the corresponding metric.

A plane region D whose complement has at least two points we call a hyperbolic plane domain. It is an important result in the geometric function theory that on a hyperbolic plane domain there is a unique complete hyperbolic density(metric) whose the Gaussian curvature is -1. For a hyperbolic plane domain D, we also denote by ρ_D (or λ_D) the hyperbolic density (and by abusing notation the hyperbolic metric occasionally), by d_D the hyperbolic metric and by σ_D the pseudo-hyperbolic metric on D. If we wish to be more specific we denote by $\mathrm{hyp}_D(z)$ the hyperbolic density at $z \in D$ and by $d_{\mathrm{hyp},D}$ (or simple hyp_D) the hyperbolic metric. To avoid abusing of notation we write $\mathrm{Hyp}_D(z_1, z_2)$ for the hyperbolic distance between $z_1, z_2 \in D$. Occasionally by λ_0 and ρ_0 we denote respectively hyperbolic metric on the unit disk and on the strip $\mathbb{S}_0$.

In this paper we choose that the hyperbolic density (metric) on the unit disk $\mathbb{U}$ is given by

$$\mathrm{hyp}_{\mathbb{U}}(z) = \frac{2}{1 - |z|^2}\,, \quad z \in \mathbb{U}\,. \tag{3.1}$$

This is motivated by the fact that then the Gaussian curvature of this metric is -1.

For the convenience of the reader we first collect and stress the following simple properties (A-1), (A-2), (I)–(IV).

(A-1) If ϕ is a conformal mapping of a planar domain D onto $\mathbb{U}$, we define the hyperbolic density on D by $\mathrm{hyp}_D(z) = \mathrm{hyp}_{\mathbb{U}}(\phi z)|\phi'(z)|$, $z \in D$.

(A-2) If G and D are simply connected domains different from $\mathbb{C}$ and ϕ conformal mapping of D onto G, then $\mathrm{hyp}_G(\phi z)|\phi'(z)| = \mathrm{hyp}_D(z)$, $z \in D$.

In particular, using a conformal mapping from the unit disk $\mathbb{U}$ onto $\mathbb{S}_0$ one can define the hyperbolic density $\mathrm{hyp}_{\mathbb{S}_0}$ of $\mathbb{S}_0$ and get the following version of Schwarz-Pick lemma(see Section 3. for more details and more general versions related to hyperbolic domains):

(I) If G is a simply connected domain different from $\mathbb{C}$ and ω a holomorphic mapping from $\mathbb{U}$ into G, then $||(\,\mathrm{d}\omega)_z|| \leq 1$, for every $z \in \mathbb{U}$, where $||(\,\mathrm{d}\omega)_z||$ is defined with respect to the hyperbolic norms on the corresponding tangent spaces $T_z\mathbb{C}$ and $T_{\omega z}\mathbb{C}$.

For a more general version of (I) (concerning the corresponding results related to hyperbolic domains) see Proposition 3.2 and (ScAh1) for the geometric form below. In particular if $G = \mathbb{S}_0$, in the statement (I), the following holds:

(I-1) (i) Suppose that ω is a holomorphic mapping from the unit disk $\mathbb{U}$ into $\mathbb{S}_0$.

a) Then $\mathrm{hyp}_{\mathbb{S}_0}(\omega z)|\omega'(z)| \leq \mathrm{hyp}_{\mathbb{U}}(z)$, $z \in \mathbb{U}$, with the equality at some $z \in \mathbb{U}$ if and only if ω is a conformal mapping of $\mathbb{U}$ onto $\mathbb{S}_0$.

b) If in addition to (i), we have $\omega(0) = 0$, then $|\omega'(0)| \leq 4/\pi$ with the equality if and only if ω is a conformal mapping of $\mathbb{U}$ onto $\mathbb{S}_0$ with $\omega(0) = 0$.

Note that by the definition given by the equation (3.1), $\mathrm{hyp}_{\mathbb{U}}(0) = 2$, and it is well known that $\mathrm{hyp}_{\mathbb{S}_0}(0) = \pi/2$ (see Example 3.2 below) and therefore b) follows from a).

We will use the following connection between harmonic and holomorphic functions:

(II). If $f = u + \mathrm{i}v$ is a complex valued harmonic and $F = U + \mathrm{i}V$ holomorphic function on a region D such that $\mathrm{Re}\, f = \mathrm{Re}\, F$ on D (in this setting we say that F is associated to f), then

(a) $F' = U_x + \mathrm{i}V_x = U_x - \mathrm{i}U_y = u_x - \mathrm{i}u_y = \overline{\nabla u}$, and $\nabla u = (u_x, u_y)$.

(b) In particular, $|F'| = |\overline{\nabla u}| = |\nabla u|$.

The following property of strip domains is crucial for our derivation of Schwarz lemma for harmonic and pluriharmonic functions:

(III) – strip property in connection with harmonic functions:

(IIIa) the hyperbolic density $\mathrm{hyp}_{\mathbb{S}(a,b)}(w)$ on $\mathbb{S}(a,b)$, $-\infty < a < b \leq \infty$, (in particular, for $\mathrm{hyp}_{\mathbb{S}_0}(w)$ on $\mathbb{S}_0$, see (3.6) below), depends only on $\mathrm{Re}\,(w)$.

(IIIb) Suppose that D is a simply connected plane domain and $f : D \to \mathbb{S}(a,b)$ is a complex harmonic function on D.

Then it is known from the standard course of complex analysis that

(i): there is an analytic function F on D such that $\mathrm{Re}\, f = \mathrm{Re}\, F$ on D, and it is clear that

(ii) $F : D \to \mathbb{S}(a,b)$.

By (I)-(III) it is readable that we have

Proposition 3.1 ([60, Proposition 2.4], [32, 17])**.** *Let $u : \mathbb{U} \to (-1,1)$ be harmonic function and let F be holomorphic function which is associated to u. Then*

$$\rho_{\mathbb{S}}(u(z))|\nabla u(z)| = \rho_{\mathbb{S}}(F(z))|F'(z)| \leq \rho_{\mathbb{U}}(z) \quad \textit{for all} \quad z \in \mathbb{U}. \tag{3.2}$$

Note the above simple method described by the properties (I)-(III) is basically based on the Schwarz-Pick lemma for holomorphic maps from $\mathbb{U}$ into $\mathbb{S}$ and it yields a proof of the above proposition to which we refer as the Schwarz-Pick lemma related to distortion for harmonic functions from $\mathbb{U}$ into $(-1,1)$.

It is convenient that the properties (I)-(III) together with Proposition 3.1 we shortly call the strip property of harmonic functions and refer to it as the *strip method*.

3.2. The Schwarz lemma for real harmonic functions

For planar domains D and G we denote by $\mathrm{Hol}(D,G)$ (respectively $\mathrm{Har}(D,G)$) the class of all holomorphic (respectively harmonic) mappings from G into D. For complex Banach manifolds X and Y we denote by $\mathcal{O}(X,Y)$ the class of all holomorphic mappings from X into Y.

We write $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. On $\mathbb{C}^n$ we define the standard Hermitian inner product by

$$\langle z, w\rangle = \sum_{k=1}^{n} z_k \overline{w_k}$$

for $z, w \in \mathbb{C}^n$ and by $|z| = \sqrt{\langle z, z\rangle}$ we denote the norm of vector z. By $\mathbb{B} = \mathbb{B}_n$ we denote the unit ball in $\mathbb{C}^n$. In particular we use also notation $\mathbb{U}$ and $\mathbb{H}$ for the unit disk and the upper half-plane in the complex plane respectively.

The following statement is useful in applications.

Lemma 3.1. *Let D and G be planar domains with metric densities σ and ρ respectively. If f is a C^1 mapping of D into G and*

$$|f'(z)|\rho(f(z)) \leq \sigma(z),\ z \in D,$$

then

$$d_\rho(fz, fw) \leq d_\sigma(z, w),\ z, w \in D.$$

Note that we do not suppose here that metric densities σ and ρ are hyperbolic.

It seems that results of this type are well known and that proofs are straightforward. For example, in particular if the densities are hyperbolic this result is used in [41] (see 3.A and 3.B there).

If G is an hyperbolic domain and $z \in G$, for a vector $\mathbf{v} \in T_z\mathbb{C}$ we define $|\mathbf{v}|_{\mathrm{hyp}} = \mathrm{hyp}(z)|\mathbf{v}|_e$. For convenience of the reader we outline some basic facts related to planar Schwarz lemma (see also [57]). For $z_1 \in \mathbb{U}$, define

$$T_{z_1}(z) = \frac{z - z_1}{1 - \overline{z_1}z},$$

$\varphi_{z_1} = -T_{z_1}$ and $\sigma = \sigma_{\mathbb{U}}$ by

$$\sigma(z_1, z_2) = |T_{z_1}(z_2)| = \left|\frac{z - z_1}{1 - \overline{z_1}z}\right|.$$

We call σ the pseudo-hyperbolic distance.

By Riemann mapping theorem simply connected domains different from $\mathbb{C}$ are conformally equivalent to $\mathbb{U}$. Using this important result one can transfer the concept of the pseudo-hyperbolic distance on simply connected domains different from $\mathbb{C}$ and it is shown that the pseudo-hyperbolic distance σ and the hyperbolic distance ρ are related by

$$\sigma = \tanh(\rho/2).$$

The following result, which we call the classical Schwarz lemma 1-the unit disk, is a corollary of the maximum modulus principle:

(Sc) Suppose that $\omega : \mathbb{U} \to \mathbb{U}$ is an analytic map and $\omega(0) = 0$. Then

(i) $|\omega(z)| \leq |z|$ and (ii) $|\omega'(0)| \leq 1$.

Using conformal authomorphisms of $\mathbb{U}$ one can derive from (i) and (ii) the following results (Sc1) and (SP1) below:

(Sc1) If $\omega : \mathbb{U} \to \mathbb{U}$ is an analytic map, then

$$|\omega'(z)| \le \frac{1 - |\omega z|^2}{1 - |z|^2},\ z \in \mathbb{U}. \tag{3.3}$$

We can rewrite this inequality in the form:

(Sc2) [Classical Schwarz lemma 2-the unit disk].

Suppose that $\omega : \mathbb{U} \to \mathbb{U}$ is an analytic map and $z \in \mathbb{U}$.

