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ALGEBRAIC DISTANCE BETWEEN SUBMODULES

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A b s t r a c t. We investigate the algebraic distance between closed and orthogonaly complemented submodules of a Hilbert C^*-module, which is defined as the norm of a difference of corresponding orthogonal projections. Some results are proved using 2×2 decompositions of adjointable operators.

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1. *Motivation*

Let X, Y be subspaces of a Banach space M. The distance between subspaces X and Y is introduced in the following way (see [1], [2], [6] and many other):

$$\theta(X,Y) = \sup\left\{ \sup_{x\in X, \|x\|=1} \mathrm{dist}(x,Y), \sup_{y\in Y, \|y\|=1} \mathrm{dist}(y,X) \right\}.$$

If M is Hilbert space, then

$$\theta(X,Y) = \|P_X - P_Y\|, \tag{1.1}$$

where P_X is the orthogonal projection on the closure of X. This topic is widely investigated, both in finite and infinite dimensional case. For example, it is used in describing convergence properties of closed operators, and continuity properties of generalized inverses of operators on Banach or Hilbert spaces.

The proof of the equality (1.1) depends on specific properties of Hilbert spaces.

In this paper we investigate the value $\|P_X - P_Y\|$ assuming that $\mathcal{M}$ is a Hilbert C^*-module, X, Y are closed submodules of $\mathcal{M}$ such that there exist orthogonal projections P_X and P_Y. Hilbert C^*-modules are Banach spaces which extend the notion of Hilbert spaces. Some important properties of Hilbert spaces are not valid for general Hilbert C^*-modules, such as the existence of the orthogonal complement of the closed submodule, and the standard form of the Pithagorean theorem. For this reason, the value $\|P_X - P_Y\|$ will be called the algebraic distance between closed and orghogonally complemeted submodules X and Y.

It seems that distance between submodules is not investigated. We prove some results in this setting, and thus extend some well-known results for Hilbert spaces.

2. *Hilbert C^*-modules*

The main references for Hilbert C^*-modules are [9] and [12]. We present results that we will use later.

Let $\mathcal{A}$ be a complex C^*-algebra and let $\mathcal{M}$ be a complex right $\mathcal{A}$- Hilbert C^*-module. The $\mathcal{A}$-vauled inner product in $\mathcal{M}$ is denoted by $\langle\cdot,\cdot\rangle$, and the norm satisfies $\|x\| = \|\langle x, x\rangle\|^{1/2}$ for every $x \in \mathcal{M}$. Then $(\mathcal{M}, \|\cdot\|)$ is a Banach space.

Let $\mathcal{N}$ also be a right $\mathcal{A}$- Hilbert C^*-module. The mapping $T : \mathcal{M} \to \mathcal{N}$ is an operator, provided that it is both linear and $\mathcal{A}$-linear. T is adjointable, if there exists an operator $T^* : \mathcal{N} \to \mathcal{M}$ such that for every $x \in \mathcal{M}$ and $y \in \mathcal{N}$ we have $\langle Tx, y\rangle = \langle x, T^*y\rangle$, and T^* is the unique adjoint of T. If T is adjointable, then both T and T^* are bounded. The set of all adjointable operators from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $\mathrm{Hom}^*_{\mathcal{A}}(\mathcal{M},\mathcal{N})$. Then $\mathrm{Hom}^*_{\mathcal{A}}(\mathcal{M}, N)$ is a Banach space with the usual operator norm. Particularly, $\mathcal{E}_{\mathcal{M}} = \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{M},\mathcal{M})$ is the set of all adjointable endomorphisms on $\mathcal{M}$, which is a C^*-algebra.

If $T \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{M}, N)$, then $\mathrm{Im}(T)$ and $\mathrm{Ker}(T)$ denote the image and the kernel of T.

In [12, Theorem 2.3.3] the following result is proved.

Lemma 2.1. *If $T \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{M}, N)$ is an operator such that $\mathrm{Im}(T)$ is closed, then $\mathrm{Im}(T^*)$ is also closed and we have the following orthogonal decompositions with respect to closed submodules:*

$$\mathcal{M} = \mathrm{Im}(T^*) \oplus \mathrm{Ker}(T), \qquad \mathcal{N} = \mathrm{Im}(T) \oplus \mathrm{Ker}(T^*).$$

In general, the question of the existence of topological complements and particularly orthogonal complements of closed submodules in a Hilbert C^*-module is not trivial.

An operator $P \in \mathcal{E}_{\mathcal{M}}$ is an orthogonal projection, if $P^2 = P$ and $\mathcal{M} = \mathrm{Im}(P) \oplus \mathrm{Ker}(P)$. It is easy to see that in this case P is selfadjoint.

