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ON HYPERCYCLICITY AND SUPERCYCLICITY OF STRONGLY CONTINUOUS SEMIGROUPS INDUCED BY SEMIFLOWS. DISJOINT HYPERCYCLIC SEMIGROUPS

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A b s t r a c t. We enquire into the basic structural properties of positively supercyclic strongly continuous semigroups induced by locally Lipschitz continuous semiflows in the setting of weighted L^p and C_0-type spaces. We also introduce and investigate disjoint hypercyclic semigroups whose index set is an appropriate sector of the complex plane. Several illustrative examples are also provided in order to justify our analysis.

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1. *Introduction and preliminaries*

Let $\alpha \in (0, \frac{\pi}{2}]$, $\delta > 0$ and $I \neq \emptyset$. Define $\Delta(\alpha) := \{re^{i\theta} : r \geq 0,\ \theta \in [-\alpha, \alpha]\}$ and suppose $\Delta \in \{[0, \infty),\ \mathbb{R},\ \mathbb{C}\}$ or $\Delta = \Delta(\alpha)$ for an appropriate $\alpha \in (0, \frac{\pi}{2}]$. Further on, put $\Delta_\delta := \{z \in \Delta : |z| \leq \delta\}$.

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Suppose that X is an infinite-dimensional separable Fréchet space over the field $\mathbb{K} \in \{\mathbb{R},\ \mathbb{C}\}$. Denote by $L(X)$ the space of all bounded linear operators from X into X and by $\mathrm{R}(T)$ the range of an operator $T \in L(X)$. It is said that an operator family $(S(\tau))_{\tau\in I}$ $(S(\tau) \in L(X),\ \tau \in I)$ is:

(i) *hypercyclic*, if there exists $x \in X$ whose orbit $\{S(\tau)x : \tau \in I\}$ is dense in X,

(ii) *topologically transitive*, if for every open non-empty subsets $U,\ V$ of X, there exists $\tau \in I$ such that $S(\tau)U \cap V \neq \emptyset$,

(iii) *supercyclic*, if there exists $x \in X$ such that its *projective orbit* $\{cS(\tau)x : c \in \mathbb{K},\ \tau \in I\}$ is dense in X,

(iv) *positively supercyclic*, if there exists $x \in X$ such that its *positive projective orbit* $\{cS(\tau)x : c \in (0,\infty),\ \tau \in I\}$ is dense in X.

An operator family $(T(t))_{t\in\Delta}$ is said to be a strongly continuous semigroup if:

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s),\ t,\ s \in \Delta$ and

(iii) the mapping $t \mapsto T(t)x,\ t \in \Delta$ is continuous for every fixed $x \in X$.

The first systematic exposition of hypercyclic strongly continuous semigroups in Banach spaces was presented by W. Desch, W. Schappacher and G. F. Webb in [10] while the notion of hypercyclicity and chaoticity of strongly continuous translation semigroups whose index set is an appropriate sector of the complex plane was introduced by J. A. Conejero and A. Peris in [7]–[8]. We also refer the reader to [2], [15]–[19] and [23]. In [15]–[16], T. Kalmes has recently analyzed the hypercyclicity of strongly continuous semigroups induced by semiflows. The underlying Banach space in his analysis is chosen to be the space $L^p(X,\mu,\mathbb{K})$, resp. $C_{0,\rho,\mathbb{K}}(X)$, where X is a locally compact, σ-compact Hausdorff space, $p \in [1,\infty)$ and μ is a locally finite Borel measure on X, resp. X is a locally compact, Hausdorff space and $\rho : X \to (0,\infty)$ is an upper semicontinuous function. For the purpose of research of strongly continuous semigroups induced by non-differentiable locally Lipschitz continuous semiflows (cf. Example 2.1 given below), we primarily deal with the space $L^p_{\rho_1}(\Omega,\mathbb{K})$, where Ω is an open non-empty subset of $\mathbb{R}^n$, $\rho_1 : \Omega \to (0,\infty)$ is a locally integrable function, m_n is the Lebesgue measure in $\mathbb{R}^n$ and the norm of an element $f \in L^p_{\rho_1}(\Omega,\mathbb{K})$ is given by $||f||_p := (\int_\Omega |f(\cdot)|^p \rho_1(\cdot)\,dm_n)^{1/p}$. Further on, let us recall that the space $C_{0,\rho}(X,\mathbb{K})$ consists of all continuous functions $f : X \to \mathbb{K}$ satisfying that, for every $\epsilon > 0$, $\{x \in X : |f(x)|\rho(x) \geq \epsilon\}$ is a compact subset of X; equipped with the norm $||f|| := \sup_{x\in X} |f(x)|\rho(x)$, $C_{0,\rho}(X,\mathbb{K})$ becomes a Banach space. Put, by common consent, $\sup_{x\in\emptyset} \rho(x) := 0$ and denote by $C_c(X,\mathbb{K})$ the space of all continuous functions $f : X \to \mathbb{K}$ whose support is a compact subset of X. Then $C_c(X,\mathbb{K})$ is dense in $L^p(X,\mu,\mathbb{K})$, and certainly, $C_c(X,\mathbb{K})$ is dense in $C_{0,\rho}(X,\mathbb{K})$, too (cf. [20, Section 13]). Let $C(\Omega,\mathbb{K})$ be the $\mathbb{K}$-vector space consisting of all continuous functions from Ω into $\mathbb{K}$. We equip $C(\Omega,\mathbb{K})$ with its usual Fréchet topology. In the sequel, it will not be confusing to write $L^p_{\rho_1}(\Omega)$, $C_{0,\rho}(X)$, $C_c(\Omega)$, and $m(\cdot)$, respectively, instead of $L^p_{\rho_1}(\Omega,\mathbb{K})$, $C_{0,\rho}(X,\mathbb{K})$, $C_c(\Omega,\mathbb{K})$, and $m_n(\cdot)$.

In Theorem 2.4, we focus our attention towards the study of positive supercyclicity of strongly continuous semigroups induced by semiflows and continue, in such a way, the research of M. Matsui, M. Yamada and F. Takeo [19]; the full importance of positive supercyclicity of strongly continuous semigroups is vividly exhibited in Example 2.2.

On the other hand, disjointness for finitely many operators has been introduced by L. Bernal-Gonzáles [3] and J. Bès, A. Peris [4]. The main objective in Section 3 is to extend the notion of disjoint hypercyclicity to strongly continuous semigroups whose index set is an appropriate sector of the complex plane. In this paper, we establish sufficient conditions for d-hypercyclicity of strongly continuous semigroups on the Fréchet space $C(\Omega)$ and on a class of weighted function spaces. The concrete construction of d-hypercyclic semigroups induced by semiflows, obtained by means of Theorem 3.1 and Theorem 3.2 given below, is the main purpose of this section.

Before we move ourselves to the next section, we feel duty bound to say that this paper has been finally rejected in another mathematical journal after a rather long peer-review process started from 2008 (the author would like to express his frank gratitude to the anonymous referee for many useful hints and suggestions). During the peer-review process, the obtained results were published in my second research monograph [17] (2015) and later expanded in a joint research studies with C.-C. Chen, S. Pilipović, D. Velinov [6] (2018) and V. Fedorov [12] (2018).

2. *Hypercyclic and supercyclic semigroups induced by semiflows*

Definition 2.1. Suppose $n \in \mathbb{N}$ and Ω is an open non-empty subset of $\mathbb{R}^n$. A continuous mapping $\varphi : \Delta \times \Omega \to \Omega$ is called a *semiflow* if $\varphi(0,x) = x,\ x \in \Omega$,

$$\varphi(t+s,x) = \varphi(t,\varphi(s,x)),\ t,\ s \in \Delta,\ x \in \Omega$$

and

$$x \mapsto \varphi(t,x) \text{ is injective for all } t \in \Delta.$$

Designate by $\varphi(t,\cdot)^{-1}$ the inverse mapping of $\varphi(t,\cdot)$, i.e.,

$$y=\varphi(t,x)^{-1} \text{ if and only if } x=\varphi(t,y), \quad t\in\Delta.$$

The following recollection of well known results from real analysis and measure theory will be helpful in our further work.

Theorem 2.1. *Suppose $k,\ n\in\mathbb{N}$ and Ω is an open non-empty subset of $\mathbb{R}^n$.*

(i) (Brouwer's theorem, [9]) *Suppose that the mapping $f:\Omega\rightarrow\mathbb{R}^n$ is continuous and injective. Then $f(\Omega)$ is an open subset of $\mathbb{R}^n$.*

(ii) (Rademacher's theorem, [5], [11]) *Suppose $f:\Omega\rightarrow\mathbb{R}^k$ is a locally Lipschitz continuous function. Then $f(\cdot)$ is differentiable at almost every point in Ω.*

(iii) (The change of variables in Lebesgue's integral, [14], [21]) *Suppose $f:\Omega\rightarrow\mathbb{R}^n$ is locally Lipschitz continuous and injective. Then for every measurable subset E of Ω, $f(E)$ is a measurable subset of $\mathbb{R}^n$. Suppose, further, that $g:f(\Omega)\rightarrow\mathbb{R}$ is a measurable function and that the function $x\mapsto g(x)$ is integrable on $f(E)$. Then the function $x\mapsto g(f(x))|\det Df(x)|$ is integrable on E and the following formula holds:*

$$\int_{f(E)} g(x)\,\mathrm{d}x=\int_E g(f(x))|\det Df(x)|\,\mathrm{d}x,$$

where $Df(\cdot)$ denotes the Jacobian of the mapping $f(\cdot)$, which exists for a.e. $x\in\Omega$.