If $\mathbf{v} \in T_z\mathbb{C}$ and $\mathbf{v}^* = d\omega_z(\mathbf{v})$, then

$$|\mathbf{v}^*|_{\lambda_0} \le |\mathbf{v}|_{\lambda_0}.$$

(SP1) Classical Schwarz-Pick lemma. If $\omega \in \mathrm{Hol}(\mathbb{U}, \mathbb{U})$, then

$$\sigma_{\mathbb{U}}(\omega z_1, \omega z_2) \le \sigma_{\mathbb{U}}(z_1, z_2), \quad z_1, z_2 \in \mathbb{U}.$$

It is straightforward to derive from (SP1):

(SP2) If G and D are simply connected domains different from $\mathbb{C}$ and $\omega \in \mathrm{Hol}(G, D)$, then

$$\rho_D(\omega z, \omega z') \le \rho_G(z, z'), \quad z, z' \in G.$$

If D is a hyperbolic domain, using holomorphic covering $\pi : \mathbb{U} \to D$, one can define the pseudo-hyperbolic and the hyperbolic metric on D; and use it to derive a generalized version of the classical planar Schwarz-Pick lemma for the unit disk and of (SP2), which holds hyperbolic domains:

Proposition 3.2 (ScAh)**.** *If G and D are hyperbolic domains and $\omega \in \mathrm{Hol}(G, D)$, then*

$$\mathrm{Hyp}_D(\omega z, \omega z') \le \mathrm{Hyp}_G(z, z'), \quad z, z' \in G.$$

We will refer to this result shortly as the Schwarz-Ahlfors-Pick lemma. This result has useful geometric form, which is an extension of (Sc2) and (I) a):

Proposition 3.3 (ScAh1)**.** a) *If G and D are hyperbolic domains and ω a holomorphic mapping from D into G, then $||(\,\mathrm{d}\omega)_z|| \le 1$, for every $z \in D$, where the norm is defined with respect to the hyperbolic norms on the corresponding tangent spaces T_z and $T_{\omega z}$. This property can be expressed in terms of hyperbolic densities:*

b) $\mathrm{hyp}_G(\omega z)|\omega'(z)| \le \mathrm{hyp}_D(z),\ z \in D$*, or in the equivalent form:*

c) *If $z \in G$, $\mathbf{v} \in T_z\mathbb{C}$ and $\mathbf{v}^* = \mathrm{d}\omega_z(\mathbf{v})$, then*

$$|\mathbf{v}^*|_{\mathrm{hyp}} \le |\mathbf{v}|_{\mathrm{hyp}}.$$

We also need the following result:

(A3) If G_1 and G_2 are planar hyperbolic domains such that $G_1 \subset G_2$, then $\mathrm{Hyp}_{G_2}(z_1,z_2) \le \mathrm{Hyp}_{G_1}(z_1,z_2)$, $z_1,z_2 \in G_1$.

In the following examples we give explicit formula for a conformal mapping of $\mathbb{U}$ onto $\mathbb{S}_0$ and use it to compute the hyperbolic density of a strip domain and a half-plane.

Example 3.1. Let $\mathbb{S}_1 = \{w : |\mathrm{Re}\, w| < \pi/4\}$. It is easy to check that $\tan$ maps $\mathbb{S}_1$ onto $\mathbb{U}$. Let $B(w) = \dfrac{\pi}{4}w$ and $f_0 = \tan \circ B$, i.e., $f_0(w) = \tan\left(\dfrac{\pi}{4}w\right)$. Then f_0 maps $\mathbb{S}_0$ onto $\mathbb{U}$. Further set

$$A_0(z) = \frac{1+z}{1-z}, \quad \text{and let} \quad \phi = \mathrm{i}\frac{2}{\pi}\log A_0;$$

that is $\phi = \phi_0 \circ A_0$, where $\phi_0 = \mathrm{i}\dfrac{2}{\pi}\log$. Let $\hat{\phi}$ be defined by $\hat{\phi}(z) = -\phi(iz)$. Note that ϕ maps $I_0 = (-1,1)$ onto y-axis and $\hat{\phi}$ maps I_0 onto itself, and that $\hat{\phi} = 4/\pi \arctan$ is the inverse function of f_0. Hence (i1): $\hat{\phi}'(0) = 4/\pi$ and if f is a conformal map of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(0) = 0$, then (i2): $|f'(0)| = 4/\pi$. If $\hat{u} = \mathrm{Re}\,\hat{\phi}$, then

$$\hat{u} = \frac{2}{\pi}\arg\left(\frac{1+\mathrm{i}z}{1-\mathrm{i}z}\right) \tag{3.4}$$

and $\hat{u}$ maps $I_0 = (-1,1)$ onto itself.

Example 3.2. If $\Pi = \{w : \mathrm{Re}\, w > 0\}$, then using A_0, defined in Example 3.1, we can compute

$$\mathrm{hyp}_{\Pi}(w) = \frac{1}{\mathrm{Re}\, w}.$$

Hence if $G = \mathbb{S}(a,\infty)$ and ρ is hyperbolic density on G, we find

$$\rho(w) = \frac{1}{\mathrm{Re}\, w - a}. \tag{3.5}$$

If we denote by ρ_0 hyperbolic density on $\mathbb{S}_0$, then using f_0, defined in Example 3.1, we can check that for $w = u + \mathrm{i}v \in \mathbb{S}_0$,

$$\rho_0(w) = \mathrm{hyp}_{\mathbb{S}_0}(w) = \frac{\pi}{2}\frac{1}{\cos(\frac{\pi u}{2})}. \tag{3.6}$$

If $a,b \in \mathbb{R}$, $a < b$, the linear map L defined by $L(w) = (2w - (a+b))/(b-a)$, maps $\mathbb{S}(a,b)$ conformally onto $\mathbb{S}_0$ and using it we find $\rho(w) = \rho_0(Lw)2/(b-a)$.

Hence for $w \in \mathbb{S}(a,b)$, we get

$$\rho(w) = \mathrm{hyp}_{\mathbb{S}(a,b)}(w) = \frac{\pi}{b-a}\cdot\frac{1}{\cos\left(\frac{\pi}{2}[(2u-(a+b))/(b-a)]\right)}. \tag{3.7}$$

Now we can rewrite (I-1)a) in more explicit form:

(I-2) If F is holomorphic map from $\mathbb{U}$ into $\mathbb{S}_0$, then by a very special case of Schwarz-Ahlfors-Pick lemma(see also the property (I)),

$$\rho_0(F(z))|F'(z)| \leq 2(1-|z|^2)^{-1}, \quad z \in \mathbb{U}, \tag{3.8}$$

where ρ_0 is given by (3.6). Thus we have

Proposition 3.4 (Sc-Ah.0)**.** a) *If F is holomorphic map from $\mathbb{U}$ into $\mathbb{S}_0$, then*

$$|F'(z)| \leq 2(1-|z|^2)^{-1}/\rho_0(F(z)) = \frac{4}{\pi}\cos\left(\frac{\pi}{2}U(z)\right)(1-|z|^2)^{-1}, \quad z \in \mathbb{U}, \tag{3.9}$$

where $U = \operatorname{Re} F$.

b) *If $G = \mathbb{S}(a,\infty)$ and F is holomorphic map from $\mathbb{U}$ into G, then*

$$|F'(z)| \leq \frac{2}{(1-|z|^2)\rho(F(z))} = 2(\operatorname{Re} F(z) - a)(1-|z|^2)^{-1}, \quad z \in \mathbb{U}, \tag{3.10}$$

where ρ is the hyperbolic density on G.

Since $\rho_0(0) = \pi/2$ and $\lambda_0(0) = 2$, (I-1)b) is a corollary of this proposition.

Definition 3.1 (H0)**.** In this paper we frequently use the hypothesis:

(H0) Let f ($f = u + iv$) be a complex-valued harmonic map from $\mathbb{U}$ into $\mathbb{S}_0$.

By Har_0 we denote the family of all complex valued harmonics maps f from $\mathbb{U}$ into the strip $\mathbb{S}_0$(that is the family of mappings which satisfy (H0)).

Set $M_0(u) = \min\{1, \frac{2}{\pi}(1+|u|)\}$. Hence

Proposition 3.5. *If u is a harmonic map from $\mathbb{U}$ into $I_0 = (-1,1)$, then*

$$|\nabla u(z)| \leq \frac{4}{\pi}\cos(\frac{\pi}{2}u(z))(1-|z|^2)^{-1}, \quad z \in \mathbb{U}. \tag{3.11}$$

$$|\nabla u(z)| \leq \frac{4}{\pi}\frac{1-|u(z)|^2}{1-|z|^2}, \quad z \in \mathbb{U}, \tag{3.12}$$

$$|\nabla u(z)| \leq 2M_0(u)\frac{1-|u(z)|}{1-|z|^2}, \quad z \in \mathbb{U}. \tag{3.13}$$

The results of this type are obtained by D. Kalaj and M. Vuorinen [32] and H. Chen [17].

For the function $\hat{u}$, defined in Example 3.1 by (3.4), holds equality in the first inequality.

Note that we do not suppose that $u(0) = 0$. By definition of $\mathrm{Hyp}_{\mathbb{S}_0}$, we have that if ϕ is a conformal map of $\mathbb{U}$ onto $\mathbb{S}_0$, then

$$|\phi'(z)| = \frac{4}{\pi}\cos\left(\frac{\pi}{2}\mathrm{Re}\,\phi(z)\right)(1-|z|^2)^{-1}, \quad z \in \mathbb{U}.$$

Hence by the property (II), the function $\hat{u}$ from Example 3.1 shows that the first inequality is sharp.

PROOF. [Outline of proof of Proposition 3.5] Since f satisfies (H0), by (IIIb) there is a holomorphic function $F = F_f$ associated to f. If $u = \mathrm{Re}F$, then by (IIb), $|\nabla u(z)| = |F'(z)|$, $z \in \mathbb{U}$. Now an application of Proposition 3.4 (Sc-Ah.0) (Schwarz-Ahlfors-Pick estimate (3.9)) yields (3.11). Set $X(x) = \sin(x)$, $A = (0, X(0))$ and $B = (1, X(1))$. Since $A = (0,0)$ and $B = (1, 1/\sqrt{2})$ the equation of line L determined with A and B is $y = k_0 x$, where $k_0 = \frac{1}{\sqrt{2}}\frac{4}{\pi}$ is the slope coefficient of the line L. Since X is convex on $[0, \frac{\pi}{4}]$, $k_0 t \le X(t)$, $t \in [0, \frac{\pi}{4}]$, i.e., $\frac{4}{\pi}t \le \sqrt{2}\sin t$, $0 \le t \le \frac{\pi}{4}$.

Hence

$$\text{(i)} \qquad \frac{x}{\sqrt{2}} \le \sin\left(\frac{\pi}{4}x\right), \quad \text{i.e.,} \quad x^2 \le 2\sin^2\left(\frac{\pi}{4}x\right), \quad x \in [0,1].$$

Using that $1 - \cos(\frac{\pi}{2}x) = 2\sin^2(\frac{\pi}{4}x)$ and the inequality (i), we prove $\cos(\frac{\pi}{2}x) \le 1 - x^2$, $|x| \le 1$, and therefore we get

$$\text{(A4)} \qquad \frac{\pi}{2}(1-u^2)^{-1} \le \rho_0(w).$$

Using $\cos(\frac{\pi}{2}u) = \sin(\frac{\pi}{2}(1-|u|)) \le \frac{\pi}{2}(1-|u|)$, we get

$$\text{(A5)} \quad (1-|u|)^{-1} \le \rho_0(w).$$

By (A4), (3.12) follows from (3.11) and now by (A5) we get (3.13).