Now, we need the result [12, Proposition 2.1.3].

Lemma 2.2. *If $T \in \mathcal{E}_{\mathcal{M}}^H$, then the following statements are equivalent:*

(1) $T \geq 0$ *in* $\mathcal{E}_{\mathcal{M}}$*;*

(2) *For every* $x \in \mathcal{M}$ *we have* $\langle Tx, x\rangle \geq 0$ *in* $\mathcal{A}$*.*

Lemma 2.3. *If $P \in \mathcal{E}_{\mathcal{M}}$ is a selfadjoint projection, then $P \in \mathcal{E}_{\mathcal{M}}^+$.*

PROOF. Take $x \in \mathcal{M}$ arbitrary. Then we have $\langle Px, x\rangle = \langle Px, Px\rangle \in \mathcal{A}^+$, implying that $P \in \mathcal{E}_{\mathcal{M}}^+$.

If P is the orthogonal projection in $\mathcal{E}_{\mathcal{M}}$ with $\mathrm{Im}(P) = X$, we write $P \equiv P_X$.

If $T \in \mathrm{Hom}_{\mathcal{A}}^*(\mathcal{M}, N)$, and $P_X \in \mathcal{E}_{\mathcal{M}}$ and $P_Y \in \mathcal{E}_{\mathcal{N}}$ are orthogonal projections, then

$$\begin{aligned} T &= P_Y T P_X + P_Y T(I - P_X) + (I - P_Y)TP_X + (I - P_Y)T(I - P_X) \\ &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} : \begin{bmatrix} X \\ X^\perp \end{bmatrix} \to \begin{bmatrix} Y \\ Y^\perp \end{bmatrix}, \end{aligned}$$

where T_{jk} are all adjointable on corresponding closed submodules. On the other hand, it is easy to see that if all T_{jk} are adjointable, then T is adjointable and

$$T^* = \begin{bmatrix} T_{11}^* & T_{21}^* \\ T_{12}^* & T_{22}^* \end{bmatrix} : \begin{bmatrix} Y \\ Y^\perp \end{bmatrix} \to \begin{bmatrix} X \\ X^\perp \end{bmatrix}.$$

3. *Generalized inverses*

Let $\mathcal{B}$ be a unital C^*-algebra. $\mathcal{B}^{qNil}$, $\mathcal{B}^+$, $\mathcal{B}^H$ and $\mathcal{B}^N$, respectively, denote the set of all quasinilpotent, positive, selfadjoint and normal elements in $\mathcal{B}$.

An element $a \in \mathcal{B}$ is generalized Drazin invertible, if and only if there exists (necessarily unique) $a^d \in \mathcal{B}$ satisfying

$$a^d a a^d = a^d,\ a a^d = a^d a,\ a(1 - a a^d) \in \mathcal{B}^{qNil},$$

and such a^d is the generalized Drazin inverse of a. The set of all generlized Drazin invertible elements in $\mathcal{B}$ is denoted by $\mathcal{B}^d$. It is well-known that $a \in \mathcal{B}^d$ if and only if $0 \notin \operatorname{acc}\sigma(a)$. Here $\operatorname{acc}\sigma(a)$ denotes the set of all accumulation points of the spectrum $\sigma(a)$.

An element $a \in \mathcal{B}$ is Moore-Penrose invertible, if there eixsts (necessarily unique) $a^\dagger \in \mathcal{B}$ satisfying

$$aa^\dagger a = a, \;\; a^\dagger aa^\dagger = a^\dagger, \;\; (aa^\dagger)^* = aa^\dagger, \;\; (a^\dagger a)^* = a^\dagger a,$$

and such $a^\dagger$ is the Moore-Penrose inverse of a. It is well-known that $a \in \mathcal{B}^\dagger$ if and only if $a \in a\mathcal{B}a$ [5]. If $\mathcal{B} = \mathcal{E}_\mathcal{M}$, then $T \in \mathcal{E}_\mathcal{M}^\dagger$ if and only if $\operatorname{Im}(T)$ is closed ([14]). The last statemenet follows also from the orthogonal decompositions of $\mathcal{M}$ and $\mathcal{N}$. If we assume that $\operatorname{Im}(T)$ is closed, then

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{Im}(T^*) \\ \operatorname{Ker}(T) \end{bmatrix} \to \begin{bmatrix} \operatorname{Im}(T) \\ \operatorname{Ker}(T^*) \end{bmatrix},$$

with T_1 invertible, and then

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{Im}(T) \\ \operatorname{Ker}(T^*) \end{bmatrix} \to \begin{bmatrix} \operatorname{Im}(T^*) \\ \operatorname{Ker}(T) \end{bmatrix},$$

as it is used in [4].

The following result is proved in [8, Proposition 2.11].