(iv) ([18], [21]) *Suppose that the mapping $f:\Omega\rightarrow\mathbb{R}^n$ is locally Lipschitz continuous. Then for every measurable set $E\subset\Omega$, we have that $m(E)=0$ implies $m(f(E))=0$.*

Given a number $t\in\Delta$, a semiflow $\varphi:\Delta\times\Omega\rightarrow\Omega$ and a function $f:\Omega\rightarrow\mathbb{K}$, define $T_\varphi(t)f:\Omega\rightarrow\mathbb{K}$ by $(T_\varphi(t)f)(x):=f(\varphi(t,x)),\ x\in\Omega$. Then $T_\varphi(0)f=f$, $T_\varphi(t)T_\varphi(s)f=T_\varphi(s)T_\varphi(t)f=T_\varphi(t+s)f,\ t,\ s\in\Delta$ and Brouwer's theorem implies $C_c(\Omega)\subset T_\varphi(t)C_c(\Omega)$. We refer the reader to [16, Theorem 2.1], resp. [16, Theorem 2.2], for the necessary and sufficient conditions stating when the composition operator $T_\varphi(t):L^p_{\rho_1}(\Omega)\rightarrow L^p_{\rho_1}(\Omega)$, resp. $T_\varphi(t):C_{0,\rho}(\Omega)\rightarrow C_{0,\rho}(\Omega)$, is well defined and continuous. In order to see when the semigroup $(T_\varphi(t))_{t\in\Delta}$ is strongly continuous in $L^p_{\rho_1}(\Omega)$, resp. $C_{0,\rho}(\Omega)$, we need the following auxiliary lemma which is inspired by [16, Proposition 3.2].

Lemma 2.1. *Suppose $\varphi : \Delta \times \Omega \to \Omega$ is a semiflow. Then for every compact set $K \subset \Omega$ and for every $\delta > 0$ with $K + B(0, \delta) \subset \Omega$, there exists $n \in \mathbb{N}$ such that:*

$$K \cap \varphi(t, (\Omega \setminus (K + B(0, \delta)))) = \emptyset \text{ for all } t \in \Delta_{1/n}.$$

Herein $B(0, \delta) = \{x \in \mathbb{R}^n : |x| \leq \delta\}$ and $K + B(0, \delta) = \{x + y : x \in K, y \in B(0, \delta)\}$.

PROOF. We will sketch the proof only in the non-trivial case $\Delta = \Delta(\alpha)$, where $\alpha \in (0, \pi/2)$. Suppose to the contrary that for every $n \in \mathbb{N}$ there exist $t_n \in \Delta_{1/n}$ and $x_n \in \Omega \setminus (K + B(0, \delta))$ such that $y_n = \varphi(t_n, x_n) \in K$. The continuity of $\varphi(\cdot, \cdot)$ implies that there exist $\tilde{t_1} \in (\Delta_1)^\circ$ and $x_1 \in \Omega \setminus (K + B(0, \delta))$ such that $\varphi(\tilde{t_1}, x_1) \in K + B(0, \delta/2)$. Put $\tilde{x_1} := x_1$ and choose a natural number $n_1 \geq 2$ such that $\tilde{t_1} - \Delta_{1/n_1} \in \Delta^\circ$. Apply again the continuity of $\varphi(\cdot, \cdot)$ in order to conclude that there exists $t'_{n_1} \in (\Delta_{1/n_1})^\circ$ such that $\varphi(t'_{n_1}, x_{n_1}) \in K + B(0, \delta/2)$. Put $\tilde{t_2} := t'_{n_1}$ and $\tilde{x_2} := x_{n_1}$. Then $\tilde{t_2} \in (\Delta_{1/2})^\circ$, $\tilde{x_2} \in \Omega \setminus (K + B(0, \delta))$, $\varphi(\tilde{t_2}, \tilde{x_2}) \in K + B(0, \delta/2)$ and $\tilde{t_1} - \tilde{t_2} \in \Delta^\circ$. Inductively, one obtains the existence of a sequence $(\tilde{t_n})$ in Δ° and a sequence $(\tilde{x_n})$ in $\Omega \setminus (K + B(0, \delta))$ such that: $\tilde{t_n} \in (\Delta_{1/n})^\circ$, $\tilde{t_n} - \tilde{t_{n+1}} \in \Delta^\circ$ and $\varphi(\tilde{t_n}, \tilde{x_n}) \in K + B(0, \delta/2)$, $n \in \mathbb{N}$. Especially, $\tilde{t_1} - \tilde{t_n} \in \Delta^\circ$, $n \in \mathbb{N}$ and, without loss of generality, we may assume that $\lim_{n\to\infty} \varphi(\tilde{t_n}, \tilde{x_n}) = x \in K + B(0, \delta/2)$. Then one gets $\lim_{n\to\infty} \varphi(\tilde{t_1}, \tilde{x_n}) = \lim_{n\to\infty} \varphi(\tilde{t_1} - \tilde{t_n}, \varphi(\tilde{t_n}, \tilde{x_n})) = \varphi(\tilde{t_1}, x)$. Since the mapping $\varphi(\tilde{t_1}, \cdot) : \Omega \to \Omega$ is continuous and injective, Brouwer's theorem implies that the inverse mapping $\varphi(\tilde{t_1}, \cdot)^{-1} : \varphi(\tilde{t_1}, \Omega) \to \Omega$ is continuous. Hence, one obtains that $\lim_{n\to\infty} \tilde{x_n} = x$ contradicting $\tilde{x_n} \in \Omega \setminus (K + B(0, \delta))$.

The following lemma is suggested by the referee and notably shorten the former proof of Theorem 2.2 given below.

Lemma 2.2. *Let $f : \Omega \to \Omega$ be locally Lipschitz continuous and injective and let f^{-1} be also locally Lipschitz continuous. Then $Df(x)Df^{-1}(f(x)) = I$ a.e. with I being the identity matrix.*

PROOF. Denote $N := \{x \in \Omega : f \text{ is not differentiable in } x\}$ and $N^- := \{x \in f(\Omega) : f^{-1} \text{ is not differentiable in } x\}$. Then

$$m(N) = m(N^-) = 0, \quad f^{-1}(N^-) = \{x \in \Omega : f^{-1} \text{ is not differentable in } f(x)\},$$

and by Theorem 2.1(iv), $m(f^{-1}(N^-)) = 0$. This implies $m(N \cup f^{-1}(N^-)) = 0$ and by the chain rule we have

$$Df(x)Df^{-1}(f(x)) = I, \quad x \in \Omega \setminus (N \cup f^{-1}(N^-)).$$

Theorem 2.2. *Suppose $\varphi : \Delta \times \Omega \to \Omega$ is a semiflow and $\varphi(t,\cdot)$ is a locally Lipschitz continuous function for all $t \in \Delta$. Then* (ii) *implies* (i)*, where*

(i) *$(T_\varphi(t))_{t\in\Delta}$ is a strongly continuous semigroup in $L^p_{\rho_1}(\Omega)$ and*

(ii) $\exists M,\ \omega \in \mathbb{R}\ \forall t \in \Delta$:

$$\rho_1(x) \le M\mathrm{e}^{\omega|t|}\rho_1(\varphi(t,x))|\det D\varphi(t,x)| \ \textit{a.e.}\ \ x \in \Omega. \tag{2.1}$$

If, additionally, $\varphi(t,\cdot)^{-1}$ is locally Lipschitz continuous for every $t \in \Delta$, then the above are equivalent.

PROOF. Suppose that (ii) holds. Then Theorem 2.1(iii) implies:

$$\begin{aligned}
\|T_\varphi(t)f\|^p &= \int_\Omega |f(\varphi(t,x))|^p\rho_1(x)\,\mathrm{d}x \\
&\le M\mathrm{e}^{\omega|t|}\int_\Omega |f(\varphi(t,x))|^p\rho_1(\varphi(t,x))|\det D\varphi(t,x)|\,\mathrm{d}x \\
&= M\mathrm{e}^{\omega|t|}\int_{\varphi(t,\Omega)} |f(x)|^p\rho_1(x)\,\mathrm{d}x \\
&\le M\mathrm{e}^{\omega|t|}\|f\|^p, \qquad t\in\Delta,\ \ f\in L^p_{\rho_1}(\Omega).
\end{aligned}$$

Hence, $T_\varphi(t) \in L(L^p_{\rho_1}(\Omega)),\ t \in \Delta$, and

$$\|T_\varphi(t)\| \le M^{1/p}\mathrm{e}^{\omega|t|/p}. \tag{2.2}$$

Furthermore, the dominated convergence theorem and Lemma 2.1 imply that $\lim_{t\to 0,\ t\in\Delta} T_\varphi(t)f = f$ for all $f \in C_c(\Omega)$; then the strong continuity of $(T_\varphi(t))_{t\in\Delta}$ follows easily from the standard limit procedure and (2.2). Suppose now that $\varphi(t,\cdot)^{-1}$ is locally Lipschitz continuous for every $t \in \Delta$ and that (i) holds. The existence of real numbers M and ω satisfying $||T_\varphi(t)|| \le M\mathrm{e}^{\omega|t|},\ t \in \Delta$, is obvious and, as a simple consequence of Theorem 2.1(iii), one obtains:

$$\int_{\varphi(t,\cdot)^{-1}(L\cap\varphi(t,\Omega))}\rho_1(\cdot)\,\mathrm{d}m = \int_L \chi_{\varphi(t,\Omega)}(\cdot)\rho_1(\varphi(t,\cdot)^{-1})|\det D\varphi(t,\cdot)^{-1}|\,\mathrm{d}m \tag{2.3}$$

for $t \in \Delta$. Then one can apply [15, Theorem 2.1] and (2.3) (cf. also [16, Appendix B]) in order to see that, for every $t \in \Delta$, the inequality:

$$\chi_{\varphi(t,\Omega)}(\cdot)\rho_1(\varphi(t,\cdot)^{-1})|\det D\varphi(t,\cdot)^{-1}| \le M\mathrm{e}^{\omega|t|}\rho_1(\cdot) \tag{2.4}$$

holds almost everywhere in Ω. By Lemma 2.2, one has

$$\det D\varphi(t,x) \times \det D\varphi(t,\cdot)^{-1}(\varphi(t,x)) = 1 \quad \text{for a.e. } x \in \Omega. \tag{2.5}$$

In view of (2.4) and (2.5), one obtains that there exists a measurable subset N of Ω such that $m(N) = 0$ and for each $y \in \Omega \setminus N$:

$$\chi_{\varphi(t,\Omega)}(y)\rho_1(\varphi(t,y)^{-1}) \leq M e^{\omega|t|}\chi_{\varphi(t,\Omega)}(y)\rho_1(y)|\det D\varphi(t,\varphi(t,y)^{-1})|. \tag{2.6}$$

By Theorem 2.1(iv), we obtain that $m(\varphi(t,\cdot)^{-1}(N)) = 0$ and an application of (2.6) implies that (2.1) holds for every $x \in \Omega \setminus (N \cup \varphi(t,\cdot)^{-1}(N))$. This completes the proof of theorem.