Concerning the proof of Proposition 3.5, note that

1. by (A3), $\mathrm{hyp}_{\mathbb{S}_0} \le \mathrm{hyp}_{\mathbb{U}}$ and by (A4), we find

   (A6) $\quad \mathrm{Hyp}_{\mathbb{U}}(x_1, x_2) \le \frac{4}{\pi}\mathrm{Hyp}_{\mathbb{S}_0}(x_1, x_2)$, $x_1, x_2 \in (-1,1)$;

2. the inequality $\cos(\frac{\pi}{2}x) \le 1 - x^2$, $|x| \le 1$, shows that $\lambda_0(u) \le \frac{4}{\pi}\rho_0(w)$, $u = \mathrm{Re}\,w$.

   $\lambda_0(w)$ tends to ∞ if $|w|$ tends 1 throughout $\mathbb{U}$. Recall $\rho_0(w) \le \lambda_0(w)$, $w \in \mathbb{U}$.

3. Set $\chi(x) = \lambda_0(x)/\rho_0(x)$, $x \in [-1, 1]$. Then $\chi(x) = \frac{4}{\pi} g(x)$, where $g(x) = \cos(\frac{\pi}{2}x)(1-x^2)^{-1}$. By l'Hospital's rule $g(x)$ tends to $\pi/4$ and therefore $\chi(x)$ tends to 1 if $x \to 1_-$.

   Hence we conclude that the inequality $\lambda_0(x) \leq \frac{4}{\pi}\rho_0(x)$, $x \in [-1, 1]$, is sharp near 0, but it is not sharp if $|x|$ is near 1.

4. If $0 < t_0 \leq \pi/2$ and $I_0 = [0, t_0]$, then using the convexity of sinus-function on I_0, we find $(\sin(t_0)/t_0)t \leq \sin t$, $t \in [0, t_0]$. If $0 < t_0 \leq \pi/4$, this inequality is more precisely than (i) on I_0.

If F is holomorphic map from $\mathbb{U}$ into $\mathbb{S}_0$ which satisfies (IIIb) (recall $u = \mathrm{Re}F$, then $\mathrm{Hyp}_{\mathbb{S}_0}(uz_1, uz_2) \leq \mathrm{Hyp}_{\mathbb{S}_0}(Fz_1, Fz_2) \leq \mathrm{Hyp}_{\mathbb{U}}(z_1, z_2)$.

Hence by (A6), we get

$$\text{(A7)} \quad \mathrm{Hyp}_{\mathbb{U}}(u(z_1), u(z_2)) \leq \tfrac{4}{\pi}\mathrm{Hyp}_{\mathbb{U}}(z_1, z_2).$$

Theorem 3.1. *Suppose that D is a hyperbolic planar domain and $v : D \to (-1, 1)$ is a real harmonic on D. Then*

$$\mathrm{Hyp}_{\mathbb{U}}(v(z_1), v(z_2)) \leq \frac{4}{\pi}\,\mathrm{Hyp}_D(z_1, z_2).$$

PROOF. If D is the unit disk $\mathbb{U}$ this result is reduced to (A7). It also follows from (3.12) and it has been proved by Kalaj and Vuorinen [32].

In general case one can use a holomorphic cover $\mathcal{P} : \mathbb{U} \to D$ and define $\hat{v} = v \circ \mathcal{P}$. For $z, w \in D$, let $z' \in \mathcal{P}^{-1}(z)$, $w' \in \mathcal{P}^{-1}(w)$. By the definition of $\hat{v}$ it is clear that $\hat{v}(z') = v(z)$ and $\hat{v}(w') = v(w)$. Since $\hat{v} : \mathbb{U} \to (-1, 1)$, then by (A7)

$$\mathrm{Hyp}_{\mathbb{U}}\big(\hat{v}(z'), \hat{v}(w')\big) \leq \frac{4}{\pi}\mathrm{Hyp}_{\mathbb{U}}(z', w').$$

Since we can choose $z', w' \in \mathbb{U}$ such that $\mathrm{Hyp}_{\mathbb{U}}(z', w') = \mathrm{Hyp}_D(z, w)$, and hence we get a proof of Theorem 3.1.

3.3. The Schwarz lemma for harmonic maps

In the introduction we present shortly the content of this subsection. Now we discuss the subject with more details.

The following theorem is known as the Schwarz lemma for harmonic maps from $\mathbb{U}$ into itself.

Theorem 3.2 (The Schwarz lemma for harmonic maps from $\mathbb{U}$ into $\mathbb{U}$, [27], [21, p. 77])**.** *Let* $f \in \mathrm{Har}(\mathbb{U}, \mathbb{U})$ *and* $f(0) = 0$. *Then*

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|, \quad \textit{for all} \quad z \in \mathbb{U}, \tag{3.14}$$

and this inequality is sharp for each point $z \in \mathbb{U}$.

In the literature this result is often attributed to E. Heinz [27]. Later, in 1977, H. W. Hethcote [28] improved the above result by removing the assumption $f(0) = 0$ and showed the following:

Theorem 3.3 ([28, Theorem 1])**.** *Let* $f \in \mathrm{Har}(\mathbb{U}, \mathbb{U})$. *Then*

$$\left| f(z) - \frac{1 - |z|^2}{1 + |z|^2} f(0) \right| \leq \frac{4}{\pi} \arctan |z|, \quad \textit{for all} \quad z \in \mathbb{U}.$$

It seems that the researchers have overlooked H. W. Hethcote result and they have had some difficulties to handle the case $f(0) \neq 0$ in this context; see [62]. By our method, we get a simple proof of an optimal version of the Schwarz lemma for real valued harmonic functions (without the assumption that 0 is mapped to 0 by the corresponding map), see Theorem 3.4 below which improves H. W. Hethcote result.

3.4. Euclidean properties of hyperbolic discs

Recently, M. Mateljević and M. Sveltik [64] proved a Schwarz lemma for real harmonic functions with values in $(-1, 1)$ using a completely different approach than B. Burgeth [14].

In this subsection we follow [64] and we improve Theorem 3.3.

Definition 3.2. Recall by d_D we denote the hyperbolic metric and by σ_D the pseudo-hyperbolic metric on a hyperbolic planar domain D. Let $\lambda > 0$ be arbitrary. By $D_\lambda(a)$ (respectively $S_\lambda(b)$) we denote the hyperbolic disc in $\mathbb{U}$ (respectively in $\mathbb{S}$) with hyperbolic center $a \in \mathbb{U}$ (respectively $b \in \mathbb{S}$) and hyperbolic radius λ. More precisely $D_\lambda(a) = \{z \in \mathbb{U} : d_{\mathbb{U}}(z, a) < \lambda\}$ and $S_\lambda(b) = \{z \in \mathbb{S} : d_{\mathbb{S}}(z, b) < \lambda\}$. Also, $\overline{D}_\lambda(a) = \{z \in \mathbb{U} : d_{\mathbb{U}}(z, a) \leq \lambda\}$ and $\overline{S}_\lambda(b) = \{z \in \mathbb{S} : d_{\mathbb{S}}(z, b) \leq \lambda\}$ are corresponding closed discs. Specially, if $a = 0$ (respectively $b = 0$) we omit a (respectively b) from the notations.

Remark 3.1. If f is a conformal isomorphism from $\mathbb{U}$ onto $\mathbb{S}$ such that $f(a) = b$ then $f(D_\lambda(a)) = S_\lambda(b)$ and $f(\overline{D}_\lambda(a)) = \overline{S}_\lambda(b)$.

Let $r \in (0,1)$ be arbitrary. By U_r we denote Euclidean disc $\{z \in \mathbb{C} : |z| < r\}$ and by $\overline{U}_r$ we denote the corresponding closed disc. Also, let

$$\lambda(r) = d_{\mathbb{U}}(r,0) = \log \frac{1+r}{1-r} = 2\,\mathrm{artanh}\, r.$$

Since $d_{\mathbb{U}}(z,0) = \log \dfrac{1+|z|}{1-|z|} = 2\,\mathrm{artanh}\,|z|$ for all $z \in \mathbb{U}$, we have

$$D_{\lambda(r)} = \{z \in \mathbb{C} : 2\,\mathrm{artanh}\,|z| < 2\,\mathrm{artanh}\, r\} = \{z \in \mathbb{C} : |z| < r\} = U_r,$$

and similarly

$$\overline{D}_{\lambda(r)} = \overline{U}_r.$$

The closed discs $\overline{D}_{\lambda(r)}$ and $\overline{S}_{\lambda(r)}$ are shown in Figure 1 and the following lemma to claim that disc $\overline{S}_{\lambda(r)}$ be contained in a Euclidean rectangle.

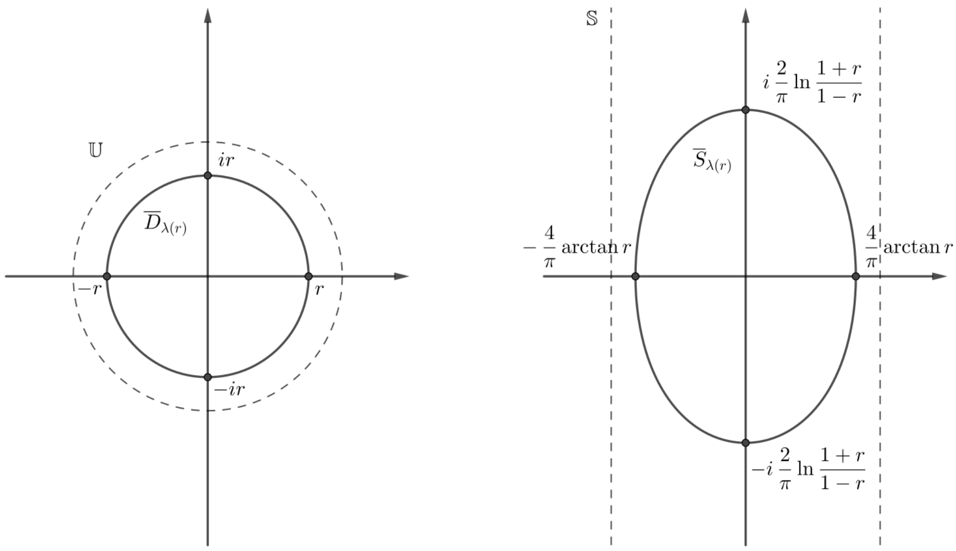

Figure 1: $\overline{D}_{\lambda(r)}$ and $\overline{S}_{\lambda(r)}$

Lemma 3.2. *Let* $r \in (0,1)$ *be arbitrary. Then*

$$\overline{S}_{\lambda(r)} \subset \left[-\frac{4}{\pi}\arctan r, \frac{4}{\pi}\arctan r\right] \times \left[-\frac{2}{\pi}\lambda(r), \frac{2}{\pi}\lambda(r)\right]. \tag{3.15}$$

In particular,

$$R_e(\overline{S}_{\lambda(r)}) = \left[-\frac{4}{\pi}\arctan r, \frac{4}{\pi}\arctan r\right]. \tag{3.16}$$

In order to appreciate the proof of the next lemma we give an example. For $s > 0$ set $R^s = [-1,1]\times[-s,s]$ and let ψ^s be conformal mapping of $\mathbb{U}$ onto R^s such that ψ^s maps $(-\mathrm{i},\mathrm{i})$ onto $(-\mathrm{i}s,\mathrm{i}s)$ with $\psi^s(0)=0$. Also, for $r\in(0,1)$ set $\overline{E}_{r,s}=\psi^s(\overline{U}_r)$. We leave to the interested reader to check that for r enough near to 1 the function e on $\overline{E}_{r,s}$ does not attain maximum at $\psi^s(\mathrm{i}r)$.