Lemma 3.1. *If $a \in \mathcal{B}^N$, then $a \in \mathcal{B}^\dagger$ if and only if $a \in \mathcal{B}^d$.*

Using properties of generalized inverses, we prove the following result.

Lemma 3.2. *Let $T \in \mathcal{E}_\mathcal{M}^H$. Then $\operatorname{Im}(T)$ is closed if and only if $0 \notin \operatorname{acc}\sigma(T)$.*

PROOF. If $T \in \mathcal{E}_\mathcal{M}^H$ and $\operatorname{Im}(T)$ is closed, then we have $\mathcal{H} = \operatorname{Im}(T) \oplus \operatorname{Ker}(T)$ and this decomposition completely reduces T. Thus,

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{Im}(T) \\ \operatorname{Ker}(T) \end{bmatrix} \to \begin{bmatrix} \operatorname{Im}(T) \\ \operatorname{Ker}(T) \end{bmatrix}$$

where T_1 is invertible. If $\operatorname{Ker}(T) = \{0\}$, then $T = T_1$ is invertible and $0 \notin \sigma(A)$. If $\operatorname{Ker}(T) \neq \{0\}$, then $\sigma(T) = \sigma(T_1) \cup \{0\}$. Since $0 \notin \sigma(T_1)$, we get $0 \notin \operatorname{acc}\sigma(T)$.

On the other hand, if $T \in \mathcal{E}_\mathcal{M}^H$ and $0 \notin \sigma(T)$, then $T \in \mathcal{E}_\mathcal{M}^d$ so $T \in \mathcal{E}_\mathcal{M}^\dagger$, implying that $\operatorname{Im}(T)$ is closed.

Corollary 3.1. *If $T \in \mathcal{E}_\mathcal{M}^H$ is singular with $\operatorname{Im}(T)$ closed, then 0 is a eigenvalue of T.*

The following result is proved in [8, Theorem 2.4.].

Lemma 3.3. *For $a \in \mathcal{B}$ the following hold:*

$$a \in \mathcal{B}^\dagger \iff a^* \in \mathcal{B}^\dagger \iff aa^* \in \mathcal{B}^\dagger \iff a^*a \in \mathcal{B}^\dagger.$$

Since $T \in \mathcal{E}_\mathcal{M}^\dagger$ if and only if $\mathrm{Im}(T)$ is closed, we have the next result.

Lemma 3.4. *For $T \in \mathcal{E}_\mathcal{M}$ the following hold:*

$$\mathrm{Im}(T) \textit{ is closed} \iff \mathrm{Im}(T^*) \textit{ is closed} \iff \mathrm{Im}(TT^*) \textit{ is closed} \iff \mathrm{Im}(T^*T) \textit{ is closed.}$$

4. *Algebraic distance between orthogonaly complemented submodules*

We continue with investigating orthogonal projections on Hilbert C^*-modules. If P_X, P_Y are ortogonal projections, then we take

$$S_{X,Y} = P_X P_Y (P_X P_Y)^* = P_X P_Y P_X.$$

If $S \in \mathcal{E}_\mathcal{M}^+$, then let

$$\mu(S) = \min \sigma(S).$$

Consider the following reductions of operators:

$$S^X = S_{X,Y} \restriction_X \colon X \to X \quad \text{and} \quad S^Y = S_{Y,X} \restriction_Y \colon Y \to Y.$$

Define

$$\Lambda(X,Y) = \min\{\mu(S^X), \mu(S^Y)\}.$$

First we prove the following lemma on decompositions of orthogonal projections. The proof is a matter of simple computation.

Lemma 4.1. *Let P_X, P_Y be orthogonal projections in $\mathcal{E}_\mathcal{M}$. Then the following hold:*

(1) $P_X = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X \\ X^\perp \end{bmatrix} \to \begin{bmatrix} X \\ X^\perp \end{bmatrix}$

and

$$P_Y = \begin{bmatrix} M_1 & M_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X \\ X^\perp \end{bmatrix} \to \begin{bmatrix} Y \\ Y^\perp \end{bmatrix},$$

where $M_1 \in \mathrm{Hom}^*_\mathcal{A}(X,Y)$ and $M_2 \in \mathrm{Hom}^*_\mathcal{A}(X^\perp, Y)$.

(2) $P_Y = P_Y^* = \begin{bmatrix} M_1^* & 0 \\ M_2^* & 0 \end{bmatrix} : \begin{bmatrix} Y \\ Y^\perp \end{bmatrix} \to \begin{bmatrix} X \\ X^\perp \end{bmatrix}$.

