Bulletin T. CLII de l'Académie serbe des sciences et des arts − 2019
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 44

ASYMPTOTICALLY ALMOST PERIODIC AND ASYMPTOTICALLY ALMOST AUTOMORPHIC SOLUTIONS OF ABSTRACT DEGENERATE MULTI-TERM FRACTIONAL DIFFERENTIAL INCLUSIONS WITH RIEMANN-LIOUVILLE DERIVATIVES

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(Presented at the 9th Meeting, held on December 26, 2019)

A b s t r a c t. In this paper, we investigate the functional analytical approach for seeking of solutions to the following abstract multi-term fractional differential inclusion:

$$\mathcal{B}D_t^{\alpha_n}u(t) + \sum_{j=1}^{n-1} \mathcal{A}_j D_t^{\alpha_j}u(t) \in \mathcal{A}D_t^{\alpha}u(t) + f(t), \quad t \in (0,\tau), \qquad (*)$$

where $n \in \mathbb{N}\setminus\{1\}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}_j$ are multivalued linear operators on a complex Banach space X $(1 \leq j \leq n-1)$, $0 \leq \alpha_1 < \cdots < \alpha_n$, $0 \leq \alpha < \alpha_n$, $0 < \tau \leq \infty$, $f(t)$ is an X-valued function, and D_t^{α} denotes the Riemann-Liouville fractional derivative of order α (see Ph.D. Thesis by E. Bazhlekova, Eindhoven University of Technology, 2001). We introduce and analyze several different types of solutions and degenerate k-regularized (C_1, C_2)-existence and uniqueness (propagation) families for $(*)$. Asymptotically almost periodic and asymptotically almost automorphic solutions of $(*)$ are sought in the case that $\mathcal{B} = I$ (the identity operator on X), $A_j \in L(X)$ for $1 \leq j \leq n-1$ and $\mathcal{A}$ is a convenable chosen translation of a C-almost sectorial multivalued linear operator.

AMS Mathematics Subject Classification (2010): 44A35, 42A75, 47D06, 34G25, 35R11.

Key Words: abstract multi-term fractional differential inclusions, Riemann-Liouville fractional derivatives, asymptotical almost periodicity, asymptotical almost automorphy, abstract Volterra integro-differential inclusions.

1. *Introduction and preliminaries*

The class of scalar-valued almost periodic functions was introduced by H. Bohr [5] (1924–1926), while the class of scalar-valued almost automorphic functions was introduced by S. Bochner [4] (1962). The study od almost periodic and almost automorphic type solutions of abstract Volterra integro-differential equations is a very active and popular field of functional analysis nowdays. For example, in [17, Section 11.4], J. Prüss has analyzed the almost periodic solutions, Stepanov almost periodic solutions and asymptotically almost periodic solutions of the following abstract nondegenerate Cauchy problem

$$u'(t) = \int_0^\infty A_0(s)u'(t-s)\,\mathrm{d}s + \int_0^\infty \mathrm{d}A_1(s)u(t-s) + f(t), \quad t \in \mathbb{R},$$

where $A_0 \in L^1([0,\infty) : L(Y,X))$, $A_1 \in BV([0,\infty) : L(Y,X))$, X and Y are Banach spaces such that Y is densely and continuously embedded into X; in [6, Chapter 10], T. Diagana has analyzed the weighted asymptotic behaviour of solutions to the abstract nonautonomous third-order differential equation

$$u''' + B(t)u' + A(t)u = h(t,u), \quad t \in \mathbb{R},$$

while, in [1], S. Abbas, V. Kavitha and R. Murugesu have examined Stepanov-like (weighted) pseudo almost automorphic solutions to the following fractional order abstract integro-differential equation:

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t), Ku(t)), \quad t \in \mathbb{R},$$

where

$$Ku(t) = \int_{-\infty}^t k(t-s)h(s,u(s))\,\mathrm{d}s, \quad t \in \mathbb{R},$$

$1 < \alpha < 2$, A is a sectorial operator with domain and range in X, of negative sectorial type $\omega < 0$, the function $k(t)$ is exponentially decaying, the functions $f : \mathbb{R} \times X \times X \to X$ and $h : \mathbb{R} \times X \to X$ are Stepanov-like weighted pseudo almost automorphic in time for each fixed elements of $X \times X$ and X, respectively, satisfying some extra conditions. For more details on the subject, we refer the reader to the monographs [6] by T. Diagana, [9] by G. M. N'Guérékata, [14] by the author, and to a great number of other scientific monographs and research papers cited therein.