Suppose $T_\varphi(t) : L^p_{\rho_1}(\Omega) \to L^p_{\rho_1}(\Omega)$ is well defined and continuous for all $t \in \Delta$. Since the range of $T_\varphi(t),\ t \in \Delta$ is dense in $L^p_{\rho_1}(\Omega)$, one can employ [13, Theorem 1, Proposition 1] in order to see that the hypercyclicity of $(T_\varphi(t))_{t\in\Delta}$ is equivalent to its topological transitivity. By [15, Theorem 2.4], $(T_\varphi(t))_{t\in\Delta}$ is hypercyclic in $L^p_{\rho_1}(\Omega)$ iff for every compact set $K \subset \Omega$ there exist a sequence of measurable subsets (L_k) of K and a sequence (t_k) in Δ such that:

$$\lim_{k\to\infty}\int_{K \setminus L_k}\rho(x)\,\mathrm{d}x = 0, \quad \lim_{k\to\infty}\int_{\varphi(t_k,L_k)}\rho(x)\,\mathrm{d}x = 0$$
$$\text{and} \quad \lim_{k\to\infty}\int_{\varphi(t_k,\cdot)^{-1}(L_k)}\rho(x)\,\mathrm{d}x = 0. \tag{2.7}$$

Example 2.1. Let $\Delta = [0,\infty)$, $\Omega = (0,\infty)$, $p \in [1,\infty)$ and let (a_n) be a decreasing sequence of positive real numbers satisfying $\sum_{n=1}^{\infty} a_n = \infty$. Put, by common consent, $\sum_{i=1}^{0} a_i := 0$ and define $f : (0,\infty) \to (0,\infty)$ by $f(x) := a_{n+1}(x-n) + \sum_{i=1}^{n} a_i$ if $x \in (n, n+1]$ for some $n \in \mathbb{N}_0$. Then $f(\cdot)$ is a strictly increasing, bijective and locally Lipschitz continuous mapping, and moreover, the inverse mapping $f^{-1} : (0,\infty) \to (0,\infty)$ possesses the same properties. Define $\varphi : \Delta \times \Omega \to \Omega$ and $\rho : \Omega \to (0,\infty)$ by $\varphi(t,x) := f^{-1}(t + f(x))$ and $\rho_1(x) := 1/(f(x)+1),\ t \in \Delta,\ x \in \Omega$. It is straightforward to see that $\varphi(\cdot,\cdot)$ is a semiflow and that the mapping $x \mapsto \varphi(t,x),\ x \in \Omega$ is locally Lipschitz continuous for every fixed $t \in \Delta$. In general, the mapping $x \mapsto \varphi(t,x),\ x \in \Omega$ need not be differentiable and one can simply verify that $\frac{\mathrm{d}}{\mathrm{d}x} f(x) = a_{n+1},\ x \in (n,n+1),\ n \in \mathbb{N}_0$ and that

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{a_{n+1}}, \quad x \in \left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n+1} a_i\right), \quad n \in \mathbb{N}_0.$$

Suppose $t \geq 0, n \in \mathbb{N}_0, k \in \mathbb{N}_0, x \in (n, n+1)$ and $t + f(x) \in (\sum_{i=1}^{k} a_i, \sum_{i=1}^{k+1} a_i)$. Then $k \geq n,\ \frac{\mathrm{d}}{\mathrm{d}x}\varphi(t,x) = \frac{a_{n+1}}{a_{k+1}} \geq 1,$

$$\frac{\rho_1(x)}{\rho_1(\varphi(t,x))} = 1 + \frac{t}{f(x)+1} \leq 1 + t \leq \mathrm{e}^t \leq \mathrm{e}^t \left| \frac{\mathrm{d}}{\mathrm{d}x}\varphi(t,x) \right|,$$

and Theorem 2.2 implies that $(T_\varphi(t))_{t\geq 0}$ is a strongly continuous semigroup in $L^p_{\rho_1}(\Omega)$. Let us prove that $(T_\varphi(t))_{t\geq 0}$ is hypercyclic whenever the sequence $(\frac{1}{a_n})$ is bounded. Suppose $K = [a,b] \subset (0,\infty)$, (t_k) is any sequence of positive real numbers satisfying $\lim_{k\to\infty} t_k = \infty$ and $M := \sup_{n\in\mathbb{N}}\{\frac{1}{a_n} : n \in \mathbb{N}\}$. Notice that for every $k \in \mathbb{N}$ and $n \in \mathbb{N}$ with $t_k + f(b) < \sum_{i=1}^{n} a_i$:

$$\left| f^{-1}(t_k + f(b)) - f^{-1}(t_k + f(a)) \right| \leq \max\left(\frac{1}{a_1}, \cdots, \frac{1}{a_{n+1}}\right) |f(b) - f(a)|.$$

This inequality implies

$$\begin{aligned}\int_{\varphi(t_k,K)} \rho_1(x)\,\mathrm{d}x &= \int_{f^{-1}(t_k+f(a))}^{f^{-1}(t_k+f(b))} \frac{1}{f(x)+1} \\ &\leq \frac{f^{-1}(t_k+f(b)) - f^{-1}(t_k+f(a))}{f(a)+t_k+1} \\ &\leq M\frac{f(b)-f(a)}{f(a)+t_k+1}\end{aligned}$$

and

$$\lim_{k\to\infty} \int_{\varphi(t_k,K)} \rho_1(x)\,\mathrm{d}x = 0.$$

Furthermore, it is clear that there exists $k_0 \in \mathbb{N}$ such that $\varphi(t_k,\cdot)^{-1}(K) = \emptyset,\ k \geq k_0$; hence, $\lim_{k\to\infty} \int_{\varphi(t_k,K)} \rho_1(x)\,\mathrm{d}x = 0$, (2.7) holds and $(T_\varphi(t))_{t\geq 0}$ is hypercyclic as claimed.

Taking into account Lemma 2.1 and the proof of [16, Theorem 3.4], one immediately obtains the following theorem which states when $(T_\varphi(t))_{t\in\Delta}$ is a strongly continuous semigroup in $C_{0,\rho}(\Omega)$.

Theorem 2.3. *Let $\varphi : \Delta \times \Omega \to \Omega$ be a semiflow. Then $(T_\varphi(t))_{t\in\Delta}$ is a strongly continuous semigroup in $C_{0,\rho}(\Omega)$ iff the following holds:*

(i) $\exists M, \omega \in \mathbb{R}\ \forall t \in \Delta,\ x \in \Omega : \rho(x) \leq M\mathrm{e}^{\omega|t|}\rho(\varphi(t,x))|$ *and*

(ii) *for every compact set* $K \subset \Omega$ *and for every* $\delta > 0$ *and* $t \in \Delta$:

$$\varphi(t, \cdot)^{-1}(K) \cap \{x \in \Omega : \rho(x) \geq \delta\} \text{ is a compact subset of } \Omega.$$

Suppose that $T_\varphi(t) : C_{0,\rho}(\Omega) \to C_{0,\rho}(\Omega)$ is well defined and continuous for all $t \in \Delta$ and that for every compact set $K \subset \Omega$, we have $\inf_{x\in K} \rho(x) > 0$. Then [15, Corollary 2.11] immediately implies that $(T_\varphi(t))_{t\in\Delta}$ is hypercyclic in $C_{0,\rho}(\Omega)$ iff for every compact set $K \subset \Omega$ there exists a sequence (t_k) in Δ such that:

$$\lim_{k\to\infty} \sup_{x\in\varphi(t_k,\cdot)^{-1}(K)} \rho(x) = \lim_{k\to\infty} \sup_{x\in\varphi(t_k,K)} \rho(x) = 0.$$

Theorem 2.4. *Let* $\varphi : \Delta \times \Omega \to \Omega$ *be a semiflow.*

(i) *Suppose* $T_\varphi(t) : L^p_{\rho_1}(\Omega) \to L^p_{\rho_1}(\Omega)$ *is well defined and continuous for all* $t \in \Delta$. *Then the following assertions are equivalent.*

(i1) $(T_\varphi(t))_{t\in\Delta}$ *is positively supercyclic in* $L^p_{\rho_1}(\Omega)$.

(i2) *For every compact set* $K \subset \Omega$ *there exist a sequence* (L_k) *of measurable subsets of* K, *a sequence* (t_k) *in* Δ *and a sequence* (c_k) *in* $(0,\infty)$ *such that:*

$$\lim_{k\to\infty} \int_{K \setminus L_k} \rho_1(x)\,\mathrm{d}x = 0 \tag{2.8}$$

and

$$\lim_{k\to\infty} c_k \int_{\varphi(t_k,\cdot)^{-1}(L_k)} \rho_1(x)\,\mathrm{d}x = \lim_{k\to\infty} \frac{1}{c_k} \int_{\varphi(t_k,L_k)} \rho_1(x)\,\mathrm{d}x = 0.$$

(i3) *For every compact set* $K \subset \Omega$ *there exist a sequence* (L_k) *of measurable subsets of* K *and a sequence* (t_k) *in* Δ *such that* (2.8) *holds and that*

$$\lim_{k\to\infty} \left[\int_{\varphi(t_k,\cdot)^{-1}(L_k)} \rho_1(x)\,\mathrm{d}x * \int_{\varphi(t_k,L_k)} \rho_1(x)\,\mathrm{d}x \right] = 0. \tag{2.9}$$

(ii) *Suppose that* $T_\varphi(t) : C_{0,\rho}(\Omega) \to C_{0,\rho}(\Omega)$ *is well defined and continuous for all* $t \in \Delta$ *and that for every compact set* $K \subset \Omega$, *we have* $\inf_{x\in K} \rho(x) > 0$. *Then the following assertions are equivalent.*

(ii1) $(T_\varphi(t))_{t\in\Delta}$ *is positively supercyclic in* $C_{0,\rho}(\Omega)$.