(3) $P_Y = P_Y^* = P_Y P_Y^* = \begin{bmatrix} M_1M_1^* + M_2M_2^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Y \\ Y^\perp \end{bmatrix} \to \begin{bmatrix} Y \\ Y^\perp \end{bmatrix}$,

with $M_1M_1^* + M_2M_2^* = I$.

(4) $P_Y = P_Y^* = P_Y^* P_Y = \begin{bmatrix} M_1^*M_1 & M_1^*M_2 \\ M_2^*M_1 & M_2^*M_2 \end{bmatrix} : \begin{bmatrix} X \\ X^\perp \end{bmatrix} \to \begin{bmatrix} X \\ X^\perp \end{bmatrix}$.

The following result can be found in [6, page 33], and it is also a consequence of a direct computation involving projections on a Banach space.

Lemma 4.2. *Let* P_X, P_Y *be projections in* $\mathcal{E}_\mathcal{M}$. *Take*

$$R = (P_X - P_Y)^2,$$

$$U' = P_Y P_X + (I - P_Y)(I - P_X),$$

$$V' = P_X P_Y + (I - P_X)(I - P_Y).$$

Then the following hold:

(1) $P_X R = R P_X, \; P_Y R = R P_Y$.

(2) $(P_X - P_Y)^2 + (I - P_X - P_Y)^2 = I$.

(3) $U'P_X = P_Y P_X = P_Y U', \; P_X V' = P_X P_Y = V' P_Y$.

Now, we prove the following result (see [6, page 56] or [3] for Hilbert space operators).

Theorem 4.1. *Let* P_X, P_Y *be orthogonal projections in* $\mathcal{E}_\mathcal{M}$. *Then:*

(1) $\|P_X - P_Y\| \leq 1$.

(2) *If* $\|P_X - P_Y\| < 1$, *then there exists a unitary* $U \in \mathcal{E}_\mathcal{M}$ *such that*

$$P_Y = U P_X U^*.$$

PROOF. (1) From Lemma 4.1 we have that $M_1M_1^* + M_2M_2^* = I$ and consequently $\max\{\|M_1M_1^*\|, \|M_2M_2^*\|\} \leq 1$. Now we get

$$\begin{aligned} 0 &\leq (P_X - P_Y)(P_X - P_Y)^* \\ &= P_X - P_X P_Y^* P_Y - P_Y^* P_Y P_X + P_Y^* P_Y \\ &= \begin{bmatrix} I - M_1^*M_1 & 0 \\ 0 & M_2^*M_2 \end{bmatrix} : \begin{bmatrix} X \\ X^\perp \end{bmatrix} \to \begin{bmatrix} X \\ X^\perp \end{bmatrix}. \end{aligned}$$

From Lemma 2.2 we obtain $0 \leq I - M_1^* M_1 \leq I$ and $\|I - M_1^* M_1\| \leq 1$. Also, $\|M_2^* M_2\| = \|M_2 M_2^*\| \leq 1$. Since

$$\sigma\Big((P_X - P_Y)(P_X - P_Y)^*\Big) = \sigma(I - M_1^* M_1) \cup \sigma(M_2^* M_2),$$

we get that

$$\|P_X - P_Y\|^2 = \max\{\|I - M_1^* M_1\|, \|M_2^* M_2\|\} \leq 1.$$

(2) Take $R = (P_X - P_Y)^2$ (as in Lemma 4.2). We have

$$I - R = \begin{bmatrix} M_1^* M_1 & 0 \\ 0 & I - M_2^* M_2 \end{bmatrix} : \begin{bmatrix} X \\ X^\perp \end{bmatrix} \to \begin{bmatrix} X \\ X^\perp \end{bmatrix}.$$

Since $\|R\| < 1$ we know that $I - R$ is invertible.

Now, we have a short proof. If $U = U'(I - R)^{-1/2}$ then $P_Y = U P_X U^*$, as it is proved in [6].

However, we can finish the proof independently from the results in [6]. We have

$$U = \begin{bmatrix} (M_1^* M_1)^{1/2} & -M_1^* M_2 (I - M_2^* M_2)^{-1/2} \\ M_2^* M_1 (M_1^* M_1)^{-1/2} & (I - M_2^* M_2)^{-1/2} \end{bmatrix}.$$

Notice that the obtained result from (1) implies $\|I - M_1^* M_1\| < 1$, so we get that $1 - \mu(M_1^* M_1) < 1$, and $M_1^* M_1$ is invertible. Since

$$\|I - M_1 M_1^*\| = \|M_2 M_2^*\| = \|M_2^* M_2\| < 1,$$

we have that $M_1 M_1^*$ is invertible. Thus, M_1 is invertible and it is now trivial to see that $P_Y = U P_X U^*$ holds.

Last lines of the previous proof imply the following result, knowing that $M_1 \in \mathrm{Hom}_{\mathcal{A}}^*(X, Y)$.