The main aim of this paper is to continue our previous joint research study with Prof. G. M. N'Guérékata concerning asymptotically almost periodic and asymptotically almost automorphic solutions of abstract degenerate multi-term fractional differential inclusions with Caputo derivatives [9]. We also provide slight generalizations of notions and results from our previous research paper [15], considering mutivalued linear operators approach here.

We use the standard notation throughout the paper. By $(X, \|\cdot\|)$ and $L(X)$ we denote a non-trivial complex Banach space and the space of all continuous linear mappings from X into X, respectively. Given $s \in \mathbb{R}$ in advance, set

$$\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}.$$

The principal branch is always used to take the powers. Set $\mathbb{N}_l := \{1, \dots, l\}$, $\mathbb{N}_l^0 := \{0, 1, \dots, l\}$, $0^\zeta := 0$, $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ ($\zeta > 0$, $t > 0$) and $g_0(t) :=$ the Dirac δ-distribution. By $\chi_S(\cdot)$ we denote the characteristic function of set S. If $\delta \in (0, \pi]$, then we define $\Sigma_\delta := \{\lambda \in \mathbb{C} : \lambda \neq 0, \ |\arg \lambda| < \delta\}$. Let $0 < \tau \leq \infty$, and let $I = (0, \tau)$. Then the Sobolev space $W^{m,1}(I : X)$ is defined in the usual way (see e.g. [3, p. 7]). We sometimes employ the condition

(P1): $h(t) : [0, \infty) \to X$ is Laplace transformable, i.e., $h \in L^1_{\text{loc}}([0, \infty) : X)$ and there exists $\beta \in \mathbb{R}$ such that

$$\tilde{h}(\lambda) := \mathcal{L}(h)(\lambda) := \lim_{b \to \infty} \int_0^b \mathrm{e}^{-\lambda t} h(t) \, \mathrm{d}t := \int_0^\infty \mathrm{e}^{-\lambda t} h(t) \, \mathrm{d}t$$

exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \beta$. Put abs$(h)$:=inf$\{\operatorname{Re} \lambda : \tilde{h}(\lambda)$ exists$\}$, and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

We refer the reader to [15] for the notion of condition (P1)-$L(X)$. Fairly complete information concerning vector-valued Laplace transform can be obtained by consulting the references [2] and [12]–[13]. Let $0 < \tau \leq \infty$ and $\mathcal{F} : [0, \tau) \to P(X)$. A single-valued function $f : [0, \tau) \to X$ is called a section of $\mathcal{F}$ if and only if $f(t) \in \mathcal{F}(t)$ for all $t \in [0, \tau)$; a continuous section of $\mathcal{F}$ is any section of $\mathcal{F}$ that is continuous on $[0, \tau)$. A multivalued map (multimap) $\mathcal{A} : X \to P(X)$ is said to be a multivalued linear operator (MLO) if and only if the following two conditions hold:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x + y)$, $x, \ y \in D(\mathcal{A})$ and $\lambda \mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

It is well known that, for every $x, \ y \in D(\mathcal{A})$ and $\lambda, \ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda \mathcal{A}x + \eta \mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. It is said that an MLO $\mathcal{A} : X \to P(X)$ is closed if and only if for any two sequences (x_n) in $D(\mathcal{A})$ and (y_n) in X such that $y_n \in \mathcal{A}x_n$ for all $n \in \mathbb{N}$ we have that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

Let Ω denote a locally compact and separable metric space and let μ denote a locally finite Borel measure defined on Ω. We need the following important lemma from [13].

Lemma 1.1. *Suppose that* $\mathcal{A} : X \to P(X)$ *is a closed* MLO. *Let* $f : \Omega \to X$ *and* $g : \Omega \to X$ *be* μ*-integrable, and let* $g(x) \in \mathcal{A}f(x)$*,* $x \in \Omega$*. Then* $\int_\Omega f\,\mathrm{d}\mu \in D(\mathcal{A})$ *and* $\int_\Omega g\,\mathrm{d}\mu \in \mathcal{A}\int_\Omega f\,\mathrm{d}\mu$*.*

Assume now that $\mathcal{A}$ is an MLO in X, $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which $R(C) \subseteq R(\lambda - \mathcal{A})$ and $(\lambda - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on X. The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is said to be the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$).