(ii2) *For every compact set* $K \subset \Omega$ *there exist a sequence* (t_k) *in* Δ *and a sequence* (c_k) *in* $(0,\infty)$ *such that:*

$$\lim_{k\to\infty} c_k \sup_{x\in\varphi(t_k,\cdot)^{-1}(K)} \rho(x) = \lim_{k\to\infty} \frac{1}{c_k} \sup_{x\in\varphi(t_k,K)} \rho(x) = 0.$$

(ii3) *For every compact set $K \subset \Omega$ there exists a sequence (t_k) in Δ such that:*

$$\lim_{k\to\infty}\left[\sup_{x\in\varphi(t_k,\cdot)^{-1}(K)}\rho(x) * \lim_{k\to\infty}\sup_{x\in\varphi(t_k,K)}\rho(x)\right] = 0.$$

PROOF. Put $I := \{(c,t) : c \in (0,\infty),\ t \in \Delta\}$, $T_\varphi(c,t) := cT_\varphi(t),\ (c,t) \in I$ and notice that the operators $T_\varphi(c,t),\ t \in \Delta$ have dense range and commute with each other. According to [13, Theorem 1, Proposition 1], one obtains that the positive supercyclicity of $(T_\varphi(t))_{t\in\Delta}$ is equivalent to the topological transitivity of $(T_\varphi(c,t))_{(c,t)\in I}$. In view of this, the equivalence of (i1) and (i2) follows automatically from an application of [15, Theorem 4.3]. Suppose now K is a compact subset of Ω. Then there exist a sequence (L_k) of measurable subsets of K and a sequence (t_k) in Δ such that (2.8) and (2.9) hold. Notice that, for two arbitrary sequences of non-negative real numbers $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}\alpha_k\beta_k = 0$ there are subsequences $(\alpha_{k_l})_{l\in\mathbb{N}}$ and $(\beta_{k_l})_{l\in\mathbb{N}}$ as well as a sequence $(c_l)_{l\in\mathbb{N}}$ of positive numbers such that $\lim_{l\to\infty} c_l\alpha_{k_l} = \lim_{l\to\infty} c_l^{-1}\beta_{k_l} = 0$ simply by choosing $(k_l)_{l\in\mathbb{N}}$ as a strictly increasing sequence of natural numbers with $k_l > l^2$ and $\alpha_k\beta_k < 1/l^2$ for all $k \geq k_l$ and by setting $c_l := l(\beta_{k_l} + k_l^{-1}(1+\alpha_{k_l})^{-1})$. The proof of implication (i3) $\Rightarrow$ (i2) follows by applying this to $\alpha k = \int_{\varphi(t_{k_l},\cdot)^{-1}(L_{k_l})}\rho_1(x)\,\mathrm{d}x$ and $\beta_k = \int_{\varphi(t_{k_l},L_{k_l})}\rho_1(x)\,\mathrm{d}x$. The proof of part (ii) is done in exactly the same way as the proof of part (i), so that it can be omitted.

Concerning Theorem 2.4(a), let us stress that it is not clear whether, as in the case of hypercyclicity (cf. [15, Example 3.19]), we can get into a situation where one must choose a sequence (L_k) of measurable subsets of K, which satisfies $L_k \neq K,\ k \geq k_0$.

The purpose of this example is to provide a positively supercyclic semigroup which is not hypercyclic.

Example 2.2. Suppose $\Delta = \Omega = \mathbb{R}$, $m \in \mathbb{N}$, $p : \mathbb{R} \to \mathbb{R}$,

$$p(x) = \sum_{i=0}^{2m+1} a_i x^i, \quad \tilde{p}(x) = \sum_{i=0}^{2m+1} |a_i| x^i,\ \ x \in \mathbb{R},$$

$a_{2m+1} > 0$ and $p'(x) \geq c > 0,\ x \in \mathbb{R}$. Then $p(\cdot)$ is bijective and strictly increasing so that we can define a semiflow $\varphi : \Delta\times\Omega \to \Omega$ by $\varphi(t,x) := p^{-1}(t+p(x)),\ t,\ x \in \mathbb{R}$. Suppose that $f : \mathbb{R} \to (0,\infty)$ is an admissible weight function in the sense of [10], i.e., $f(\cdot)$ is a measurable function and there exist appropriate numbers $M' \in [1,\infty)$ and $\omega' \in \mathbb{R}$ so that $f(x) \leq M'\mathrm{e}^{\omega'|t|}f(x+t),\ x,\ t \in \mathbb{R}$. Define a locally integrable function $\rho_1 : \mathbb{R} \to (0,\infty)$ by $\rho_1(x) := f(p(x))$, $x \in \mathbb{R}$. Further on, let us prove that there exists $C_1 \in (0,\infty)$ such that for every $t,\ x_0 \in \mathbb{R}$:

$$|\det D\varphi(t,x_0)| \geq \frac{1}{C_1\left(1+|t|^{\frac{2m}{2m+1}}\right)}. \tag{2.10}$$

To do that, notice that

$$|\det D\varphi(t,x_0)| = \frac{p'(x_0)}{p'(\varphi(t,x_0))}$$

and define $q : \mathbb{R} \to \mathbb{R}$ by

$$q(x) := \frac{p(x) - (t + p(x_0))}{a_{2m+1}}, \quad x \in \mathbb{R}.$$

It is well known from the elementary courses of numerical analysis ([22]) that every zero ξ of a real polynomial $r(x) = x^s + \sum_{i=0}^{s-1} b_i x^i,\ b_0 \neq 0,\ s \geq 2$ satisfies

$$|\xi| < 2\max\left\{|b_i|^{\frac{1}{s-i}} : 0 \leq i \leq s-1\right\}.$$

Since $q(\varphi(t,x_0)) = 0$, this assertion enables one to deduce that there exists $C \in (0,\infty)$, independent of t and x_0, such that

$$\begin{aligned}
|\varphi(t,x_0)| &\leq 2\max\left\{\left|\frac{a_{2m}}{a_{2m+1}}\right|, \left|\frac{a_{2m-1}}{a_{2m+1}}\right|^{\frac{1}{2}}, \dots, \left|\frac{a_1}{a_{2m+1}}\right|^{\frac{1}{2m}}, \left|\frac{a_0 - t - p(x_0)}{a_{2m+1}}\right|^{\frac{1}{2m+1}}\right\} \\
&\leq 2\sum_{i=0}^{2m}\left\{\left|\frac{a_i}{a_{2m+1}}\right|^{\frac{1}{2m+1-i}} + \left|\frac{t}{a_{2m+1}}\right|^{\frac{1}{2m+1}} + \left|\frac{p(x_0)}{a_{2m+1}}\right|^{\frac{1}{2m+1}}\right\} \\
&\leq C\left(1 + |t|^{\frac{1}{2m+1}} + |p(x_0)|^{\frac{1}{2m+1}}\right).
\end{aligned}$$

Taken together, this estimate and the elementary inequalities $|p(x_0)| \leq \tilde{p}(|x_0|)$,

$$1 + |t|^{\frac{i}{2m+1}} \leq 2(1 + |t|^{\frac{2m}{2m+1}}), \quad 0 \leq i \leq 2m,$$

$(a+b+c)^i \leq 3^{i-1}(a^i + b^i + c^i),\ i \in \mathbb{N},\ a,\ b,\ c \geq 0$, imply the existence of positive real number $\overline{C}$, independent of t and x_0, such that

$$\begin{aligned}|p'(\varphi(t,x_0))| &\leq \sum_{i=1}^{2m}(i+1)|a_{i+1}|C^{2i}\Big(1+|t|^{\frac{1}{2m+1}}+|p(x_0)|^{\frac{1}{2m+1}}\Big)^{2i}+|a_1| \\ &\leq \sum_{i=1}^{2m}(i+1)|a_{i+1}|C^{2i}3^{2i-1}\Big(1+|t|^{\frac{2i}{2m+1}}+|p(x_0)|^{\frac{2i}{2m+1}}\Big)+|a_1| \\ &\leq \overline{C}\left(1+|t|^{\frac{2m}{2m+1}}+\sum_{i=0}^{2m}|p(x_0)|^{\frac{i}{2m+1}}\right) \\ &\leq \overline{C}\left(1+|t|^{\frac{2m}{2m+1}}+\sum_{i=0}^{2m}|\tilde{p}(|x_0|)|^{\frac{i}{2m+1}}\right) \\ &\leq 2\overline{C}\Big(1+|t|^{\frac{2m}{2m+1}}\Big)\sum_{i=0}^{2m}|\tilde{p}(|x_0|)|^{\frac{i}{2m+1}}.\end{aligned}$$

Hence,

$$|\det D\varphi(t,x_0)|=\frac{p'(x_0)}{p'(\varphi(t,x_0))}\geq\frac{|p'(x_0)|}{2\overline{C}\Big(1+|t|^{\frac{2m}{2m+1}}\Big)\sum_{i=0}^{2m}\tilde{p}(|x_0|)^{\frac{i}{2m+1}}}. \tag{2.11}$$