Corollary 4.1. *If P_X and P_Y are orthogonal projections in $\mathcal{E}_{\mathcal{M}}$ with*

$$\|P_X - P_Y\| < 1,$$

then there exists an isomorphism from X onto Y.

The previous result is proved in [6, pages 199–200] in the setting of Banach spaces provided that at least one of X, Y is a finite dimensional subspace. In the case of closed subspaces of a Hilbert space, the results is proved in [1, page 70].

Theorem 4.2. *Let P_X, P_Y be orthogonal projections in $\mathcal{E}_{\mathcal{M}}$. Then*

$$\|P_X - P_Y\|^2 = 1 - \Lambda(X,Y).$$

PROOF. We use decompositions in Lemma 4.1 and Theorem 4.1. We consider several cases.

Case 1. Let $X^\perp = \{0\}$ and $Y \neq \{0\}$ (or the opposite way). We have

$$P_X = I, \quad P_Y = \begin{bmatrix} M_1 \\ 0 \end{bmatrix} : X \to \begin{bmatrix} Y \\ Y^\perp \end{bmatrix}, \quad S_{X,Y} = M_1^* M_1 : X \to X,$$

$$S_{Y,X} = P_Y P_X P_Y^* = \begin{bmatrix} M_1 M_1^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Y \\ Y^\perp \end{bmatrix} \to \begin{bmatrix} Y \\ Y^\perp \end{bmatrix},$$

$$S^X = M_1^* M_1, \quad S^Y = I.$$

Since $S_{X,Y} = P_X P_Y P_X$ and $Y^\perp \neq \{0\}$, we conclude that $S_{X,Y}$ is singular, i.e. $M_1^* M_1$ is singular. Thus,

$$\Lambda(X,Y) = \Lambda(\mathcal{M}, Y) = \min\{\mu(M_1^* M_1), \mu(I)\} = 0.$$

On the other hand,

$$\|P_X - P_Y\| = \|I - P_Y\| = 1.$$

Case 2. If $X = Y = \mathcal{M}$, then the equality trivially holds.

Case 3. Let $X^\perp \neq \{0\} \neq Y^\perp$. We have

$$S_{X,Y} = P_X P_Y P_X = P_X P_Y^* P_Y P_X = \begin{bmatrix} M_1^* M_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X \\ X^\perp \end{bmatrix} \to \begin{bmatrix} X \\ X^\perp \end{bmatrix}$$

and

$$S_{Y,X} = P_Y P_X P_Y = P_Y P_X P_Y^* = \begin{bmatrix} M_1 M_1^* & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Y \\ Y^\perp \end{bmatrix} \to \begin{bmatrix} Y \\ Y^\perp \end{bmatrix}.$$

Since X, Y are non-trivial, we have that

$$\sigma(S_{X,Y}) = \sigma(M_1^* M_1) \cup \{0\} = \sigma(M_1 M_1^*) \cup \{0\} = \sigma(S_{Y,X}).$$

Using previous equality of spectrums, Lemma 3.2 and Lemma 3.4, we have the following:

$$\begin{aligned}&\mathrm{Im}(S_{X,Y}) \text{ is closed } \iff 0 \notin \operatorname{acc}\sigma(S_{X,Y}) \iff 0 \notin \operatorname{acc}\sigma(M_1^*M_1)\\ &\iff \mathrm{Im}(M_1^*M_1) \text{ is closed } \iff \mathrm{Im}(M_1) \text{ is closed } \iff \mathrm{Im}(M_1M_1^*) \text{ is closed}\\ &\iff 0 \notin \operatorname{acc}\sigma(M_1M_1^*) \iff 0 \notin \operatorname{acc}\sigma(S_{Y,X}) \iff \mathrm{Im}(S_{Y,X}) \text{ is closed.}\end{aligned}$$

Notice that

$$S^X = M_1^*M_1 : X \to X, \quad S^Y = M_1M_1^* : Y \to Y.$$

We consider several subcases.

Subcase 3.1. Assume that $\mathrm{Im}(S_{X,Y})$ is not closed. Then $\mathrm{Im}(M_1^*M_1)$ is not closed, so $\mathrm{Im}(M_1)$ is not closed. It follows that M_1 is not left invertible nor right inveritble, implying that both $M_1^*M_1$ and $M_1M_1^*$ are singular with non-closed ranges. Thus, $0 \in \operatorname{acc}\sigma(M_1^*M_1) = \operatorname{acc}\sigma(M_1M_1^*)$ and $\Lambda(X,Y) = 0$. On the other hand, we have

$$\begin{aligned}\|P_X - P_Y\| &= \max\{\|I - M_1^*M_1\|, \|M_2^*M_2\|\}\\ &= \max\{1 - \mu(M_1^*M_1), \|M_2^*M_2\|\}\\ &= 1 = 1 - \Lambda(X,Y).\end{aligned}$$

Subcase 3.2. Suppose that $\mathrm{Im}(S_{X,Y})$ is closed. Again we have several subcases.