Fractional calculus and fractional differential equations are rapidly growing fields of research of many mathematicians due to their invaluable applications in engineering, physics, chemistry, biology and other sciences (for more details about fractional calculus and fractional differential equations, the reader may consult [3], [7], [11]–[13], [17], [18], and references cited therein; for abstract degenerate differential equations, one may refer e.g. to [8], [13]–[14], [16], [19] and references cited therein).

In this paper, we use the Riemann-Liouville fractional derivatives. Let $\alpha > 0$ and $m = \lceil \alpha \rceil$, and let $I = (0, \tau)$, where $\tau \in (0, \infty]$. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined for any function $f \in L^1(I : X)$, by

$$J_t^\alpha f(t) := \big(g_\alpha * f\big)(t), \quad t > 0.$$

The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined for any function $f \in L^1(I : X)$ satisfying $g_{m-\alpha} * f \in W^{m,1}(I : X)$, by

$$D_t^\alpha f(t) := \frac{\mathrm{d}^m}{\mathrm{d}t^m}\big(g_{m-\alpha} * f\big)(t) = D_t^m J_t^{m-\alpha} f(t), \quad t > 0.$$

By [3, Theorem 1.5], for every $f \in L^1(I : X)$ with $g_{m-\alpha} * f \in W^{m,1}(I : X)$, we have:

$$J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0) g_{\alpha+k+1-m}(t), \quad t > 0. \tag{1.1}$$

2. *Concepts of solutions and degenerate* k*-regularized* (C_1, C_2)*-existence and uniqueness propagation families for an abstract multi-term fractional differential inclusion*

Now, we investigate the functional analytical approach for seeking of solutions to the following abstract multi-term fractional differential inclusion:

$$\mathcal{B}D_t^{\alpha_n} u(t) + \sum_{j=1}^{n-1} \mathcal{A}_j D_t^{\alpha_j} u(t) \in \mathcal{A}D_t^{\alpha} u(t) + f(t), \quad t \in (0, \tau), \tag{2.1}$$

where $n \in \mathbb{N} \setminus \{1\}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}_j$ are multivalued linear operators on a complex Banach space X $(1 \leq j \leq n-1)$, $0 \leq \alpha_1 < \cdots < \alpha_n$, $0 \leq \alpha < \alpha_n$, $0 < \tau \leq \infty$, $f(t)$ is an X-valued function, and D_t^{α} denotes the Riemann-Liouville fractional derivative of order α (see [3]). Set $\alpha_0 := \alpha$, $m := \lceil \alpha \rceil$, $\mathcal{A}_0 := -\mathcal{A}$, $\mathcal{A}_n := \mathcal{B}$ and $m_i := \lceil \alpha_i \rceil$ for $0 \leq i \leq n$. The notion of a strong solution of (2.1) is introduced as follows:

Definition 2.1. Let $f \in L^1((0,\tau) : X)$. A strong solution of (2.1) is any function $u \in L^1((0,\tau) : X)$ such that the Riemann-Liouville fractional derivatives $D_t^{\alpha_j} u(t)$ are well defined for $j \in \mathbb{N}_n^0$ and there exist functions $u_j \in L^1((0,\tau)) \cap \mathcal{A}_j D_t^{\alpha_j} u(\cdot)$ for $j \in \mathbb{N}_n^0$ so that

$$f(t) = \sum_{j \in \mathbb{N}_n^0} u_j(t) \ \text{ for a.e. } \ t \in (0,\tau).$$

The notion of strong solution of (2.1) extends the notion of strong solution of problem [15, (1); see Definition 2.1] in the case that the operators $\mathcal{A}_j$ are single-valued and linear $(0 \leq j \leq n)$.