Using positivity of $x\mapsto p'(x)-c,\ x\in\mathbb{R}$, (2.11) and the following obvious equality

$$\lim_{x\to\infty}\frac{|p'(x)|}{\sum_{i=0}^{2m}\tilde{p}(|x|)^{\frac{i}{2m+1}}}=(2m+1)a_{2m+1}^{\frac{1}{2m+1}},$$

one immediately yields (2.10) with a suitable positive constant C_1. Now the condition (ii) given in the formulation of Theorem 2.2 follows from the admissibility of $f(\cdot)$ and (2.10); in conclusion, one gets that $(T_\varphi(t))_{t\in\mathbb{R}}$ is a strongly continuous group in $L^p_{\rho_1}(\Omega)$. Since $\varphi(t,x)^{-1}=p^{-1}(p(x)-t),\ t,\ x\in\mathbb{R}$, we obtain analogously that there exists $C_2\in(0,\infty)$ such that:

$$|\det D\varphi(t,x_0)^{-1}|\geq\frac{1}{C_2\Big(1+|t|^{\frac{2m}{2m+1}}\Big)},\quad t,\ x_0\in\mathbb{R}. \tag{2.12}$$

Using Theorem 2.1(iii), (2.10) and (2.12), it follows immediately that for every measurable subset E of $\mathbb{R}$:

$$m(\varphi(t,E)) = \int_{\varphi(t,E)} \mathrm{d}x = \int_E |\det D\varphi(t,x)| \,\mathrm{d}x$$
$$\in \left[\frac{m(E)}{C_1(1+|t|^{\frac{2m}{2m+1}})}, \frac{1}{c}\int_E p'(x)\,\mathrm{d}x \right] \tag{2.13}$$

and

$$m(\varphi(t,\cdot)^{-1}(E)) \in \left[\frac{m(E)}{C_2(1+|t|^{\frac{2m}{2m+1}})}, \frac{1}{c}\int_E p'(x)\,\mathrm{d}x \right], \quad t \in \Delta. \tag{2.14}$$

Suppose now that $\beta \geq 2m/(2m+1)$ and that a bounded measurable function $h : \mathbb{R} \to (0,\infty)$ is defined by:

$$h(s) := \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\log[(s+1)^\beta + 1], & s \geq 0, \\ 1, & s < 0. \end{cases} \tag{2.15}$$

Put now $f(x) := \exp\left(\int_0^x h(s)ds\right),\ x \in \mathbb{R}$; then

$$\frac{f(x)}{f(x+t)} = \mathrm{e}^{\int_x^{x+t} h(s)\,\mathrm{d}s} \leq \mathrm{e}^{\sup_{s\in\mathbb{R}} h(s)}|t|, \quad x,\ t \in \mathbb{R},$$

$f(\cdot)$ is admissible and $\rho_1(x) = \mathrm{e}^{\int_0^{p(x)} h(s)\,\mathrm{d}s},\ x \in \mathbb{R}$. We will prove that $(T_\varphi(t))_{t\in\mathbb{R}}$ is positively supercyclic in $L^p_{\rho_1}(\mathbb{R})$ and that $(T_\varphi(t))_{t\in\mathbb{R}}$ is not hypercyclic in $L^p_{\rho_1}(\mathbb{R})$. To this end, let $-\infty < a < b < \infty$, $K = [a,b]$ and let (t_k) be an arbitrary sequence of positive real numbers such that $\lim_{k\to\infty} t_k = \infty$. It is clear that there exists $k_0 \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$ with $k \geq k_0$, $p(a) + t_k \geq 0$ and $p(b) - t_k \leq 0$. The assumption $x \in \varphi(t_k,\cdot)^{-1}(K)$, resp. $x \in \varphi(t_k,K)$ is equivalent to $p(x) \in [p(a) - t_k, p(b) - t_k]$, resp. $p(x) \in [p(a) + t_k, p(b) + t_k]$. Thus, $\rho_1(x) = \mathrm{e}^{\int_0^{p(x)} h(s)ds} = \mathrm{e}^{p(x)},\ k \geq k_0, x \in \varphi(t_k,\cdot)^{-1}(K)$ and

$$\rho_1(x) = \mathrm{e}^{\int_0^{p(x)} h(s)ds} \leq \mathrm{e}^{\int_0^{t_k+p(b)}[\frac{\mathrm{d}}{\mathrm{d}s}\log((s+1)^\beta+1)]\,\mathrm{d}s}$$
$$= \frac{1}{2}\left((t_k + p(b) + 1)^\beta + 1\right), \quad k \geq k_0,\ x \in \varphi(t_k,K).$$

Having in mind these inequalities as well as (2.13)–(2.14), one gets:

$$\int_{\varphi(t_k,\cdot)^{-1}(K)} \rho_1(x)\,\mathrm{d}x \leq \mathrm{e}^{p(b)}\mathrm{e}^{-t_k}\left(\frac{1}{c}\int_K p'(x)\,\mathrm{d}x\right), \quad k \geq k_0 \tag{2.16}$$

and

$$\int_{\varphi(t_k,K)} \rho_1(x)\,\mathrm{d}x \le \frac{1}{2}\Big((t_k+p(b)+1)^{\beta}+1\Big)\Big(\frac{1}{c}\int_K p'(x)\,\mathrm{d}x\Big), \quad k \ge k_0. \tag{2.17}$$

Now one can employ (2.16)-(2.17) and Theorem 2.4(a) with $L_k = K,\ k \in \mathbb{N}$ to conclude that $(T_{\varphi}(t))_{t\in\mathbb{R}}$ is positively supercyclic in $L^p_{\rho_1}(\mathbb{R})$. Suppose that $(T_{\varphi}(t))_{t\in\mathbb{R}}$ is hypercyclic in $L^p_{\rho_1}(\mathbb{R})$ and that K is a compact subset of $\mathbb{R}$ such that $\inf K \ge \zeta$, where ζ is a unique real zero of the polynomial $p(\cdot)$. Then we have the existence of a sequence of measurable subsets (L_k) of K and a sequence (t_k) in $\mathbb{R}$ such that (2.7) holds. It can be straightforwardly proved that (t_k) must be unbounded and, without loss of generality, we may assume that $\lim_{k\to\infty} t_k = +\infty$. Since $p(x) \ge 0,\ x \in K$ one gets $\rho_1(x) = \frac{1}{2}((1+p(x))^{\beta}+1) \ge 1,\ x \in K$, $\lim_{k\to\infty} m(K \setminus L_k) = 0$, and consequently, there exists $k_1 \in \mathbb{N},\ k_1 \ge k_0$ such that $m(L_k) \ge \frac{1}{2}m(K),\ k \ge k_1$. Then (2.13) implies:

$$m(\varphi(t,L_k)) \ge \frac{m(L_k)}{C_1\Big(1+|t|^{\frac{2m}{2m+1}}\Big)} \ge \frac{m(K)}{2C_1\Big(1+|t|^{\frac{2m}{2m+1}}\Big)}, \quad t \in \mathbb{R},\ k \ge k_1. \tag{2.18}$$

Since $\beta \ge 2m/(2m+1)$ and

$$\begin{aligned}\rho_1(x) &= \mathrm{e}^{\int_0^{p(x)} h(s)\,\mathrm{d}s} \ge \mathrm{e}^{\int_0^{t_k+p(a)}[\frac{\mathrm{d}}{\mathrm{d}s}\log((s+1)^{\beta}+1)]\,\mathrm{d}s}\\ &= \frac{1}{2}((t_k+p(a)+1)^{\beta}+1), \quad k \ge k_0,\ x \in \varphi(t_k,L_k),\end{aligned}$$

(2.18) yields:

$$\int_{\varphi(t_k,L_k)} \rho(x)\,\mathrm{d}x \ge \frac{m(K)}{2C\Big(1+t_k^{\frac{2m}{2m+1}}\Big)}\Big[\frac{1}{2}(t_k+p(a)+1)^{\beta}+\frac{1}{2}\Big] \nrightarrow 0,\ k \to \infty.$$

The last estimate proves that $(T_{\varphi}(t))_{t\in\mathbb{R}}$ is not hypercyclic in $L^p_{\rho_1}(\mathbb{R})$.

3. *Disjoint hypercyclic semigroups induced by semiflows*

Definition 3.1. Let $n \in \mathbb{N},\ n \ge 2$ and let $(T_i(t))_{t\in\Delta}$ be hypercyclic strongly continuous semigroups in X, $i = 1,2,\dots,n$. It is said that the semigroups $(T_i(t))_{t\in\Delta},\ i = 1,2,\dots,n$ are:

(i) *disjoint hypercyclic*, in short *d-hypercyclic*, if there exists $x \in X$ such that

$$\overline{\{(T_1(t)x,\dots,T_n(t)x) \mid t \in \Delta\}} = X^n. \tag{3.1}$$

An element $x \in X$ which satisfies (3.1) is called a *d-hypercyclic vector* associated to the semigroups $(T_1(t))_{t\in\Delta}$, $(T_2(t))_{t\in\Delta}, \cdots,$ $(T_n(t))_{t\in\Delta}$;

(ii) *disjoint topologically transitive*, in short *d-topologically transitive*, if for any open non-empty subsets $V_0, V_1, \ldots, V_n$ of X, there exists $t \in \Delta$ such that $V_0 \cap T_1(t)^{-1}(V_1) \cap \cdots \cap T_n(t)^{-1}(V_n) \neq \emptyset$.

It follows immediately from Definition 3.1 that d-hypercyclicty of $(T_i(t))_{t\in\Delta},\ i = 1, 2, \cdots, n$ implies that, for every $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, there exists $t \in \Delta \setminus \{0\}$ such that $T_i(t) \neq T_j(t)$.

Suppose $(T_i(t))_{t\in\Delta},\ i = 1, 2, \ldots, n$ are strongly continuous semigroups. Arguing as in the proofs of [4, Proposition 2.3] and [13, Satz 1.2.2], one obtains that d-topological transitivity of $(T_i(t))_{t\in\Delta},\ i = 1, 2, \ldots, n$ implies that $(T_i(t))_{t\in\Delta},\ i = 1, 2, \ldots, n$ are d-hypercyclic and that the set of all d-hypercyclic vectors associated to $(T_1(t))_{t\in\Delta}, (T_2(t))_{t\in\Delta}, \ldots,$ $(T_n(t))_{t\in\Delta}$ is a dense G_δ-subset of X.