By d_e we denote euclidean distance in the complex plane.

Lemma 3.3. *Let $\lambda > 0$ be arbitrary. Then*

$$\max\{d_e(z,0) : z\in\overline{S}_\lambda\} = \frac{2}{\pi}\lambda. \tag{3.17}$$

Example 3.3. For $a\in\mathbb{U}$, define $\varphi_a(z)=\dfrac{a-z}{1-\overline{a}z}$. It is well known that φ_a is a conformal automorphism of $\mathbb{U}$. Specially, for $a\in(-1,1)$, the mapping φ_a has the following properties:

i) it is decreasing on $(-1,1)$ and maps $(-1,1)$ onto itself;

ii) for $r\in[0,1)$ it holds $\varphi_a([-r,r]) = [\varphi_a(r),\varphi_a(-r)] = \left[\dfrac{a-r}{1-ar},\dfrac{a+r}{1+ar}\right]$.

Here we denote the strip $\mathbb{S}_0$ simple with $\mathbb{S}$.

Example 3.4. Let $b\in\mathbb{S}$ be arbitrary and let ϕ_b be a conformal isomorphism from $\mathbb{U}$ onto $\mathbb{S}$ such that $\phi_b(0)=b$ and $\phi_b'(0)>0$. It is straightforward to check that $\phi_b=\phi\circ\varphi_a$, where $a=\tan\dfrac{b\pi}{4}$ and φ_a is defined in Example 3.3. Specially, for $b\in(-1,1)$, the mapping ϕ_b has the following properties:

i) it is decreasing on $(-1,1)$ and maps $(-1,1)$ onto itself;

ii) for $r\in[0,1)$ it holds

$$\phi_b([-r,r]) = [m_b(r), M_b(r)],$$

where

$$m_b(r)=\phi_b(r)=\frac{4}{\pi}\arctan\frac{a-r}{1-ar}$$

and

$$M_b(r)=\phi_b(-r)=\frac{4}{\pi}\arctan\frac{a+r}{1+ar}.$$

Recall recently, M. Mateljević and M. Sveltik [64] proved a Schwarz lemma for real harmonic functions with values in $(-1,1)$ using a completely different approach than B. Burgeth [14].

Theorem 3.4 ([64])**.** *Let* $u : \mathbb{U} \to (-1, 1)$ *be a harmonic function such that* $u(0) = b$. *Then*

$$m_b(|z|) \le u(z) \le M_b(|z|) \text{ for all } z \in \mathbb{U}.$$

Moreover, this inequality is sharp for each $z \in \mathbb{U}$, *where*

$$M_b(r) := \frac{4}{\pi} \arctan \frac{a + r}{1 + ar}, \quad m_b(r) := \frac{4}{\pi} \arctan \frac{a - r}{1 - ar},$$

and $a = \tan(b\pi/4)$.

This result also is proved by Szapiel [77]. Clearly Theorem 3.4 improves Theorem 3.3 for real harmonic functions, as one can check the following elementary proposition.

Proposition 3.6. *Let* b *be in* $(-1, 1)$ *and* $r \in [0, 1)$. *Then we have*

1. $$M_b(r) \le \frac{1 - r^2}{1 + r^2} b + \frac{4}{\pi} \arctan r =: A_b(r)$$

and

$$m_b(r) \ge \frac{1 - r^2}{1 + r^2} b - \frac{4}{\pi} \arctan r.$$

2. *The mapping* $b \mapsto M_b(r)$ *is increasing on* $(-1, 1)$.

Using a standard rotation, we can extend Theorem 3.4 for *complex* harmonic functions from the unit disc into itself.

Theorem 3.5. *Let* $f : \mathbb{U} \to \mathbb{U}$ *be a harmonic function from the unit disc into itself. Then*

$$|f(z)| \le M_{|f(0)|}(|z|)$$

holds for all $z \in \mathbb{U}$.

PROOF. Fix z_0 in the unit disc and choose unimodular λ such that $\lambda f(z_0) = |f(z_0)|$. Define $u(z) = \Re(\lambda f(z))$. Hence, using Theorem 3.4, we get

$$|f(z_0)| = u(z_0) \le M_{u(0)}(|z_0|) \le M_{|f(0)|}(|z_0|),$$

as the mapping $b \mapsto M_b(|z_0|)$ is increasing.

For further results see [62, 36, 67, 76].

3.5. Burgeth method

Definition 3.3. Recall $\sigma = \sigma_p$ denotes the usual surface (Haar-) measure on the sphere $\mathbb{S}_{p-1}$, $d\sigma$ corresponding area element, $\sigma^0 = \sigma/\sigma_p$ and $\|.\|$ is the Euclidean norm.. By ω_p or $V(\mathbb{B}^p)$ we denote p-volume of the ball $\mathbb{B}^p$ and by σ_p $p-1$-dimensional volume of sphere $\mathbb{S}^{p-1}$. Thus σ^0 is the unique rotation invariant normalized Borel measure on $\mathbb{S}^{p-1}$ such that $\sigma^0(\mathbb{S}^{p-1}) = 1$.

In this paper expressions ω_{p-1}/ω_p and σ_{p-1}/σ_p we shortly denote by $\omega_*(p)$ and $\sigma_*(p)$ respectively on some places.

Burgeth [14] used the following notations.

$$\mathcal{H}^c = H_p^c := \Big\{ h \text{ is harmonic (hyperbolic-harmonic) on } \mathbb{B}^p : h(0) = c, 0 < h < 1 \Big\}.$$

$\mathcal{K}^c$ denotes the Lebesgue space of essentially bounded functions f such that $0 \leq f \leq 1$ σ-a.e. with $\int_{\mathbb{S}_{p-1}} f d\sigma^0 = c$. More precisely its elements are equivalence classes $\tilde{f}$ of all functions g with $g = f$ σ-a.e. By $\mathbb{I}_A$ we denote the characteristic function of A. Set

$$M_c^p(|x|) = 2\int_{\mathbb{S}_{p-1}} \mathbb{I}_{S(c,\hat{x})} P_x d\sigma^0 - 1, \quad (3.18)$$

$$m_c^p(|x|) = 2\int_{\mathbb{S}_{p-1}} \mathbb{I}_{S(c,-\hat{x})} P_x d\sigma^0 - 1, \quad (3.19)$$

where $x \in \mathbb{B}_p$ and $S(c, \hat{x})$ denotes the polar cap with center $\hat{x}$ and σ-measure c. Instead of $S(c, \hat{x})$ we also use the notation $S(\hat{x}, \gamma)$.

Recall we use the following notations $\hat{x} = x/|x|$ and $\hat{0} = e_1$, $S(\hat{x}, \gamma) = \{y \in \mathbb{S}_2 : \langle y, \hat{x}\rangle \geq \cos\gamma\}$ for the polar cap with center $\hat{x}$, where γ is the spherical angle of it.

Now we illustrate how M-method leads to the Burgeth method. Let us to consider the inequality

(B1) $h \in H^c$ then $h(x) \leq \overline{M}_c^p(|x|) := \int_{\mathbb{S}_{p-1}} \mathbb{I}_{S(c,\hat{x})} P_x d\sigma^0$.

At first glance it is not clear whether the inequality (1) holds.

But we can use the trick that

$$h(x) = P[h^*](x) = \int_{\mathbb{S}_{p-1}} (P_x - t) h^*(y) d\sigma^0(y) + t h(0)$$

and choose t such that $P_x(y) = t$ for $y \in S(\hat{x}, \gamma)$. Then

$$h(x) \leq \int_{S(\hat{x},\gamma)} (P_x - t) h^*(y) d\sigma(y) + t h(0) \leq \int_{S(\hat{x},\gamma)} (P_x - t) d\sigma^0(y) + t h(0).$$

Hence, since $\sigma(S(\hat{x},\gamma)) = h(0)$, we get (B1).

Check that

$$M_c^p(|x|) = \frac{2}{\sqrt{\pi}}\frac{\Gamma(p/2)}{\Gamma(\frac{p-1}{2})}(1-|x|^2))^{\nu}\int_0^{\alpha(c)}\frac{\sin^{p-2}\varphi}{(1-2|x|\cos\varphi+|x|^2)^{\mu}}d\varphi,$$

$$m_c^p(|x|) = \frac{2}{\sqrt{\pi}}\frac{\Gamma(p/2)}{\Gamma(\frac{p-1}{2})}(1-|x|^2))^{\nu}\int_{\pi-\alpha(c)}^{\pi}\frac{\sin^{p-2}\varphi}{(1-2|x|\cos\varphi+|x|^2)^{\mu}}d\varphi,$$

where $(\nu,\mu) = (1, p/2)$ in the harmonic case resp. $(\nu,\mu) = (p-1, p-1)$ in the hyperbolicharmonic case and $\alpha(c)$ is the spherical angle of $S(c,\hat{x})$. For the sake of simplicity we consider $g = (1+h)/2 \in \mathcal{H}^c$ $(g(0) = a)$. Our aim is to find the extreme values of the integral $\int_{\mathbb{S}_{p-1}} P_x g^* d\sigma^0$ where g is varying in $\mathcal{H}^c$.

Define

$$U_c^x(z) := 2\int_{\mathbb{S}_{p-1}}\chi_{S(c,\hat{x})}P_z - 1 \quad \text{and} \quad u_c^x(z) := 2\int_{\mathbb{S}_{p-1}}\chi_{S(c,-\hat{x})}P_z - 1.$$

Note that for a fixed x the functions U_c^x and U_c^t (respectively u_c^x and u_c^t) are equal as function of z if $t \in [0,\hat{x})$. Hence we can write $U_c^{\hat{x}}$ instead of U_c^x and in a similar way $u_c^{\hat{x}}$ instead of u_c^x. For a fixed x the functions $U_c^{\hat{x}}(z)$ and $u_c^{\hat{x}}(z)$ are harmonic in z.