Assume now that $u(\cdot)$ is a strong solution of (2.1) and $\mathcal{A}_j$ is a closed MLO for $0 \leq j \leq n$. Let $k \in \mathbb{N}_n^0$ be fixed. Then we can integrate the equation (2.1) α_n-times by using Lemma 1.1 and the formula (1.1). In such a way, we get that

$$\begin{aligned}\big(g_{\alpha_n} * f\big)(t) \in & \sum_{j=0, j\neq k}^{n} \big(g_{\alpha_n} * u_j\big)(t) \\ & + g_{\alpha_n-\alpha_k} * \mathcal{A}_k \left[u(\cdot) - \sum_{i=0}^{m_k-1} \big(g_{m_k-\alpha_k} * u\big)^{(i)}(0) g_{\alpha_k+i+1-m_k}(\cdot) \right](t)\end{aligned}$$

for all $t \in [0,\tau)$. Hence, the function

$$t \mapsto v_k(t) := (g_{\alpha_n} * f)(t) - \sum_{j=0, j\neq k}^{n} (g_{\alpha_n} * u_j)(t), \quad t \in [0,\tau)$$

is a continuous section of multivalued mapping

$$\begin{aligned} t \mapsto & \mathcal{V}_k(t) := \mathcal{A}_k \Bigg\{ g_{\alpha_n-\alpha_k} \\ & * \left[u(\cdot) - \sum_{i=0}^{m_k-1} \big(g_{m_k-\alpha_k} * u\big)^{(i)}(0) g_{\alpha_k+i+1-m_k}(\cdot) \right] \Bigg\}(t), \quad t \in [0,\tau). \end{aligned} \tag{2.2}$$

Furthermore, we have:

$$\sum_{k=0}^{n} v_k(t) = (n+1)\big(g_{\alpha_n} * f\big)(t) - n\sum_{j=0}^{n}\big(g_{\alpha_n} * u_j\big)(t) = \big(g_{\alpha_n} * f\big)(t), \;\; t \in [0,\tau).$$

This motivates us to introduce the following notion of a mild solution of (2.1):

Definition 2.2. Suppose $0 < \tau \leq \infty$ and $f \in L^1((0,\tau) : X)$. By a mild solution of (2.1) we mean any function $u \in L^1((0,\tau) : X)$ such that the Riemann-Liouville fractional derivatives $D_t^{\alpha_j} u(t)$ are well defined for $j \in \mathbb{N}_n^0$, as well as that for each $k \in \mathbb{N}_n^0$ there exists a continuous section $v_k(\cdot)$ of multivalued mapping $\mathcal{V}_k(\cdot)$ given by (2.2), so that

$$\sum_{k=0}^{n} v_k(t) = \big(g_{\alpha_n} * f\big)(t), \quad t \in [0,\tau). \tag{2.3}$$

The notion of mild solution of (2.1) extends the notion of mild solution of problem [15, (1); see Definition 2.2] in the case that the operators $\mathcal{A}_j$ are single-valued and linear ($0 \leq j \leq n$). In our previous analysis, we have actually proved that any strong solution of (2.1) is a mild solution of the same problem; even in single-valued linear case, the converse statement fails to be true.

In the sequel, we shall primarily use the following notion, which generalizes the notion of a mild (strong) solution of the abstract Cauchy problems [15, (8)–(9); see Definition 2.4]:

Definition 2.3. Suppose $0 < \tau \leq \infty$ and $f \in L^1((0,\tau) : X)$. Consider the following abstract multi-term integral inclusion:

$$\big(g_{\alpha_n} * f\big)(t) \in \sum_{k=0}^{n} \mathrm{V}_k(t), \quad t \in [0,\tau), \tag{2.4}$$

where

$$\mathrm{V}_k(t) := \mathcal{A}_k\Bigg\{ g_{\alpha_n-\alpha_k} * \Bigg[u(\cdot) - \sum_{i=0}^{m_k-1} g_{\alpha_k+i+1-m_k}(\cdot)x_{i,k} \Bigg]\Bigg\}(t), \;\; t \in [0,\tau),$$

for some elements $x_{i,k} \in X$ ($k \in \mathbb{N}_n^0$, $0 \leq i \leq m_k - 1$).

(i) By a mild solution of (2.4), we mean any continuous function $u \in C([0,\tau) : X)$ such that, for every $k \in \mathbb{N}_n^0$, there exists a section $v_k(\cdot)$ of multivalued mapping $\mathcal{V}_k(\cdot)$, and (2.3) holds true.