Now we are in a position to clarify the following theorem which concerns sufficient conditions for d-topological transitivity of strongly continuous semigroups on a class of weighted function spaces.

Theorem 3.1. *Suppose $p \in [1, \infty)$, $n \in \mathbb{N} \setminus \{1\}$, $\varphi_i : \Delta \times \Omega \to \Omega$ is a semiflow for all $i = 1, 2, \cdots, n$, $\rho : \Omega \to (0, \infty)$ is an upper semicontinuous function and $\rho_1 : \Omega \to (0, \infty)$ is a locally integrable function.*

(i) *Suppose that $X = C_{0,\rho}(\Omega)$ and that $(T_{\varphi_i}(t))_{t\in\Delta},\ i = 1, 2, \ldots, n$ are strongly continuous semigroups in X. If for every compact set $K \subset \Omega$ there exists a sequence (t_k) in Δ which satisfies the following conditions:*

(A) $\lim\limits_{k\to\infty} \sup\limits_{\varphi_i(t_k,x)\in\varphi_j(t_k,K)} \rho(x) = 0,\ \ i,\ j \in \{1, 2, \ldots, n\},\ i \neq j$, *and*

(B) $\lim\limits_{k\to\infty} \sup\limits_{x\in\varphi_i(t_k,\cdot)^{-1}(K)} \rho(x) = \lim\limits_{k\to\infty} \sup\limits_{x\in\varphi_i(t_k,K)} \rho(x) = 0,\ i = 1, 2, \ldots, n,$

then the semigroups $(T_{\varphi_i}(t))_{t\in\Delta},\ i = 1, 2, \ldots, n$, are d-topologically transitive.

(ii) *Suppose that $X = L^p_{\rho_1}(\Omega)$ and that and that $(T_{\varphi_i}(t))_{t\in\Delta},\ i = 1, 2, \ldots, n$ are strongly continuous semigroups in X. If for every compact set $K \subset \Omega$ there exist a sequence of measurable subsets (L_k) of K and a sequence (t_k) in Δ which satisfies the following conditions for $i, j \in \{1, 2, \ldots, n\}$,*

(A1) $\lim\limits_{k\to\infty} \int_{K \setminus L_k} \rho_1(x)\, \mathrm{d}x = 0;$

(B1) $\lim\limits_{k\to\infty} \int_{\varphi_i(t_k,\cdot)^{-1}(\varphi_j(t_k,L_k))} \rho_1(x)\, \mathrm{d}x = 0, \quad i \neq j;$

(C1) $\lim_{k\to\infty} \int_{\varphi_i(t_k,\cdot)^{-1}(L_k)} \rho_1(x)\,\mathrm{d}x = \lim_{k\to\infty} \int_{\varphi_i(t_k,L_k)} \rho_1(x)\,\mathrm{d}x = 0,$

then the semigroups $(T_{\varphi_i}(t))_{t\in\Delta},\ i = 1, 2, \ldots, n$, are d-topologically transitive.

PROOF. To prove (i), notice that Theorem 2.3 and the suppositions (A) and (A') imply that $(T_{\varphi_i}(t))_{t\in\Delta}$ is a locally equicontinuous semigroup in X for all $i = 1, 2, \ldots, n$. In order to prove that $(T_{\varphi_i}(t))_{t\in\Delta},\ i = 1, 2, \ldots, n$ are d-topologically transitive, let us suppose $\epsilon > 0,\ u, v_1, \ldots, v_n \in C_c(\Omega)$ and $K = \mathrm{supp} u \cup \mathrm{supp} v_1 \cup \cdots \cup \mathrm{supp} v_n$. The prescribed assumption implies that, for the compact set K, one can find a sequence (t_k) in Δ satisfying (A)–(B). Define, for every $k \in \mathbb{N}$, a function $\omega_k : \Omega \to \mathbb{K}$ by setting:

$$\omega_k(x) := u + \sum_{i=1}^{n} v_i(\varphi_i(t_k,\cdot)^{-1})\chi_{\varphi_i(t_k,\mathrm{supp} v_i)}.$$

Clearly, $\mathrm{supp}\omega_k$ is a compact set for every $k \in \mathbb{N}$ and Brouwer's theorem implies that $\omega_k \in C_c(\Omega),\ k \in \mathbb{N}$. Hence, the proof of (i) follows immediately if one prove that there exist $k_0 \in \mathbb{N}$ and $t \in \Delta$ which fulfill the next condition:

$$\max\left(||\omega_{k_0} - u||,\ ||T_{\varphi_1}(t)\omega_{k_0} - v_1||, \ldots,\ ||T_{\varphi_n}(t)\omega_{k_0} - v_n||\right) < \epsilon. \tag{3.2}$$

The definition of $\omega_k(\cdot)$ gives the next inequality:

$$||\omega_k - u|| \leq \sum_{i=1}^{n} ||v_i||_\infty \sum_{i=1}^{n} \sup_{x\in\varphi_i(t_k,\mathrm{supp} v_i)} \rho(x), \quad k \in \mathbb{N}. \tag{3.3}$$

Owing to (A) and (3.3), there exists $k_{0,0} \in \mathbb{N}$ such that:

$$||\omega_k - u|| < \epsilon,\ \ k \geq k_{0,0}. \tag{3.4}$$

Proceeding in a similar way, one gets that, for every $k \in \mathbb{N}$ and $i = 1, 2, \ldots, n$:

$$\begin{aligned} ||T_{\varphi_i}(t_k)\omega_k - v_i|| \leq{}& ||u||_\infty \sup_{x\in\varphi_i(t_k,\cdot)^{-1}(K)} \rho(x) \\ &+ \sum_{\substack{1\leq j\leq n \\ j\neq i}} ||v_j||_\infty \sum_{\substack{1\leq j\leq n \\ j\neq i}} \sup_{x\in\varphi_i(t_k,\cdot)^{-1}(\varphi_j(t_k,\mathrm{supp} v_j))} \rho(x). \end{aligned}$$

Now an application of (A)–(B) shows that, for every $i = 1, 2, \ldots, n$, there exists $k_{0,i} \in \mathbb{N}$ such that:

$$||T_{\varphi_i}(t_k)\omega_k - v_i|| < \epsilon, \quad k \geq k_{0,i}. \tag{3.5}$$

Put $k_0 := \max(k_{0,0}, \ldots, k_{0,n})$ and notice that (3.4) and (3.5) imply the validity of (3.2) with $t = t_{k_0}$. To prove (ii), suppose $\epsilon > 0$, $u, v_1, \ldots, v_n \in C_c(\Omega)$ and $K = \operatorname{supp} u \cup \operatorname{supp} v_1 \cup \cdots \cup \operatorname{supp} v_n$. For this compact set $K \subset \Omega$, one can find a sequence of measurable subsets (L_k) of K and a sequence (t_k) in Δ satisfying (A1)-(C1). Define, for every $k \in \mathbb{N}$, a function $\omega_k : \Omega \to \mathbb{K}$ as follows:

$$\omega_k := u\chi_{L_k} + \sum_{i=1}^{n} v_i(\varphi_i(t_k, \cdot)^{-1})\chi_{\varphi_i(t_k, L_k)}.$$

It can be simply verified that $\omega_k \in L^p_{\rho_1}(\Omega),\ k \in \mathbb{N}$. Proceeding as in the proof of (i), we have the existence of a positive real number c such that, for every $k \in \mathbb{N}$ and $i = 1, 2, \ldots, n$:

$$||\omega_k - u||^p \leq c[||u||^p_\infty \int_{K \setminus L_k} \rho_1(x)\,\mathrm{d}x + \sum_{i=1}^{n} ||v_i||^p_\infty \int_{\varphi_i(t_k, L_k)} \rho_1(x)\,\mathrm{d}x]$$

and

$$||T_{\varphi_i}(t_k)\omega_k - v_i||^p \leq c[||v_i||^p_\infty \int_{K \setminus L_k} \rho_1(x)\,\mathrm{d}x + ||u||^p_\infty \int_{\varphi_i(t_k, \cdot)^{-1}(L_k)} \rho_1(x)\,\mathrm{d}x$$
$$+ \sum_{\substack{1 \leq j \leq n \\ j \neq i}} ||v_j||^p_\infty \int_{\varphi_i(t_k, \cdot)^{-1}(\varphi_j(t_k, L_k))} \rho_1(x)\,\mathrm{d}x].$$

By (A1)–(C1), one gets that the semigroups $(T_{\varphi_i}(t))_{t \in \Delta},\ i = 1, 2, \ldots, n$ are d-topologically transitive in $L^p_{\rho_1}(\Omega)$, as required.

Problem (2008). Suppose K is a compact subset of Ω and the strongly continuous semigroups $(T_{\varphi_i}(t))_{t \in \Delta},\ i = 1, 2, \ldots, n$, are d-topologically transitive in $C_{0,\rho}(\Omega)$, resp. $L^p_{\rho_1}(\Omega)$. Does there exist a sequence (t_k) in Δ satisfying (A)–(B), resp. a sequence of measurable subsets (L_k) of K and a sequence (t_k) in Δ satisfying (A1)–(C1)?

Repeating literally the arguments given in the proof of Theorem 3.1 (i), one can prove the following assertion concerning d-topological transitivity of strongly continuous semigroups on the Fréchet space $C(\Omega)$.

Theorem 3.2. *Suppose that $\varphi_i : \Delta \times \Omega \to \Omega$ is a semiflow for all $i = 1, 2, \ldots, n$, and that for every compact set $K \subset \Omega$ there exists a sequence (t_k) in Δ satisfying the following condition: For every compact set $K' \subset \Omega$ there exists $k_0(K') \in \mathbb{N}$ such that:*

(A2) $\varphi_i(t_k,\cdot)^{-1}(\varphi_j(t_k,K)) \cap K' = \emptyset,\ i,\ j \in \{1,2,\ldots,n\},\ i \neq j,\ k \geq k_0(K')$ *and*

(B2) $\varphi_i(t_k,K) \cap K' = \varphi_i(t_k,\cdot)^{-1}(K) \cap K' = \emptyset,\ i = 1,2,\ldots,n,\ k \geq k_0(K')$.