Theorem 3.6 ([14]). *Let h be a harmonic or hyperbolic-harmonic function with $|h| \le 1$ and $h(0) = a$, $-1 < a < 1$. Then for c $= (a+1)/2$ and all $x \in \mathbb{B}^p$*

$$m_c^p(|x|) \le h(x) \le M_c^p(|x|). \tag{3.20}$$

Equality on the right (resp.,left-) hand side for some $z \in \mathbb{B}^p \setminus \{0\}$ implies

$$h(x) = U_c^x(x) := 2\int_{\mathbb{S}_{p-1}}\chi_{S(c,\hat{x})}P_x - 1, \tag{3.21}$$

$$h(x) = u_c^x(x) := 2\int_{\mathbb{S}_{p-1}}\chi_{S(c,-\hat{x})}P_x - 1. \tag{3.22}$$

Note first that we can restate theorem in the form that for every $x \in \mathbb{B}_p$ and $z \in [o,\hat{x})$, we have

(B2) $u_c^{\hat{x}}(z) \le h(x) \le U_c^{\hat{x}}(z)$. A proof can be based on (B1), but we prefer to illustrate the Burgeth approach.

PROOF. For simplicity we introduce $g := (1+h)/2$ and note that $g(0) = c \in (0,1)$ and $g \in \mathcal{H}^c$. Set $A(t,x) = \{P_x > t\}$, $A^-(t,x) = \{P_x < t\}$ and $\mu(t,x) =$

$\sigma(A(t,x))$. First, note that the function defined by $t \mapsto \mu(t,x)$ ($t \in \mathbb{R}^+, x \in \mathbb{B}^p$) is continuous and strictly decreasing, so to each $c \in]0,1[$ there exists a unique number t_c such that $\mu(t_c,x) = c$. Further note that $\mathbb{I}_{S(c,\hat{x})} = \mathbb{I}_{A(t_c,x)} \in K^c$. Since $g \in \mathcal{H}^c$ we note that there is $g^* \in \mathcal{K}^c$ such that

$$g(x) = P[g^*](x) = \int P_x g^* \,\mathrm{d}\sigma.$$

Fixing $x \in \mathbb{B}^p$ and note that

$$\int_{\mathbb{S}_{p-1}} g^*(P_x - t_c)\mathbb{I}_{A^-(t_c,x)} \,\mathrm{d}\sigma^0 = \int_{A^-(t_c,x)} g^*(P_x - t_c) \,\mathrm{d}\sigma^0 < 0.$$

Hence if in addition we suppose that $g^* \neq \mathbb{I}_{S(c,\hat{x})}$ we conclude

$$\begin{aligned} g(x) &= \int_{\mathbb{S}_{p-1}} P_x g^* \,\mathrm{d}\sigma = \int (P_x - t_c) g^* \,\mathrm{d}\sigma^0 + c t_c \\ &= \int g^*(P_x - t_c)\mathbb{I}_{A(t_c,x)} \,\mathrm{d}\sigma + c t_c + \int g^*(P_x - t_c)\mathbb{I}_{A^-(t_c,x)} \,\mathrm{d}\sigma^0 \\ &< \int (P_x - t_c)\mathbb{I}_{A(t_c,x)} \,\mathrm{d}\sigma^0 + c t_c \\ &= \int_{\mathbb{S}_{p-1}} \mathbb{I}_{S(c,\hat{x})} P_x \,\mathrm{d}\sigma = \frac{1}{2}\big(M_c^p(|x|) + 1\big). \end{aligned}$$

Theorem 1 [14] stated here as Theorem 3.6 yields some corollaries. In particular, we have

Theorem 3.7 (Corollary 2 in [14])**.** *Let h be harmonic or hyperbolic-harmonic on $\mathbb{B}_p$ with $|h| \leq 1$ and $h(0) = 0$. Then the estimate*

$$|\nabla h(0)| \leq \begin{cases} \tau_p := \dfrac{2}{\sqrt{\pi}} \dfrac{p}{p-1} c_p & \text{if } h \text{ is harmonic}, \\ \tau_p^1 := \dfrac{4}{\sqrt{\pi}} c_p & \text{if } h \text{ is hyperbolic harmonic} \end{cases}$$

where $c_p = \Gamma(p/2)/\Gamma(p/2 - 1/2)$. By (1)*, $\tau_p = 2\omega_{p-1}/\omega_p$ and therefore if h is harmonic, $|\nabla h| \leq 2\omega_{p-1}/\omega_p$.*

For the background related to the Khavinson conjecture see Kresin-Ma'zya book [42]. In the paper [45] Liu proved the Khavinson conjecture for harmonic functions on n-dimensional the unit ball (previously M. Marković gave a proof for the points near the boundary of the unit ball, D. Kalaj for $n = 4$ and P. Melentijević for $n = 3$), which says for bounded harmonic functions on the unit ball of $\mathbb{R}^n$ the sharp constants in the estimates for their radial derivatives and for their gradients coincide. In the paper [46], He further prove the following

Theorem 3.8. *Let u be a real-valued bounded harmonic function on the unit ball $\mathbb{B}_n$ of $\mathbb{R}^n$.* (i) *When $n = 2$ or $n \geq 4$, the following sharp inequality holds:*

$$|\nabla u(x)| \leq \frac{2V(\mathbb{B}^{n-1})}{V(\mathbb{B}^n)} \frac{1}{1-|x|^2} |u|_\infty, \quad x \in \mathbb{B}^n, \tag{3.23}$$

where m_n denotes the Lebesgue measure on $\mathbb{R}^n$. Equality holds if and only if $x = 0$ and $u = U \circ T$ for some orthogonal transformation T, where U is the Poisson integral of the function that equals 1 on a hemisphere and -1 on the remaining hemisphere.

(ii) *When $n = 3$ we have*

$$|\nabla u(x)| \leq \frac{8}{3\sqrt{3}} \frac{1}{1-|x|^2} |u|_\infty, \quad x \in \mathbb{B}^3. \tag{3.24}$$

The constant $\frac{8}{3\sqrt{3}}$ here is the best possible.

Remark 3.2. Note that the inequality (3.23) holds when $n = 3$ at $x = 0$. Curiously, the inequality (3.23) fails when $n = 3$ in general. Note that $\frac{8}{3\sqrt{3}} \approx 1.5396$, while the constant $2V(\mathbb{B}^{n-1})/V(\mathbb{B}^n)$ in (3.23) equals $3/2$ when $n = 3$.

Conjecture 3.1 ([46])**.** *If $u : \mathbb{B}_n \to [-1, 1]$ is a harmonic function,*

$$|\nabla u(x)| \leq \frac{2V(\mathbb{B}^{n-1})}{V(\mathbb{B}^n)} \frac{1-u^2(x)}{1-|x|^2}. \tag{3.25}$$

We confirmed the conjecture for $n = 3$ and disprove it in general.

By $A = A_n^{cap}(\gamma)$ we denote the normalized $n - 1$-dimensional volume of the spherical cap with contact angle γ. For $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, the generalized Poisson kernel is defined by

$$P_{\alpha,\beta}(x, y) = \frac{(1-|x|^2)^\alpha}{|x-y|^{2\beta}}, \quad x \in \mathbb{B}^n \text{ and } y \in \mathbb{S}.$$

Theorem 3.9 ([36])**.** *Suppose (h1) Let $h^* : \mathbb{S} \to [-1, 1]$ be a bounded function on $\mathbb{S}$ with values in $[-1, 1]$ and $h = P_{\alpha,\beta}[h^*]$. Then,*

$$|\nabla h(0)| \leq D_n(\gamma, \beta) := \frac{4\beta}{n} \frac{V(\mathbb{B}^{n-1})}{V(\mathbb{B}^n)} (\sin\gamma)^{n-1}, \tag{3.26}$$

where γ is the unique angle in $[0,\pi]$ such that

$$A = A_n^{cap}(\gamma) = \frac{1+h(0)}{2}.$$

The estimate (3.26) *is sharp and* $h^0 := 2P_{\alpha,\beta}\left[1_{S(e_n,\gamma)}\right] - U_{\alpha,\beta;n}$ *is an extremal function.*

It is readable that $D_n(\gamma,\beta)$ is $\sup|\nabla h(0)|$ where the suprimum is taken under the class $B_1 H^a_{\alpha,\beta}$ of function h which satisfies (h1) with constraint $h(0) = a$, where γ is determined by $A = A_n^{cap}(\gamma) = \frac{1+a}{2}$. It is interesting that $(\sin\gamma)^{n-1} \geq 1 - a^2$ and therefore

$$D_n(\gamma,\beta) \geq (1-a^2)\frac{4\beta}{n}\omega_*(n)$$

for $n \geq 4$. So the ratio volume $\omega_*(n)$ appears in our research. During our work on the subject we realized that $\omega_*(n)$ has very interesting properties and that there are remarkable upper and lower bounds for the ratio, see for example Borgwardt [11] and Alzer [5].

Since

$$\sqrt{\frac{n+1/2}{2\pi}} \leq \omega_*(n) \leq \sqrt{\frac{n-1+\pi/2}{2\pi}},$$

we conclude that in harmonic case ($\beta = n/2$) (B) $D_n(\gamma, n/2) \geq 2\omega_*(n)(1-a^2)$, and therefore (B1) $D_n(\gamma, n/2) \to \infty$ as $n \to \infty$. Note that (B) also disprove Liu conjecture.

For $n = 2$, $D_2(\gamma, 1/2) \geq 2\omega(2)(1-a^2)$, where $\omega(2) = 2/\pi$.

Remark 3.3. 1. We use the formula for the normalized area of spherical cap

$$A(\gamma) = \frac{(n-1)\omega_{n-1}}{n\omega_n}A_0,$$

where $A_0 = \int_0^\gamma (\sin\theta)^{n-2}d\theta$. In particular if under the hypothesis of Theorem 3.9, h is harmonic and Conjecture 3.1 is true then

(A0) $\quad (\sin\gamma)^{n-1} \leq 1 - h^2(0) = 4A(1-A).$

2. For $n = 3$ in (A0) holds equality.

3. For $n = 2$, $A = \gamma/\pi$, and (A0) can be written as

$$h^2(0) \leq (\sin(\gamma/2) + \cos(\gamma/2))^2,$$

4. i.e., $|h(0)| \le a(\gamma) := \sqrt{2}\sin(\frac{\pi}{4}+\frac{\gamma}{2})$. If $h(0)<0$, then

$$|h(0)| = b(\gamma) := 1-2A = 1-\frac{2}{\pi}\gamma.$$

Since $a(0)=b(0)=1$, $a'(\gamma)=\frac{\sqrt{2}}{2}\cos(\frac{\pi}{4}+\frac{\gamma}{2})$, $b'(\gamma)=-\frac{2}{\pi}$, (A0) holds.