Subcase 3.2.1. Assume that $S^X = M_1^*M_1$ and $S^Y = M_1M_1^*$ are both invertible. We get $0 \notin \sigma(M_1^*M_1) = \sigma(M_1M_1^*)$ and

$$\Lambda(X,Y) = \mu(M_1^*M_1) = \mu(M_1M_1^*) > 0.$$

Knowing that $M_1M_1^* + M_2M_2^* = I$, we calculate as follows:

$$\begin{aligned}\|P_X - P_Y\| &= \max\{\|I - M_1^*M_1\|, \|M_2^*M_2\|\}\\ &= \max\{1 - \mu(M_1^*M_1), \|M_2^*M_2\|\}\\ &= \max\{1 - \mu(M_1M_1^*), \|M_2^*M_2\|\}\\ &= \max\{\|I - M_1M_1^*\|, \|M_2^*M_2\|\}\\ &= \|M_2M_2^*\|\\ &= \|I - M_1M_1^*\|\\ &= 1 - \mu(M_1M_1^*)\\ &= 1 - \mu(M_1^*M_1)\\ &= 1 - \Lambda(X,Y).\end{aligned}$$

Subcase 3.2.2. Assume that $S^X = M_1^*M_1$ is singular. Then $\Lambda(X,Y) = 0$, $0 \in \sigma(M_1^*M_1)$, and we calculate as follows:

$$\begin{aligned}\|P_X - P_Y\| &= \max\{\|I - M_1^*M_1\|, \|M_2^*M_2\|\} \\ &= \max\{1 - \mu(M_1^*M_1), \|M_2^*M_2\|\} \\ &= 1 = 1 - \Lambda(X,Y).\end{aligned}$$

Subcase 3.2.3. If we assume that S^Y is singular, then we simply change places of X and Y in *Subcase* 3.2.2, and obtain the requested result.

Previous result is an extension of [13, Lemma 2.2], where a finite dimensional case is considered. If X, Y are subspaces of $\mathbb{C}^n$ with the same dimension, then our definition of $\Lambda(X,Y)$ is equal to the smallest non-zero eigenvalue of $S_{X,Y}$ and this is exactly $\cos^2 \varphi_{\max}(X,Y)$, where $\varphi_{\max}$ is the maximum canonical angle between X and Y. This means that $\|P_X - P_Y\| = \sin \varphi_{\max}$.

We prove the following result (see [1, pages 70-71] for Hilbert spaces).

Theorem 4.3. *Let P_X, P_Y be orthogonal projections in $\mathcal{E}_{\mathcal{M}}$. Then*

$$\begin{aligned}\|P_X - P_Y\| &= \max\left\{\|(I - P_X)P_Y\|, \|(I - P_X)P_Y\|\right\} \\ &= \max\left\{\|(I - P_X)\restriction_Y\|, \|(I - P_Y)\restriction_X\|\right\}.\end{aligned}$$

PROOF. Notice that in Theorem 4.1 we proved that

$$\|P_X - P_Y\|^2 = \max\{\|I - M_1^*M_1\|, \|M_2^*M_2\|\}.$$

Also

$$I - P_Y = I - P_Y^*P_Y = \begin{bmatrix} I - M_1^*M_1 & -M_1^*M_2 \\ -M_2^*M_1 & I - M_2^*M_2 \end{bmatrix} : \begin{bmatrix} X \\ X^\perp \end{bmatrix} \to \begin{bmatrix} X \\ X^\perp \end{bmatrix},$$

and consequently

$$\begin{bmatrix} I - M_1^*M_1 & 0 \\ 0 & 0 \end{bmatrix} = P_X(I - P_Y)P_X.$$

Since all operators are positive, we have

$$\|I - M_1^*M_1\| = \|P_X(I - P_Y)P_X\| = \|(I - P_Y)P_X\|^2.$$

In the same way we have

$$(I - P_X)P_Y(I - P_X) = (I - P_X)P_Y^*P_Y(I - P_X) = \begin{bmatrix} 0 & 0 \\ 0 & M_2^*M_2 \end{bmatrix},$$

and consequently

$$\|M_2^*M_2\| = \|(I - P_X)P_Y(I - P_X)\| = \|(I - P_X)P_Y\|^2.$$

To prove the second part of this theorem, we take short notations:

$$T = (I - P_Y) \restriction_X = \begin{bmatrix} I - M_1^*M_1 \\ -M_2^*M_1 \end{bmatrix}, \qquad T^* = \begin{bmatrix} I - M_1^*M_1 & -M_1^*M_2 \end{bmatrix},$$

$$S = (I - P_Y)P_X = \begin{bmatrix} I - M_1^*M_1 & 0 \\ -M_2^*M_1 & 0 \end{bmatrix}, \quad S^* = \begin{bmatrix} I - M_1^*M_1 & -M_1^*M_2 \\ 0 & 0 \end{bmatrix},$$

and obtain

$$T^*T = \begin{bmatrix} (I - M_1^*M_1)^2 + M_1^*M_2M_2^*M_1 \end{bmatrix}$$

and

$$S^*S = \begin{bmatrix} (I - M_1^*M_1)^2 + M_1^*M_2M_2^*M_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Using the connection between the norm and the spectrum of a positive operator, we obtain

$$\|T\|^2 = \|T^*T\| = \|S^*S\| = \|S\|^2.$$

If we assume that $\mathcal{M}$ is a Hilbert space and $x \in X$, $\|x\| = 1$, then $\|(I - P_Y)x\| = \mathrm{dist}(x, Y)$. Then

$$\|(I - P_Y) \restriction_X \| = \sup_{x \in X, \|x\|=1} \mathrm{dist}(x, Y)$$

and

$$\|(I - P_X) \restriction_Y \| = \sup_{y \in Y, \|y\|=1} \mathrm{dist}(y, X).$$

Thus, Theorem 4.3 implies the natural definition of the distance between subspaces in a Banach space (as it is concluded in [1]). However, the standard form of the Pithagorean theorem does not hold in a general Hilbert C^*-module (see [11, Example 2.7]).

Thus, if X, Y are closed and orthogonaly complemented submodules of $\mathcal{M}$, then it is natural to say that $\|P_X - P_Y\|$ is the algebraic distance between X and Y.

5. *Adjointable oblique projections*

In this section we consider oblique projections which have their adjoints. If X and Y are closed submodules of $\mathcal{M}$ such that $X \dot{+} Y = \mathcal{M}$ (the sum is topological and not necessarily orthogonal), then $P_{X,Y}$ denotes the projection from $\mathcal{M}$ onto X parallel to Y. Notice that $P_{X,Y}$ is bounded by the closed graph theorem, but it is not necessary adjointable [12, page 22].

If X, Y are closed submodules of $\mathcal{M}$ such that $\mathcal{M} = X \oplus Y$ and $P_{X,Y}$ is adjointable, then let

$$T_{X,Y} = P_{X,Y} P_{X,Y}^*.$$

We prove the following result, which is an extension of some results from [10] and [13, Lemma 2.1]. We use $\operatorname{card}(Z)$ to denote the cardinality of the set Z.

Theorem 5.1. *Let X, Y be non-trivial closed submodules $\mathcal{M}$, such that $\mathcal{M} = X \dot{+} Y$ and $P_{X,Y}, P_{Y,X}$ are adjointable operators. Then P_X, P_Y exist and the following holds:*

(1) $1 \notin \sigma(S_{Y,X})$*;*

(2) *The mapping $\lambda \mapsto (1-\lambda)^{-1} = \nu$ is a bijection from $\sigma(S_{Y,X}) \setminus \{0\}$ onto $\sigma(T_{Y,X}) \setminus \{0\}$.*

(3) *If λ and ν are the same as in the part* (2)*, then*

$$\operatorname{card}\Big([\lambda, \|S_{Y,X}\|] \cap \sigma(S_{Y,X})\Big) = \operatorname{card}\Big([\nu, \|T_{Y,X}\|] \cap \sigma(T_{Y,X})\Big).$$

(4) $(1 - \|P_X P_Y\|^2)\|P_{Y,X}\|^2 = 1$.

PROOF. (1) We recall notations and results form Lemma 4.1 and Theorem 4.1. Since $P_{X,Y}$ is adjointable and has a closed range, we have the orthogonal decomposition of closed submodules

$$\mathcal{M} = \operatorname{Im}(P_{X,Y}^*) \oplus \operatorname{Ker}(P_{X,Y}) = \operatorname{Im}(P_{X,Y}^*) \oplus Y.$$

Thus, Y has the orthogonal complement, so $\operatorname{Im}(P_{X,Y}^*) = Y^\perp$ and P_Y exists. From the same reason we have $\operatorname{Im}(P_{Y,X}^*) = X^\perp$ and P_X exists.