Then $(T_{\varphi_i}(t))_{t\in\Delta}$ is a strongly continuous semigroup in $C(\Omega)$ for every $i \in \{1,\ldots,n\}$ and $(T_{\varphi_1}(t))_{t\in\Delta},\ldots,(T_{\varphi_n}(t))_{t\in\Delta}$ are d-topologically transitive in $C(\Omega)$.

Example 3.1. (i) Suppose $p \in [1,\infty)$, $\alpha \in (0,\frac{\pi}{2}]$, $\Delta \in \{[0,\infty),\ \Delta(\alpha)\}$, $\Omega = (1,\infty)$, $n \in \mathbb{N}\setminus\{1\}$ and $0 < \alpha_1 < \cdots < \alpha_n \leq 1$. Define $\varphi_i : \Delta \times \Omega \to \Omega$, $i = 1,2,\ldots,n$, and $\rho_1 : \Omega \to (0,\infty)$ by:

$$\varphi_i(t,x) := (\mathrm{Re}(t) + x^{\alpha_i})^{1/\alpha_i} \text{ and } \rho_1(x) := \mathrm{e}^{-x^{\alpha_1}},\ t \in \Delta,\ x \in \Omega.$$

It is straightforward to check that $\varphi_i(\cdot,\cdot)$ is a semiflow for all $i = 1,2,\ldots,n$. We will prove that the semigroups $(T_{\varphi_i}(t))_{t\in\Delta}$, $i = 1,2,\ldots,n$, are d-topologically transitive in $L^p_{\rho_1}(\Omega)$; without loss of generality, we may assume that $\Delta = [0,\infty)$. The existence of numbers $M \in [1,\infty)$ and $\omega \in \mathbb{R}$ satisfying:

$$\rho_1(x) \leq M\mathrm{e}^{\omega|t|}\rho_1(\varphi_i(t,x)),\ \ t \geq 0,\ x \in \Omega,\ i = 1,2,\ldots,n, \tag{3.6}$$

is obvious. Furthermore, we have that, for every $t \geq 0$, $x \in \Omega$ and $i = 1,2,\ldots,n$:

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\varphi_i(t,x)\right| = \left(1 + \frac{t}{x^{\alpha_i}}\right)^{\frac{1-\alpha_i}{\alpha_i}} \in \left[1,(1+t)^{\frac{1-\alpha_i}{\alpha_i}}\right]. \tag{3.7}$$

We infer easily from (3.6) and (3.7) that the condition (ii) of Theorem 2.2 is fulfilled so that $(T_{\varphi_i}(t))_{t\in\Delta}$, $i = 1,2,\ldots,n$, are strongly continuous semigroups in $L^p_{\rho_1}(\Omega)$. (Suppose $1 < a < b < \infty$ and $K = [a,b]$. Clearly, there exists $t_0 \in (0,\infty)$ such that:

$$(t + x^{\alpha_n})^{\frac{1}{\alpha_n}} < (t + x^{\alpha_{n-1}})^{\frac{1}{\alpha_{n-1}}} < \cdots < (t + x^{\alpha_1})^{\frac{1}{\alpha_1}},\ t \geq t_0,\ x \in [a,b].$$

Let $L_k = K$, $k \in \mathbb{N}$ and let (t_k) be any increasing sequence of positive real numbers satisfying $\lim_{k\to\infty} t_k = \infty$ and $t_1 \geq \max(b^{\alpha_1},\ldots,b^{\alpha_n},t_0)$. Then (A1) holds and there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ and $i,j \in \{1,2,\ldots,n\}$ with $i < j$:

$$\varphi_i(t_k,\cdot)^{-1}(K) = \varphi_i(t_k,\cdot)^{-1}(\varphi_j(t_k,K)) = \emptyset. \tag{3.8}$$

Furthermore, one can simply verify that $\lim_{k\to\infty}\int_{\varphi_i(t_k,K)}\rho_1(x)\,\mathrm{d}x = 0$ for all $i = 1,2,\cdots,n$. Now one can employ (3.8) in order to conclude that (C1) holds and that (B1) holds with $i < j$. So, it is enough to prove the validity of (B1) with $i > j$; to this end, define $f : [a, b] \to \mathbb{R}$ by $f(x) := ((t_k + x^{\alpha_j})^{\frac{\alpha_i}{\alpha_j}} - t_k)^{\frac{1}{\alpha_i}},\ x \in [a, b]$. Then

$$\begin{aligned} f'(x) &= x^{\alpha_j - 1}((t_k + x^{\alpha_j})^{\frac{\alpha_i}{\alpha_j}} - t_k)^{\frac{1}{\alpha_i} - 1}(t_k + x^{\alpha_j})^{\frac{\alpha_i}{\alpha_j} - 1} \\ &\leq a^{\alpha_j - 1}(t_k + b^{\alpha_j})^{\frac{\alpha_i}{\alpha_j} - 1}(t_k + b^{\alpha_j})^{\frac{\alpha_i}{\alpha_j}\frac{1-\alpha_i}{\alpha_i}} \\ &= a^{\alpha_j - 1}(t_k + b^{\alpha_j})^{\frac{1-\alpha_j}{\alpha_j}}, \quad x \in [a, b], \end{aligned}$$

and the Lagrange mean value theorem implies that, for every $k \in \mathbb{N}$:

$$\frac{((t_k + b^{\alpha_j})^{\frac{\alpha_i}{\alpha_j}} - t_k)^{\frac{1}{\alpha_i}} - ((t_k + a^{\alpha_j})^{\frac{\alpha_i}{\alpha_j}} - t_k)^{\frac{1}{\alpha_i}}}{(b-a)a^{\alpha_j - 1}} \leq (t_k + b^{\alpha_j})^{\frac{1-\alpha_j}{\alpha_j}}.$$

In other words,

$$\text{meas}(\varphi_i(t_k, \cdot)^{-1}(\varphi_j(t_k, K))) \leq (b-a)a^{\alpha_j - 1}(t_k + b^{\alpha_j})^{\frac{1-\alpha_j}{\alpha_j}}. \tag{3.9}$$

The existence of an integer $k_{i,j} \in \mathbb{N}$ satisfying $\rho_1(x) \leq \mathrm{e}^{-t_k^{\alpha_1/\alpha_i}}$ for all $x \in \varphi_i(t_k, \cdot)^{-1}(\varphi_j(t_k, K))$ and $k \geq k_{i,j}$ is clear. Thereby, we have the following:

$$\int_{\varphi_i(t_k,\cdot)^{-1}(\varphi_j(t_k,K))} \rho_1(x)\,\mathrm{d}x \leq \text{meas}(\varphi_i(t_k, \cdot)^{-1}(\varphi_j(t_k, K)))\mathrm{e}^{-t_k^{\alpha_1/\alpha_i}}, \tag{3.10}$$

for $k \geq k_{i,j}$. Now (B1) follows from (3.9)–(3.10) and Theorem 3.1 implies that the semigroups $(T_{\varphi_i}(t))_{t\geq 0},\ i = 1, 2, \ldots, n$ are d-topologically transitive in $L^p_{\rho_1}(\Omega)$, as claimed.

(ii) Suppose $\alpha \in (0, \frac{\pi}{2}]$, $m \in \mathbb{N}$, $\Delta \in \{[0, \infty),\ \Delta(\alpha)\}$, $\Omega = (0, \infty)^m$, $\Theta = [1, \infty)^m$, $n \in \mathbb{N} \setminus \{1\}$, $[\alpha_{ij}]_{1\leq i\leq n,\ 1\leq j\leq m}$ is a matrix whose elements are positive real numbers and $c = \min_{1\leq i\leq n,\ 1\leq j\leq m} \alpha_{ij}$. Suppose, in addition, that for every $i,\ j \in \{1, 2, \ldots, n\}$ with $i \neq j$, there exists $l \in \{1, \ldots, m\}$ such that $\alpha_{il} \neq \alpha_{jl}$. Define $\varphi_i : \Delta \times \Omega \to \Omega$, $i = 1, 2, \ldots, n$, and $\rho : \Omega \to (0, \infty)$ by setting:

$$\tilde{\varphi}_i(t, x) := \Big((\text{Re}(t) + x_1^{\alpha_{i1}})^{1/\alpha_{i1}}, \ldots, (\text{Re}(t) + x_m^{\alpha_{im}})^{1/\alpha_{im}}\Big)$$

and

$$\tilde{\rho}(x) := \mathrm{e}^{-(x_1^c + \cdots + x_m^c)}, \quad t \in \Delta,\ x = (x_1, \ldots, x_m) \in \Omega.$$

Notice that, for every $i = 1, 2, \ldots, n$ and $t \in \Delta$, $T_{\tilde{\varphi}_i}(t) \notin L(C_{0,\tilde{\rho}}(\Omega))$ since the condition (ii) given in the formulation of Theorem 2.3 does not hold. Define $\rho : \Theta \to (0, \infty)$ and $\varphi_i : \Delta \times \Theta \to \Theta$ by $\rho(x) := \tilde{\rho}(x)$ and $\varphi_i(t, x) := \tilde{\varphi_i}(t, x),\ t \in \Delta,\ x \in \Theta$. Let us show that, for every fixed $i = 1, 2, \ldots, n$ and $t \in \Delta$, the mapping $T_{\varphi_i}(t) : C_{0,\rho}(\Theta) \to C_{0,\rho}(\Theta)$ is well defined and continuous. The simple calculation

$$s^c - (\mathrm{Re}(t) + s^{\alpha_{ij}})^{\frac{c}{\alpha_{ij}}} \geq s^c - (\mathrm{Re}(t)^{\frac{c}{\alpha_{ij}}} + s^{\alpha_{ij}\frac{c}{\alpha_{ij}}}) = -\mathrm{Re}(t)^{\frac{c}{\alpha_{ij}}} \geq -1 - \mathrm{Re}(t),$$

for $s \geq 1,\ 1 \leq j \leq m$ implies

$$-(x_1^c + \cdots + x_m^c) \leq -\left((\mathrm{Re}(t) + x_1^{\alpha_{i1}})^{c/\alpha_{i1}} + \cdots + (\mathrm{Re}(t) + x_m^{\alpha_{im}})^{c/\alpha_{im}}\right) + (m + \mathrm{Re}(t)), \quad x \in \Theta,$$

i.e.,

$$\rho(x) \leq \mathrm{e}^m \mathrm{e}^{\mathrm{Re}(t)} \rho(\varphi_i(t, x)), \quad x \in \Theta.$$