4. *Negative curvature and strip codomain*

In this section we discuss some versions of the Schwarz lemma related to curvature and several variables.

First we need some definitions and results.

4.1. The Riemann surfaces

In this subsection we collect some basic definitions and properties of Riemann surfaces.

1. A Riemann surface R is a complex manifold of complex dimension one. This means that R is a connected Hausdorff topological space endowed with an atlas: for every point $p \in R$ there is a neighbourhood U containing p homeomorphic to an open subset of the complex plane $\mathbb{C}$ (the unit disk of the complex plane) by a map z defined on U. The map z (we use also notation (U,z)) carrying the structure of the complex plane to the Riemann surface is called a chart. Additionally, the transition maps between two overlapping charts are required to be holomorphic:

   (i) If $U_\alpha \cap U_\beta \neq \emptyset$, then $z_{\alpha\beta} = z_\beta \circ z_\alpha^{-1}$ is complex analytic on $z_\alpha(U_\alpha \cap U_\beta)$.

   If $\{U_i\}$ form an open covering of R, the system is said to define a *conformal (analytic) structure* on R. A point $p \in U_i$ and $z_i = \phi_i$ is referred to as a *local variable* or *local parameter* (the terms coordinate patch, local coordinate, local chart are also used in the literature). Sometimes is convenient to work with ϕ_i^{-1} which we call the inverse of the coordinate chart.

   We speak of a surface if instead of (i) we require that the mappings $z_{\alpha\beta}$ are only topological and of *differentiable surface* if they are of class C^∞.

   A function $f : M \to N$ between Riemann surfaces is analytic at $p \in M$ if there exist coordinate charts ϕ_1 and ϕ_2 about p and $f(p)$ respectively such that $\phi_2 \circ f \circ (\phi_1)^{-1}$ is analytic at $\phi_1(p)$. this definition is invariant under transition maps. We say that $f : M \to N$ is analytic if it is analytic at each point $p \in M$.

2. A smooth covering of a surface S is a pair (W, f), where W is a surface and $f : W \to S$ is a local homeomorphism (in the literature usually surjective is required).

3. Let γ be a path on S. A path $\tilde{\gamma}$ with initial point $a = \tilde{\gamma}(0)$ and with the property $f \circ \tilde{\gamma} = \gamma$ is called a *lift* of γ from a. It is easy to prove that on a smooth covering of a surface the lift from a fixed initial point is unique.

   From now on all covering surfaces are understood to be regular.

   We also suppose that S is a surface and (W, f) its covering surface. If in addition S and W are Riemann surfaces and f analytic, we call (W, f) analytic covering.

4. A smooth covering of a surface S is said to be regular if every path on S has a lift to W from each point which is over its initial points.

   In this case $f : W \to S$ is surjective, and the cardinality of the set $f^{-1}(p)$ is the same for all $p \in S$; show that the set of points that are covered exactly n times is open and closed.

   Let W and W^* be surfaces, and consider a mapping $f : W^* \to W$. Let V be open set on W. We shall say that V is *evenly covered* by (W^*, f) if every component of the inverse image $f^{-1}(V)$ is in one-to-one (homeomorphic) correspondence with V. A covering surface (W^*, f) of W is said to be complete if every point $a \in W$ has an evenly covered open neighborhoods V_a. One can show that a covering surface is complete if and only if it is regular.

   So, from now on all covering surfaces are understood to be complete.

5. The Poincaré-Koebe uniformization theorem (a generalization of the Riemann mapping theorem) states that every simply connected Riemann surface is conformally equivalent to one of the following: The Riemann sphere $\widehat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$, which is isomorphic to the $\mathbf{P}^1(\mathbf{C})$; The complex plane $\mathbf{C}$; The open disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ which is isomorphic to the upper half-plane. A Riemann surface is elliptic, parabolic or hyperbolic according to whether its universal cover is isomorphic to $\mathbf{P}^1, \mathbf{C}$ or $\mathbf{D}$.

6. Every Riemann surface is the quotient of free, proper and holomorphic action of a discrete group on its universal covering and this universal covering is holomorphically isomorphic (one also says: "conformally equivalent" or "biholomorphic") to one of the following:

   the Riemann sphere, the complex plane, the unit disk in the complex plane.

On an oriented 2-manifold, a Riemannian metric induces a complex structure using the passage to isothermal coordinates. The existence of isothermal coordinates can be proved by other methods, for example using the general theory of the Beltrami equation, as in Ahlfors [3] (2006), or by direct elementary methods, as in Jost [30] (2006). From this correspondence with compact Riemann surfaces, a classification of closed orientable Riemannian 2-manifolds follows. Each such is conformally equivalent to a unique closed 2-manifold of constant curvature, so a quotient of one of the following by a free action of a discrete subgroup of an isometry group: the sphere (curvature $+1$), the Euclidean plane (curvature 0), the hyperbolic plane (curvature -1).

7. There have been many investigations into the properties of surfaces with negative Gaussian curvature. One of the principal results in this area is due to Efimov which states that no surface S with Gaussian curvature $K \leq d < 0$ for any constant $d < 0$ can be C^2 immersed into $\mathbb{R}^3$ so that S is complete in the induced Riemannian metric; For a full exposition of Efimov's theorem and its proof see [68] and for some related results [66].

4.2. Yau-Royden result

There are many results related to the Schwarz lemma and holomorphic functions of several variables, see for example [38, 1, 13, 24, 37, 39, 40, 60, 70, 71, 85, 86, 82, 88] and literature cited there. Here we will mention only a few of them.
Yau [88] proved the following generalization of Schwarz lemma.

Theorem 4.1 (Yau). *Let M be a complete $Kähler$ manifold with Ricci curvature bounded from below by a constant, and N be another Hermitian menifold with holomorphic bisectional curvature bounded from above by a negative constant. Then any holomorphic mapping f from M into N decrease distances up to a constant depending only on the curvature of M and N.*

Royden [70] improved the estimate in Yau theorem.

Theorem 4.2 (Royden). *Let M be a complete Hermitian manifold with holomorphic sectional curvature bounded from below by a constant $-k$, $k \geq 0$, and N be another Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant $-K$, $K > 0$. Assume either that M has Riemann sectional curvature bounded from below or that M is $Kähler$ with holomorphic bisectional curvature bounded from below. Then any holomorphic mapping f from M into N satisfies*

$$\| \mathrm{d}f \|^2 \leq \frac{k}{K} \, .$$

4.3. The Kobayashi metric

Definition 4.1. 1. Let G be bounded connected open subset of complex Banach space, $p \in G$ and $\mathbf{v} \in T_pG$. We define $k_G(p, \mathbf{v}) = \inf\{|\mathbf{h}|\}$, where infimum is taking over all $\mathbf{h} \in T_0\mathbb{C}$ for which there exists a holomorphic function such that $\phi : \mathbb{U} \to G$ such that $\phi(0) = p$ and $\mathrm{d}\phi(\mathbf{h}) = \mathbf{v}$.

Let G be an open subset of $\mathbb{C}^n$, $p \in G$ and $v \in T_pG$.

2. If ϕ maps $\mathbb{D}$ into G, we define

$$L_G\phi(p, v) = \sup\big\{\lambda : \ \phi(0) = p, \ \ \mathrm{d}\phi_0(1) = \lambda v\big\},$$

and $L_G(p, v) = \sup L_G\phi(p, v)$, where the supremum is taken over all maps $\phi : \mathbb{D} \to G$ which are analytic in $\mathbb{D}$ with $\phi(0) = p$. Note that

$$L_G(p, v)k_G(p, v) = 1.$$

If G is the unit ball, we write $L_\phi(p, v)$ instead of $L_G\phi(p, v)$. We also use the notation kob_G instead of k_G. We call kob_G Kobayashi-Finsler norm on tangent bandle.

For some particular domains, we can explicitly compute Kobayashi norm of a tangent vector by the corresponding angle.

We define the distance function on G by integrating the pseudometric k_G: for $z, z_1 \in G$,

$$\mathrm{Kob}_G(z, z_1) = \inf_{\gamma} \int_0^1 k_G(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t \tag{4.1}$$

where the infimum is taken over all piecewise paths $\gamma\colon [0, 1] \to G$ with $\gamma(0) = z$ and $\gamma(1) = z_1$. In complex geometry, the Kobayashi metric is a pseudometric intrinsically associated to any complex manifold. It was introduced by S. Kobayashi in 1967 [37]. Kobayashi hyperbolic manifolds are an important class of complex manifolds, defined by the property that the Kobayashi pseudometric is a metric. Kobayashi hyperbolicity of a complex manifold X implies that every holomorphic map from the complex line $\mathbb{C}$ to X is constant. Here we list some properties: Every bounded domain in $\mathbb{C}^n$ is hyperbolic.

A hermitian manifold M whose holomorphic sectional curvature is bounded above by a negative constant is hyperbolic.

In general we say that a metric space M is complete if for each point $p \in M$ and each positive number r the closed ball of radius r around p is a compact subset of M. If M is complete in this sense, then every Cauchy sequence of M converges, but not conversely in general.

If M is a complex manifold with complete Carathéodory distance c_M, then M is a complete hyperbolic manifold.

4.4. The monodromy theorem

A topological space X is called simply connected (or 1-connected, or 1-simply connected) if it is path-connected and every closed path can be continuously transformed in X to a point.

There every path between two points can be continuously transformed (intuitively for embedded spaces, staying within the space) into any other such path while preserving the two endpoints in question.

If, however, one can continuously deform one of the curves into another while keeping the starting points and ending points fixed, and analytic continuation is possible on each of the intermediate curves, then the analytic continuations along the two curves will yield the same results at their common endpoint. This is called the monodromy theorem and its statement is made precise below. A sufficient criterion for the single-valuedness of a branch of an analytic function. Let D be a simply-connected domain in the complex space $\mathbb{C}^n$, $n = 1$. Now, if an analytic function element $\mathcal{F}(z_0; r_0) = (f_0, B(z_0, r_0)$, with centre $z_0 \in D$, can be analytically continued along any path in D, then the branch of an analytic function $f(z)$, $z = (z_1, \ldots, z_n)$, arising by this analytic continuation is single-valued in D, see for example [80]. In other words, the branch of the analytic function $f(z)$ defined by the simply-connected domain D and the element$S(z_0; r_0)$ with centre $z_0 \in D$ must be single-valued. Another equivalent formulation is: If an element $S(z_0; r_0)$ can be analytically continued along all paths in an arbitrary domain $D \subset \mathbb{C}^n$, then the result of this continuation at any point $z^* \in D$ (that is, the element $\mathcal{F}(z^*; r^*) = (f_*, B(z^*, r^*))$ with centre z^*) is the same for all homotopic paths in D joining z_0 to z^*. The monodromy theorem is valid also for analytic functions $f(z)$ defined in domains D on Riemann surfaces or on Riemann domains.