(ii) By a strong solution of (2.4), we mean any continuous function $u \in C([0,\tau) : X)$ such that, for every $k \in \mathbb{N}_n^0$, there exists a continuous section $w_k(\cdot)$ of multivalued mapping

$$t \mapsto \mathcal{W}_k(t) := \mathcal{A}_k\Bigg[u(\cdot) - \sum_{i=0}^{m_k-1} g_{\alpha_k+i+1-m_k}(\cdot)x_{i,k}\Bigg](t), \ \ t \in [0,\tau) \quad (2.5)$$

and

$$\sum_{k=0}^{n}\big(g_{\alpha_n-\alpha_k} * w_k\big)(t) = \big(g_{\alpha_n} * f\big)(t), \quad t \in [0,\tau).$$

In order to subject initial values to (2.1), we first define

$$\mathcal{T}_{(2.1)} := \begin{cases} 1, & \text{if there exists } j \in \mathbb{N}_n^0 \text{ such that } \alpha_j \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

and $S := \{j \in \mathbb{N}_n^0 : \alpha_j \in \mathbb{N}\}$. As in single-valued linear case, we distinguish the following three subcases of (2.1):

(SC1) $\alpha_n > 1$: Then for each integer $i \in \mathbb{N}_{m_n-1}$ we set

$$\mathcal{D}_i := \{j \in \mathbb{N}_n^0 : m_j - 1 \geq i\}, \quad S_i := \{m_j - \alpha_j : j \in \mathcal{D}_i\}$$

and $s_i :=$ card(S_i). Then we have $S_i \subseteq [0,1)$ and

$$S_i = \{a_{i,1}, \ldots, a_{i,s_i}\},$$

where $0 \leq a_{i,1} < \cdots < a_{i,s_i} \leq 1$ $(i \in \mathbb{N}_{m_n-1})$. Define

$$\mathcal{D}_i^l := \{j \in \mathcal{D}_i : m_j - \alpha_j = a_{i,l}\} \quad (i \in \mathbb{N}_{m_n-1},\ 1 \leq l \leq s_i).$$

For each integer $i \in \mathbb{N}_{m_n-1}$ we introduce s_i initial values $x_{i,1}, \ldots, x_{i,s_i}$ for terms $(g_{m_j-\alpha_j} * u)^{(i)}(0)$, where $j \in \mathcal{D}_i$. Additionally, if there exists $j \in \mathbb{N}_n^0$ such that $\alpha_j \in \mathbb{N}$, i.e., if $S \neq \emptyset$, then we introduce a new initial value x_0 for the term $(g_0 * u)(0) \equiv u(0)$.

(SC2) $\alpha_n = 1$: In this case, we introduce only one initial value for term $(g_0 * u)(0) \equiv u(0)$.

(SC3) $\alpha_n < 1$: In this case, we consider the equation (2.1) without initial conditions.

Put

$$\mathcal{B}_{(2.1)} := \begin{cases} s_1 + \cdots + s_{m_n-1} + \mathcal{T}_{(2.1)}, & \text{if } \alpha_n > 1, \\ 1, & \text{if } \alpha_n = 1, \\ 0, & \text{if } \alpha_n < 1. \end{cases}$$

By the foregoing, there will be exactly $\mathcal{B}_{(2.1)}$ initial conditions for (2.1).

The subcase (SC3) will not be examined henceforth. For the subcases (SC1) and (SC2), we will use the following definition (cf. [13, Section 2.4] and [15] for more details about single-valued linear case).

Definition 2.4. Let $0 < \tau \leq \infty$, $k \in C([0, \tau))$, $C,\ C_1,\ C_2 \in L(X)$, and let C and C_2 be injective.