Thereby, the condition (ii) (a) quoted in the formulation of [16, Theorem 2.2, p. 1601] holds. On the other hand, it is checked at once that for every compact set K of Θ and for every $t \in \Delta$ and $\sigma > 0$, $\varphi(t, \cdot)^{-1}(K) \cap \{x \in \Theta : \rho(x) \geq \delta\}$ is a compact subset of Θ, and this implies that the condition (ii)(b) quoted in the formulation of [16, Theorem 2.2] also holds. By [16, Theorem 2.2], one gets that $T_{\varphi_i}(t) \in L(C_{0,\rho}(\Theta))$. Furthermore, the proof of [16, Theorem 2.2] (cf. also Lemma 2.1) implies that, for every $i = 1, 2, \ldots, n$, $(T_{\varphi_i}(t))_{t \in \Delta} \subset L(C_{0,\rho}(\Theta))$ is a strongly continuous semigroup in $C_{0,\rho}(\Theta)$ and the analysis given in (i) implies that the semigroups $(T_{\varphi_i}(t))_{t \in \Delta},\ i = 1, 2, \ldots, n$, are d-topologically transitive in $C_{0,\rho}(\Theta)$.

(iii) Suppose that every element of a real matrix $[a_{ij}]_{1 \leq i \leq n,\ 1 \leq j \leq m}$ is a positive real number and that for every $i,\ j \in \{1, 2, \ldots, n\}$, with $i \neq j$, there exists $l \in \{1, \ldots, m\}$ such that $a_{il} \neq a_{jl}$. Let $p \geq 1$, $q > m/2$, $\Delta = [0, \infty)$ and let Ω be as in (ii). Define semiflows $\varphi_i : \Delta \times \Omega \to \Omega$, $i = 1, 2, \ldots, n$ and $\rho_1 : \Omega \to (0, \infty)$ as follows:

$$\varphi_i(t, x_1, \ldots, x_m) := (\mathrm{e}^{a_{i1}t} x_1, \ldots, \mathrm{e}^{a_{im}t} x_m) \tag{3.11}$$

and

$$\rho_1(x_1, \ldots, x_m) := \frac{1}{(1 + |x|^2)^q}, \quad t \in \Delta,\ x = (x_1, \ldots, x_m) \in \Omega. \tag{3.12}$$

One can simply verify that $(T_{\varphi_i}(t))_{t \geq 0}$ is a strongly continuous semigroup in $L^p_{\rho_1}(\Omega),\ 1 \leq i \leq n$. Suppose $K = [a_1, b_1] \times \cdots \times [a_m, b_m]$ is a compact subset of Ω and set $L_k := K,\ k \in \mathbb{N}$. Let (t_k) be a sequence in Δ such that t_1 is sufficiently large and that $\lim_{k\to\infty} t_k = \infty$. It can be simply checked that (A1) and (C1) hold. To see that (B1) also holds, suppose $i,\ j \in \{1, 2, \ldots, n\}$, $i \neq j$, $a_{il} \neq a_{jl}$, $x = (x_1, \cdots, x_m) \in \varphi_i(t_k, \cdot)^{-1}(\varphi_j(t_k, K))$ and notice that:

$$\lim_{r\to\infty} \int_{|x|\geq r} \frac{\mathrm{d}x}{(1 + |x|^2)^q} = 0. \tag{3.13}$$

Obviously, $x_s \in [\mathrm{e}^{(a_{js}-a_{is})t_k} a_s, \mathrm{e}^{(a_{js}-a_{is})t_k} b_s],\ s = 1, \ldots, m$. In the case $a_{il} < a_{jl}$, (3.13) immediately leads us to the following:

$$\int_{\varphi_i(t_k,\cdot)^{-1}(\varphi_j(t_k,K))} \frac{\mathrm{d}x}{(1 + |x|^2)^q} \leq \int_{|x|\geq \mathrm{e}^{(a_{jl}-a_{il})t_k} a_l} \frac{\mathrm{d}x}{(1 + |x|^2)^q} \to 0,\ k \to \infty.$$

Suppose now $a_{il} > a_{jl}$. Then the inequality:

$$(1 + |x|^2)^q \geq (1 + x_1^2)^{q/m} \cdots (1 + x_m^2)^{q/m}$$

and (3.13) imply the existence of an appropriate positive real number C, depending only on K, p, m and $[a_{ij}]_{1\leq i\leq n,\ 1\leq j\leq m}$, so that:

$$\begin{aligned}
&\int_{\varphi_i(t_k,\cdot)^{-1}(\varphi_j(t_k,K))} \frac{\mathrm{d}x}{(1 + |x|^2)^q} \\
&\leq \int_{\mathrm{e}^{(a_{jl}-a_{il})t_k} a_l}^{\mathrm{e}^{(a_{jl}-a_{il})t_k} b_l} \frac{\mathrm{d}x}{(1 + x_l^2)^{q/m}} \prod_{\substack{1\leq s\leq m \\ s\neq l}} \int_{\mathrm{e}^{(a_{js}-a_{is})t_k} a_s}^{\mathrm{e}^{(a_{js}-a_{is})t_k} b_s} \frac{\mathrm{d}x}{(1 + x_s^2)^{q/m}} \\
&\leq \mathrm{e}^{(a_{jl}-a_{il})t_k}(b_l - a_l) \prod_{\substack{1\leq s\leq m \\ s\neq l}} \int_{\mathrm{e}^{(a_{js}-a_{is})t_k} a_s}^{\mathrm{e}^{(a_{js}-a_{is})t_k} b_s} \frac{\mathrm{d}x}{(1 + x_s^2)^{1/2}} \\
&= \mathrm{e}^{(a_{jl}-a_{il})t_k}(b_l - a_l) \prod_{\substack{1\leq s\leq m \\ s\neq l}} \log \frac{\mathrm{e}^{(a_{js}-a_{is})t_k} b_s + \sqrt{\mathrm{e}^{2(a_{js}-a_{is})t_k} b_s^2 + 1}}{\mathrm{e}^{(a_{js}-a_{is})t_k} a_s + \sqrt{\mathrm{e}^{2(a_{js}-a_{is})t_k} a_s^2 + 1}} \\
&\leq C\mathrm{e}^{(a_{jl}-a_{il})t_k}(b_l - a_l) \to 0, \quad k \to \infty.
\end{aligned} \tag{3.14}$$

Hence, Eq. (3.14) shows that (B1) holds and the semigroups $(T_{\varphi_i}(t))_{t\geq 0},\ i = 1, 2, \ldots, n$, are d-topologically transitive in $L^p_{\rho_1}(\Omega)$ (cf. also [15, Example 3.19] and [16, Theorem 6.22]). Define $\widetilde{\varphi}_i : \Delta \times \mathbb{R}^m \to \mathbb{R}^m,\ i = 1, 2, \ldots, n$

and $\tilde{\rho_1} : \mathbb{R}^m \to (0, \infty)$ through (3.11) and (3.12). In this case, the strongly continuous semigroups $(T_{\widetilde{\varphi_i}}(t))_{t\geq 0}$, $i = 1, 2, \ldots, n$, are d-topologically transitive in $L^p_{\tilde{\rho_1}}(\mathbb{R}^m)$. This fact follows from the previous computations and Theorem 3.1; notice only that we must use an appropriate sequence (L_k) of measurable subsets of K satisfying $0 \notin L_k^\circ,\ k \in \mathbb{N}$. Herein we point out that an employment of [15, Theorem 3.7] implies that, for every $i = 1, 2, \ldots, n$, $(T_{\varphi_i}(t))_{t\geq 0}$, resp. $(T_{\widetilde{\varphi_i}}(t))_{t\geq 0}$, is a non-hypercyclic strongly continuous semigroup in $C_{0,\rho_1}(\Omega)$, resp. $C_{0,\tilde{\rho_1}}(\mathbb{R}^m)$.

Example 3.2. Suppose $\Delta = [0, \infty)$, $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$, $|(x, y)| = \sqrt{x^2 + y^2}$, $(x, y) \in \mathbb{R}^2$, $n \in \mathbb{N} \setminus \{1\}$, $0 < p_1 < \cdots < p_n < \infty$, $q_i \in \mathbb{R}$, $1 \leq i \leq n$, K is a compact subset of Ω and, for $1 \leq i \leq n$:

$$\varphi_i(t, x, y) = \mathrm{e}^{p_i t}(x \cos q_i t - y \sin q_i t, x \sin q_i t + y \cos q_i t),\ t \geq 0,\ (x, y) \in \Omega.$$

Since $|\varphi_i(t, x, y)| = \mathrm{e}^{p_i t}|(x, y)|,\ t \geq 0,\ (x, y) \in \Omega,\ 1 \leq i \leq n$, one can simply check that, for every $i \in \{1, \ldots, n\}$, $\varphi_i : \Delta \times \Omega \to \Omega$ is a semiflow. Let (t_k) be a sequence in Δ such that $\lim_{k\to\infty} t_k = \infty$. Then for an arbitrary compact subset K' of Ω, it is straightforward to verify that (A2) and (B2) hold. According to Theorem 3.2, the strongly continuous semigroups $(T_{\varphi_i}(t))_{t\geq 0},\ i = 1, 2, \ldots, n$ are d-topologically transitive in $C(\Omega)$.

Finally, let us notice that Example 2.1 can be used for the construction of d-topologically transitive semigroups induced by non-differentiable semiflows.

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