4.5. New results

In Section 3. we described the strip method, see also [64] and [60]. Recall that here by λ_0 and σ_0 we denote respectively hyperbolic metric on the unit disk and on the strip $\mathbb{S}_0 := (-1, 1) \times \mathbb{R}$.

The main result of [47] is there Theorem 3.1 stated here as

Theorem 4.3. *Let S be a surface with a conformal metric density $\rho = \rho_S$ and assume that its curvature is bounded by a negative constant from below, i.e., let*

(h1)*: $K_S(p) \geq -k$, $p \in S$, where $k > 0$.*

Let d_S be a distance on S induced by the conformal metric ρ_S.

(i) *For an analytic mapping f on S which has its the real part in $(-1,1)$, we have*

$$\mathrm{Hyp}_{\mathbb{U}}(u(z_1), u(z_2)) \leq k\frac{4}{\pi}d_S(z_1, z_2),$$

$z_1, z_2 \in S$, *where* $u = \mathrm{Re}\, f$.

(ii) *For an analytic mapping $f : S \to \mathbb{U}$, we have*

$$\mathrm{Hyp}_{\mathbb{U}}(u(z_1), u(z_2)) \leq k d_S(z_1, z_2), \qquad z_1, z_2 \in S.$$

The result caught our attention. It seems that the hypotheses need to be clarified. In particular we need to check the following:

1. It seems here that we need to suppose in addition that (h2): (S, d_S) is complete metric space, and to write
2. (h3): $K_S(p) \geq -k^2$, instead of (h1).
3. If S is hyperbolic Riemman surface, whether $\mathrm{hyp}_S \leq k\rho_S$ on S?

A proof can be deduced from Schwarz lemma for analytic functions using the following:

(A1) $\lambda_0(|z|, |w|) \leq \lambda_0(z, w), \quad z, w \in \mathbb{D}$.

(A2) $\sigma_0(\mathrm{Re}\, z, \mathrm{Re}\, w) \leq \sigma_0(z, w), \quad z, w \in \mathbb{S}_0$.

(A3) $\frac{\pi}{4}\lambda_0(x, y) \leq \sigma_0(x, y)$ on $(-1, 1)$.

Proof of (A1) and (A2) follow from simple geometric consideration related to hyperbolic densities of $\mathbb{D}$ and $\mathbb{S}_0$.

Proof of (A3) is based on the inequality $\cos(\pi/2x) \leq 1 - x^2$ on $(-1, 1)$.

Now we prove a generalization of Theorem 3.1 and M. Marković result stated the above as Theorem 4.3.

Theorem 4.4. (h1) *Let S be Riemann surface with complete metric ρ curvature bounded from below by a constant $-k^2, k > 0$, and* (h2) $u : S \to (-1, 1)$ harmonic. If σ_0 is the hyperbolic distance on the strip $\mathbb{S}_0$, we have

$$\text{(i)} \qquad \sigma_0(u(p), u(q)) \leq k d_S(p, q), \qquad p, q \in S.$$

First we give some comments:

1. If S is hyperbolic surface and satisfies (h1) then $hyp_S \leq k\rho$ on S.

2. Note that the hypothesis (h1) that the metric is complete is crucial here. For example if $S = \mathbb{U}$, $u(z) = x$, and density $e = 1$ is euclidean density, then $K(e) = 0$, but there is no $k > 0$ such that $\sigma_0 \leq ke$ on $\mathbb{D}$.

3. Note that that (h1) does not implies that S is hyperbolic Riemann surface in general. Namely, if $S = \mathbb{C}$ with euclidean e metric on $\mathbb{C}$ then $(\mathbb{C}, e)$ satisfy hypothesis (h1), but $\mathbb{C}$ is not hyperbolic.

PROOF. If S is not hyperbolic, then u is a constant. If S is hyperbolic and $\pi : \mathbb{D} \to S$, then $\pi^*\rho$ is complete on $\mathbb{D}$ and therefore $hyp_{\mathbb{D}} \leq k\pi^*\rho$. Hence (1) $hyp_S \leq kd_\rho$. Further $U = u \circ \pi$ is harmonic and there is holomorphic F on $\mathbb{D}$ such that $U = \operatorname{Re} F$ and therefore $\sigma_0(Uz, Uw) \leq \sigma_0(Fz, Fw) \leq \operatorname{Hyp}(z, w)$, $z, w \in \mathbb{D}$. Hence if $z \in \pi^{-1}(p)$ and $w \in \pi^{-1}(q)$, we have $\sigma_0(u(p), u(q)) \leq \operatorname{Hyp}_S(p, q)$ and by (1) we get (i).

The author communicated at Belgrade seminar in June 2021, the following results:

Theorem 4.5. *Let G be an open subset of $\mathbb{C}^n$ (more generally complex manifold) and $u : G \to (-1, 1)$ pluriharmonic function. Then $\sigma_0(u(p), u(q)) \leq \operatorname{Kob}_G(p, q)$, $p, q \in G$.*

Theorem 4.6. *Let M be a complete simply connected Hermitian manifold with holomorphic sectional curvature bounded from below by a constant $-k^2$, $k \geq 0$. Assume either that M has Riemann sectional curvature bounded from below or that M is $K\ddot{a}hler$ with holomorphic bisectional curvature bounded from below. Then for any $u : M \to (-1, 1)$ pluriharmonic, we have $\sigma_0(u(p), u(q)) \leq kd_M(p, q)$.*

Note that A. Khalfallah [35] considered pseudo-distances defined by pluriharmonic functions.

Concerning further research for the beginning we can try to generalize Theorem 4.5 and consider the following question:

Question 2. *Can we reduce proof of a version of Yau-Royden Theorem to a version of Schwarz lemma for surfaces?*

In connection with Royden's theorem 4.2 we have the following question:

Question 3. *Let (M, g) be a complete Hermitian manifold with holomorphic sectional curvature bounded from below by a constant $-k^2$, $k \geq 0$. Whether $kd_M \geq d_{Kob}$?*

We suggest an idea to consider here the pull back metric $\phi^* g$ on $\mathbb{D}$, where $\phi : \mathbb{D} \to M$ is analytic and try to compare it with λ/k. But here we have a problem because in general case of a Hermitian manifold (M, g), $\phi^* g$ is not complete on $\mathbb{D}$. It seems that it is true if M is Riemann surface.

5. *The Schwarz lemma for harmonic and quasiconformal maps*

Assume that f is a h-qch mapping from the unit disc $\mathbb{U}$ onto itself. Recall that by λ we denote hyperbolic density on $\mathbb{U}$. Let $\lambda^* = \lambda^*_f = \lambda \circ f\,|f_z|$ and $\mathbf{K}_0 = K_{\lambda^*}$ be the Gaussian curvature of λ^*. Let $\sigma = (1-k)^2\lambda^*$. Hence, the Gaussian curvature of the conformal metric $\mathrm{d}s^2 = \sigma(z)|\,\mathrm{d}z|^2$ satisfies

$$\mathbf{K}(\sigma)(z) = -\frac{1}{(1-k)^2}\left(1 + |\mu(z)|^2 + 2\mathrm{Re}\left(\frac{(f(z))^2\,\overline{f_z(z)}\,\overline{f_{\bar z}(z)}}{|f_z(z)|^2}\right)\right), \qquad (5.1)$$

for all $z \in \mathbb{U}$. If we set

$$A(z) = \left(\frac{(f(z))^2\,\overline{f_z(z)}\,\overline{f_{\bar z}(z)}}{|f_z(z)|^2}\right) = \left(\frac{(f(z))^2\overline{q}}{p}\right)$$

and $B(z) = 2\,\mathrm{Re}\,A(z)$, then (cf. [41])

$$\mathbf{K}(\sigma)(z) = -\frac{1}{(1-k)^2}\left(1 + |\mu(z)|^2 + B(z)\right).$$

Wan [81] showed that

Theorem 5.1 (Wan)**.** *Every harmonic quasi-conformal diffeomorphism from Δ onto itself with respect to $Poincaré$ metric is a quasi-isometry of $Poincaré$ disk.*

Let $\rho_0 = \sigma \circ f\,|f_z|$ and $K_0 = K_{\rho_0}$ the Gaussian curvature of ρ. In his proof Wan [81] used the method of sub-solutions and super-solutions and the fact that ρ_0 is complete metric. Recall, we will show in a forthcoming paper that we can use Ahlfors-Schwarz lemma and Theorem 6 instead of the method of sub-solutions and super-solutions and, in particular, that a proof of Wan's result can be based on these results.

Definition and properties of Harmonic and quasiregular maps. Let R and S be two surfaces. Let $\sigma(z)|\,\mathrm{d}z|^2$ and $\rho(w)|\,\mathrm{d}w|^2$ be the metrics with respect to the isothermal coordinate charts on R and S respectively, and let f be a C^2-map from R to S.

It is convenient to use notation in local coordinates $\mathrm{d}f = p\,\mathrm{d}z + q\,\mathrm{d}\overline{z}$, where $p = f_z$ and $q = f_{\overline{z}}$. Also we introduce the complex (Beltrami) dilatation

$$\mu_f = Belt[f] = \frac{q}{p},$$

where it is defined.

The energy integral of f is

$$E(f,\rho) = \int_R \rho\circ f\,(|p|^2 + |q|^2)\,\mathrm{d}x\,\mathrm{d}y\,.$$

A critical point of the energy functional is called a harmonic mapping. The Euler-Lagrange equation for the energy functional is

$$\tau(f) = f_{z\overline{z}} + (\log\rho)_w \circ f\,p\,q = 0\,.$$

Thus, we say that a C^2-map f from R to S is ρ-harmonic (shortly harmonic) if f satisfies the above equation. For basic properties of harmonic maps and for further information on the literature we refer to Jost [29] and Schoen-Yau [73].

The following facts and notation are important in our approach:

A1 If f is a harmonic mapping then

$$\varphi\,\mathrm{d}z^2 = \rho\circ f\,p\,\overline{q}\,\mathrm{d}z^2$$

is a quadratic differential on R, and we say that φ is the *Hopf differential* of f and we write φ =Hopf(f).

A2 The Gaussian curvature on S is given by

$$K_S = -\frac{1}{2}\frac{\boldsymbol{\Delta}\log\rho}{\rho}\,.$$

A3 We will use the following notation $\mu = Belt[f] = q/p$ and $\tau = \log(1/|\mu|)$,

$$|\partial f|^2 = \frac{\rho}{\sigma}|p|^2, \quad J_e(f) = |p|^2 - |q|^2, \quad J(f) = \frac{\rho}{\sigma}J_e(f)$$

and *Bochner* formula (see [73]).