Notice that

$$S_{Y,X} = \begin{bmatrix} M_1 M_1^* & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Y \\ Y^\perp \end{bmatrix} \to \begin{bmatrix} Y \\ Y^\perp \end{bmatrix}.$$

Since $\operatorname{Im}(P_{Y,X}) = Y$ and $\operatorname{Ker}(P_{Y,X}) = X$, we conclude that

$$P_{Y,X} = \begin{bmatrix} 0 & P_1 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X \\ X^\perp \end{bmatrix} \to \begin{bmatrix} Y \\ Y^\perp \end{bmatrix},$$

where $P_1 \in \mathrm{Hom}^*_{\mathcal{A}}(X^\perp, Y)$ is invertible. From $P_{Y,X}P_Y = P_Y$ we have

$$P_{Y,X}P_Y^* = \begin{bmatrix} P_1M_2^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Y \\ Y^\perp \end{bmatrix} \to \begin{bmatrix} Y \\ Y^\perp \end{bmatrix}.$$

We conclude that M_2 is invertible and $P_1 = (M_2^*)^{-1}$. Hence

$$P_{Y,X} = \begin{bmatrix} 0 & (M_2^*)^{-1} \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X \\ X^\perp \end{bmatrix} \to \begin{bmatrix} Y \\ Y^\perp \end{bmatrix}, P_{Y,X}^* = \begin{bmatrix} 0 & 0 \\ M_2^{-1} & 0 \end{bmatrix} : \begin{bmatrix} Y \\ Y^\perp \end{bmatrix} \to \begin{bmatrix} X \\ X^\perp \end{bmatrix}.$$

We obtain the following

$$T_{Y,X} = \begin{bmatrix} (M_2M_2^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Y \\ Y^\perp \end{bmatrix} \to \begin{bmatrix} Y \\ Y^\perp \end{bmatrix}.$$

Since X, Y are non-trivial, we have $\sigma(S_{Y,X}) = \sigma(M_1M_1^*) \cup \{0\}$ and

$$\sigma(T_{Y,X}) = \sigma\big((M_2M_2^*)^{-1}\big) \cup \{0\}.$$

Using $M_1M_1^* + M_2M_2^* = I$ and invertibility of $M_2M_2^*$, we obtain

$$\|S_{Y,X}\| = \|M_1M_1^*\| = \|I - M_2M_2^*\| = 1 - \mu(M_2M_2^*) < 1.$$

Thus, $1 \notin \sigma(S)$.

(2) Let $\lambda \in \sigma(S_{Y,X}) \setminus \{0\}$ be arbitrary. Then $\lambda \in \sigma(M_1M_1^*)$. Since

$$M_1M_1^* + M_2M_2^* = I,$$

there exists the unique $\xi \in \sigma(M_2M_2^*)$ such that $\lambda = 1 - \xi$. Obviously, there exists the unique $\nu \in \sigma(M_2M_2^*)^{-1}$ such that $\nu^{-1} = \xi$. Hence, $(1-\lambda)\nu = 1$.

(3) Using the spectral mapping theorem for bijective functions $x \mapsto x^{-1}$ $(x > 0)$, $x \mapsto 1 - x$ $(x \in \mathbb{R})$ and bijective correspondence between $\lambda, \xi, \nu > 0$ established

above, we compute as follows:

$$\begin{aligned}
&\operatorname{card}\Big([\nu, \|T_{Y,X}\|] \cap \sigma(T_{Y,X})\Big)\\
&\quad= \operatorname{card}\Big(\big[\nu, \|(M_2M_2^*)^{-1}\|\big] \cap \sigma\big((M_2M_2^*)^{-1}\big)\Big)\\
&\quad= \operatorname{card}\Big(\big[\xi^{-1}, \|(M_2M_2)^{-1}\|\big] \cap \sigma\big((M_2M_2^*)^{-1}\big)\Big)\\
&\quad= \operatorname{card}\Big(\big[\mu(M_2M_2^*), \xi\big] \cap \sigma(M_2M_2^*)\Big)\\
&\quad= \operatorname{card}\Big(\big[\mu(I-M_1M_1^*), 1-\lambda\big] \cap \sigma(I-M_1M_1^*)\Big)\\
&\quad= \operatorname{card}\Big(\big[1-\|M_1M_1^*\|, 1-\lambda\big] \cap \sigma(I-M_1M_1^*)\Big)\\
&\quad= \operatorname{card}\Big(\big[\lambda, \|M_1M_1^*\|\big] \cap \sigma(M_1M_1^*)\Big)\\
&\quad= \operatorname{card}\Big(\big[\lambda, \|S_{Y,X}\|\big] \cap \sigma(S_{Y,X})\Big).
\end{aligned}$$

(4) Take $\lambda = \|S_{Y,X}\| = \|P_XP_Y\|^2$ and corresponding

$$\nu = \|T_{Y,X}\| = \|P_{Y,X}\|^2.$$

Thus, Theorem 5.1 is proved.

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