(i) (SC1) Suppose that, for every $i \in \mathbb{N}_{m_n-1}$ and $l \in \mathbb{N}_{s_i}$, $(R_{i,l}(t))_{t\in[0,\tau)} \subseteq L(X)$ is strongly continuous, as well as that, for every $t \in [0, \tau)$, $x \in X$, $i \in \mathbb{N}_{m_n-1}$, $l \in \mathbb{N}_{s_i}$ and $j \in \mathbb{N}_n^0$, one has

$$[g_{\alpha_n-\alpha_j} * (R_{i,l}(\cdot)x - \chi_{\mathcal{D}_i^l}(j)(k * g_{\alpha_j+i-m_j})(\cdot)C_1x)](t) \in D(\mathcal{A}_j)$$

and

$$\begin{aligned} &\mathcal{B}\big[R_{i,l}(t)x - \chi_{\mathcal{D}_i^l}(n)\big(k * g_{\alpha_n+i-m_n}\big)(t)C_1x\big] \\ &+ \sum_{j=1}^{n-1} \mathcal{A}_j\Big[g_{\alpha_n-\alpha_j} * \Big(R_{i,l}(\cdot)x - \chi_{\mathcal{D}_i^l}(j)\big(k * g_{\alpha_j+i-m_j}\big)(\cdot)C_1x\Big)\Big](t) \\ &\in \mathcal{A}\Big[g_{\alpha_n-\alpha} * \Big(R_{i,l}(\cdot)x - \chi_{\mathcal{D}_i^l}(0)\big(k * g_{\alpha+i-m}\big)(\cdot)C_1x\Big)\Big](t) \end{aligned} \tag{2.6}$$

holds. If $S \neq \emptyset$, then we also introduce a strongly continuous family

$$(R_{0,1}(t))_{t\in[0,\tau)} \subseteq L(X)$$

satisfying that, for every $t \in [0, \tau)$, $x \in X$ and $j \in \mathbb{N}_n^0$, one has

$$[g_{\alpha_n-\alpha_j} * (R_{0,1}(\cdot)x - \chi_S(j)k(\cdot)C_1x)](t) \in D(\mathcal{A}_j)$$

and

$$\begin{aligned} &\mathcal{B}\big[R_{0,1}(t)x - \chi_S(n)k(t)C_1x\big] \\ &\qquad + \sum_{j=1}^{n-1} \mathcal{A}_j\Big[g_{\alpha_n-\alpha_j} * \Big(R_{0,1}(\cdot)x - \chi_S(j)k(\cdot)C_1x\Big)\Big](t) \\ &\qquad \in \mathcal{A}\Big[g_{\alpha_n-\alpha} * \Big(R_{0,1}(\cdot)x - \chi_S(0)k(\cdot)C_1x\Big)\Big](t). \end{aligned} \tag{2.7}$$

Then the sequence $((R_{i,l}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ if $S=\emptyset$, resp.,

$$((R_{i,l}(t))_{t\in[0,\tau)},\ (R_{0,1}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$$

if $S\neq\emptyset$, is said to be a (local, if $\tau<\infty$) k-regularized C_1-existence propagation family for (2.1).

(SC2) A strongly continuous family $(R(t))_{t\in[0,\tau)}\subseteq L(X)$ satisfying that, for every $t\in[0,\tau)$, $x\in X$ and $j\in\mathbb{N}_n^0$, one has $[g_{\alpha_n-\alpha_j}*R(\cdot)x](t)\in D(\mathcal{A}_j)$ and

$$\mathcal{B}\big[R(t)x-k(t)C_1x\big]+\sum_{j=1}^{n-1}\mathcal{A}_j\big(g_{\alpha_n-\alpha_j}*R(\cdot)x\big)(t)\in\mathcal{A}\big(g_{\alpha_n-\alpha}*R(\cdot)x\big)(t),$$

is said to be a (local, if $\tau<\infty$) k-regularized C_1-existence propagation family for (2.1).

(ii) (SC1) Suppose that, for every $i\in\mathbb{N}_{m_n-1}$ and $l\in\mathbb{N}_{s_i}$, $(W_{i,l}(t))_{t\in[0,\tau)}\subseteq L(X)$ is strongly continuous, as well as that $\bigcap_{j=0}^n D(\mathcal{A}_j)\neq\emptyset$ and

$$\begin{aligned}W_{i,l}(\cdot)x_n&-\chi_{\mathcal{D}_i^l}(n)\big(k*g_{\alpha_n+i-m_n}\big)(\cdot)C_2x_n\\&+\sum_{j=1}^{n-1}g_{\alpha_n-\alpha_j}*\Big[W_{i,l}(\cdot)x_j-\chi_{\mathcal{D}_i^l}(j)\big(k*g_{\alpha_j+i-m_j}\big)(\cdot)C_2x_j\Big]\\&=g_{\alpha_n-\alpha}*\Big[W_{i,l}(\cdot)x_0-\chi_{\mathcal{D}_i^l}(0)\big(k*g_{\alpha+i-m}\big)(\cdot)C_2x_0\Big],\end{aligned}$$