If f is ρ-harmonic then

$$\boldsymbol{\Delta}_R \log|\partial f| = -K_S\,J(f) + K_R\,,$$

$$\boldsymbol{\Delta}_R \log|\bar{\partial} f| = K_S\,J(f) + K_R\,,$$

$$\mathbf{\Delta}\tau = -K_S\,|\varphi|\,\sinh\tau\,.$$

A4 Definition of quasiregular function. Let R and S be two Riemann surfaces and $f : R \to S$ be a C^2-mapping. If P is a point on R, $\tilde{P} = f(P) \in S$, ϕ a local parameter on R defined near P and ψ a local parameter on S defined near $\tilde{P}$, then the map $w = h(z)$ defined by $h = \psi \circ f \circ \phi^{-1}|_V$ (V is a sufficiently small neighborhood of P) is called a local representer of f at P. The map f is called k-quasiregular if there is a constant $k \in (0,1)$ such that for every representer h, at every point of R, $|h_{\overline{z}}| \le k|h_z|$.

Ahlfors-Schwarz lemma for harmonic-quasiregular maps. Let $\sigma^0 = \sqrt{\rho}\circ f\,|p|$ and $K_0 = K_{\sigma^0}$ the Gaussian curvature of σ^0.

If we take euclidean metric on R by Bochner formula $K_0 = K_S\,(1 - |\mu|^2)$

Further suppose that the Gaussian curvature of the metric $\mathrm{d}s^2 = \rho(w)|\,\mathrm{d}w|^2$ be uniformly bounded from above on S by the negative constant $-a^2, a > 0$ and that f is k-qc.

Then $K_0 \le a^2(k^2 - 1)$ and $b\sigma^0$, where $b = a\sqrt{1-k^2}$, is ultrahyperbolic and therefore by Ahlfors-Schwarz lemma $b\sigma^0 \le \lambda$. Next if $\sigma^+ = \sqrt{\rho}\circ f\,(|p| + |q|)$, then $\sigma^+ \le (1+k)\sigma$ and hence $\sigma^+ \le \sqrt{K}/a$.

Hence we can prove the following result.

Theorem 5.2 (Theorem 5.1 [41])**.** *Let R be hyperbolic surfaces with Poincare metric densities λ and S be another with Poincare metric densities ρ and let the Gaussian curvature of the metric $\mathrm{d}s^2 = \rho(w)|\,\mathrm{d}w|^2$ be uniformly bounded from above on S by the negative constant $-a^2,\ a > 0$. Then any ρ-harmonic k-quasiregular map f from R into S decreases distances up to a constant $\sqrt{K}/a$ depending only on a and k.*

Proposition 5.1. *Let R be hyperbolic surfaces with Poincare metric densitiy λ and S be another with metric densitiy ρ. Suppose that (S,ρ) is complete and let the Gaussian curvature of the metric $\mathrm{d}s^2 = \rho(w)|\,\mathrm{d}w|^2$ be uniformly bounded from below on S by the negative constant $-b^2,\ a > 0$.*

Then any ρ-harmonic k-quasiconformal map f from R into S,

$$\lambda(z_1, z_2) \le \frac{b}{1-k} d_\rho(f(z_1), f(z_2)). \tag{5.2}$$

PROOF. If $-b^2 \le K_S \le -a^2$, then $K_0 \ge -b^2$ and f is k-qc, $l_f \ge (1-k)|f_z|$. Next by opposite Ahlfors lemma $b\sigma^0 \ge \lambda$ and therefore $b\sqrt{\rho}\circ f l_f \ge (1-k)\lambda$. Hence (5.2) holds.

M is a complete simply-connected Riemannian manifold whose curvature satisfies $-b^2 \le K \le -a^2$ for some positive constants $0 < a < b$. In this setting, we say pinched Hadamard manifold with constants $0 < a < b$.

For instance, the hyperbolic disk $\mathbb{D}$ is a pinched Hadamard surface with constant curvature -1.

Let S with metric density ρ be a pinched Hadamard surface with constants $0 < a < b$ and λ Poincaré density. Then $\lambda_b \leq \rho \leq \lambda_a$. Y. Benoist & D. Hulin [8] prove

Theorem 5.3. *Let $h : S \to R$ be a harmonic quasi-isometric map between pinched Hadamard surfaces. Then h is a quasi-conformal diffeomorphism.*

They give a short new proof the injectivity result:

(B) Any harmonic quasi-isometric map between hyperbolic disks d is quasi-conformal harmonic diffeomorphism, and used (B) as a starting point to prove Theorem 5.3. Note that (B) is an output of Markovic solution [49] of the Schoen Conjecture:

Theorem 5.4. *every quasisymmetric homeomorphism of the circle $b\mathbb{H}^2$ admits a harmonic quasiconformal extension to the hyperbolic plane $\mathbb{H}^2$. This proves the Schoen Conjecture.*

The author communicated at Belgrade seminar in June 2021, the following result:

Theorem 5.5. *Let $h : S \to R$ be a harmonic quasi-isometric map between pinched Hadamard surfaces. Then h is bi-Lip.*

The proof is based on Proposition 5.1.

6. *Applications*

Uniformly bounded maximal φ-disks, Bers space and harmonic maps. Let φ be an analytic function on the unit disk Δ. Then φ belongs to *Bers space* $Q = Q(\Delta)$ if

$$\text{ess sup}\, \omega(z)^2 |\varphi(z)| < +\infty\ ,$$

where $\omega(z) = 1 - |z|^2$.

In this section we will give an uniform estimate of radius of maximal φ-disks of the Hopf differential of a quasiregular harmonic map with respect to strongly negatively curved metric (see below Theorem 6.1). As an application we show that the Hopf differential of a quasiregular harmonic map with respect to strongly negatively curved metric belongs to Bers space. First we define maximal φ-disks.

Maximal φ-disk. Let φ be an analytic function on the unit disk Δ and let z_0 be a regular point of φ, i.e. $\varphi(z_0) \neq 0$. Let Φ_0 be a single valued branch of

$$w = \Phi(z) = \int \sqrt{\varphi(z)}\, \mathrm{d}z$$

near z_0, $\Phi(z_0) = 0$. There is a neighborhood U of z_0 which is mapped one-to-one conformally onto an open set V in the w-plane. We can assume, by restriction, that V is a disk around $w = 0$. The inverse Φ_0^{-1} is a conformal homeomorphism of V into Δ and evidently there is a largest open disk V_0 around $w = 0$ such that the analytic continuation of Φ_0^{-1} (which is still denoted by Φ_0^{-1}) is homeomorphic, and that $\Phi_0^{-1}(V_0) \subset \Delta$. The image $U_0 = \Phi_0^{-1}(V_0)$ is called the *maximal φ-disk* around z_0; its φ-radius (injectivity radius) r_0 is the Euclidean radius of V_0.

For the definition of φ-disks and a discussion of their important role in the theory of holomorphic quadratic differentials we refer the interested reader to Strebel's book [75].

Theorem 6.1 ([6]). *Let ρ be the metric on Δ with Gaussian curvature K uniformly bonded from above on Δ by the negative constant $-a$, and let f be a harmonic k-quasiregular map from Δ into itself with respect to the metric ρ. If $R = R_z$ is the radius of the maximal φ-disk around z, where $\varphi =$*Hopf(f), *then R is bounded from above by the constant C which depends only on k and a.*

PROOF. Let $R = R_z$ be the radius of the maximal φ-disk $U = U_z$ around $z \in \Delta$. Since f is k-qusiregular then $\tau \geq m$, where $m = \log \frac{1}{k}$, $m > 0$. Let $\zeta = \Phi(z)$ be the natural parameter in U and $\Phi(U) = V = B(0, R)$ With respect to the parameter ζ the Bochner formula takes the simple form

$$\boldsymbol{\Delta}\tau = -K \sinh \tau\,.$$

Since $K \leq -a$ and $\tau \geq m$, we conclude that

$$\boldsymbol{\Delta}\tau \geq \delta \mathrm{e}^{\tau} \text{ on } V, \tag{6.1}$$

where $\delta = a \sinh m/\mathrm{e}^m$. Let $\mathrm{d}s = \lambda(\zeta)|\,\mathrm{d}\zeta|$, where $\lambda(\zeta) = 2R/(R^2 - |\zeta|^2)$ is the hyperbolic metric on V and let $\tilde{\lambda}(\zeta) = \left(\delta \mathrm{e}^{\tau(\zeta)}/2\right)^{1/2}$. From (6.1) we have for the Gaussian curvature of the metric $\mathrm{d}\tilde{s} = \tilde{\lambda}(\zeta)|\,\mathrm{d}\zeta|$ on V that $\tilde{K} \leq -1$ and then we can use the Ahlfors-Schwarz Lemma 1 (see also [2]) to obtain

$$\frac{\delta}{2k} \leq \tilde{\lambda}^2(\zeta) \leq \lambda^2(\zeta). \tag{6.2}$$

Setting $\zeta = 0$ in (6.2) one obtains $R^2 \leq 8k/\delta$.

In [6], I. Anić, V. Marković and M. Mateljević characterize Bers space by means of maximal φ-disks. As an application, using Theorem 6.1, they show that the Hopf differential of a quasiregular harmonic map with respect to strongly negatively curved metric belongs to Bers space. Also they give further sufficient or necessary conditions for a holomorphic function to belong to Bers space.

Let φ be a quadratic differential on a hyperbolic Riemann surface R with Poincaré metric $\mathrm{d}s^2 = \rho(z)|\,\mathrm{d}z|^2$. Let $p \in R$ and let z be a local parameter near p. We will define

$$\|\varphi\|(p) = \rho^{-1}(z(p))|\varphi(z(p))|.$$

We say that φ belongs to the *Bers space* of R (notation $Q(R)$) if $\|\varphi\|$ is a uniformly bounded function on R.

Theorem 6.2. ([6]) *Let R and S be hyperbolic surfaces with metric densities σ and ρ respectively and let the Gaussian curvature of the metric $\mathrm{d}s^2 = \rho(w)|\,\mathrm{d}w|^2$ be uniformly bounded from above on S by the negative constant $-a$. If f is a harmonic k-quasiregular map from R into S with Hopf differential φ, then $\varphi \in Q(R)$.*

PROOF. Let $\tilde{f}$ be the lifting of f which maps Δ into itself and let $\tilde{\varphi}$ be the lifting of the quadratic differential φ. Let $\tilde{\rho}$ be the lifting of the density ρ. Since $\tilde{f}$ is harmonic with respect to the metric $\tilde{\rho}(\tilde{w})|\,\mathrm{d}\tilde{w}|^2$ on Δ and k-quasiregular then, by Theorem 2 [6], $\tilde{\varphi} \in Q(\Delta)$. Hence $\varphi \in Q(R)$.

Acknowledgement. The author is indebted to M. Arsenović, M. Svetlik, A. Khalfallah and A. Seepi for interesting discussions on this paper.

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Faculty of Mathematics
University of Belgrade
Studentski trg 16
11000 Beograd
Republic of Serbia
e-mail: miodrag@matf.bg.ac.rs