whenever $i\in\mathbb{N}_{m_n-1}$, $l\in\mathbb{N}_{s_i}$ and $(x,x_j)\in\mathcal{A}_j$ for all $j\in\mathbb{N}_n^0$. If $S\neq\emptyset$, then we also introduce a strongly continuous family $(W_{0,1}(t))_{t\in[0,\tau)}\subseteq L(X)$ satisfying that

$$\begin{aligned}W_{0,1}(\cdot)x_n-\chi_S(n)k(\cdot)C_2x_n+\sum_{j=1}^{n-1}g_{\alpha_n-\alpha_j}*\Big[W_{0,1}(\cdot)x_j-\chi_S(j)k(\cdot)C_2x_j\Big]\\=g_{\alpha_n-\alpha}*\Big[W_{0,1}(\cdot)x_0-\chi_S(0)k(\cdot)C_2x_0\Big],\end{aligned}$$

whenever $(x,x_j)\in\mathcal{A}_j$ for all $j\in\mathbb{N}_n^0$. Then the sequence

$$((W_{i,l}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$$

if $S=\emptyset$, resp.,

$$((W_{i,l}(t))_{t\in[0,\tau)},\ (W_{0,1}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$$

if $S \neq \emptyset$, is said to be a (local, if $\tau < \infty$) k-regularized C_2-uniqueness propagation family for (2.1). If, in addition to the above, $\mathcal{A}_j$ for $0 \leq j \leq n$, any operator family $W_{i,l}(\cdot)$, the operator family $W_{0,1}(\cdot)$ if $S \neq \emptyset$, and the operator C_2, all commute with each other, then $((W_{i,l}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ if $S = \emptyset$, resp., $((W_{i,l}(t))_{t\in[0,\tau)},\ (W_{0,1}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ if $S \neq \emptyset$, is said to be a k-regularized C_2-resolvent propagation family for (2.1).

(SC2) A strongly continuous family $(W(t))_{t\in[0,\tau)} \subseteq L(X)$ satisfying that

$$W(t)x_n - k(t)C_2x_n + \sum_{j=1}^{n-1}\big(g_{\alpha_n-\alpha_j} * W(\cdot)C_2x_j\big)(t) = \big(g_{\alpha_n-\alpha} * W(\cdot)C_2x_0\big)(t), \quad t \in [0,\tau),$$

whenever $(x, x_j) \in \mathcal{A}_j$ for all $j \in \mathbb{N}_n^0$, is said to be a (local, if $\tau < \infty$) k-regularized C_2-uniqueness propagation family for (2.1). If, additionally, $\mathcal{A}_j$ for $0 \leq j \leq n$, $W(\cdot)$ and the operator C_2, all commute with each other, then $(W(t))$ is said to be a k-regularized C_2-resolvent propagation family for (2.1).

The notions of k-regularized C_2-uniqueness propagation families for (2.1) and k-regularized C_2-resolvent propagation families for (2.1) coincide with the corresponding notions introduced in single-valued linear case (cf. [15, Definition 2.3(ii)–(iii)]). Furthermore, the notion of a k-regularized C_1-existence propagation family for (2.1) is slightly weaker from that one introduced in [15, Definition 2.3(i)] because, in the case that the operator $\mathcal{B} = B$ is single-valued, we do not assume here the strong continuity of families $(BR_{i,l}(t))_{t\in[0,\tau)} \subseteq L(X)$ and $(BR_{0,1}(t))_{t\in[0,\tau)} \subseteq L(X)$ (the subcase (SC1)).

In the case that $k(t) = g_{\zeta+1}(t)$, where $\zeta \geq 0$, then we say that a k-regularized C_1-existence propagation family for (2.1) is also a ζ-times integrated C_1-existence propagation family for (2.1); 0-times integrated C_1-existence propagation family for (2.1) is further abbreviated to C_1-existence propagation family for (2.1); a similar language is used for the classes of C_2-uniqueness propagation families for (2.1) and C-resolvent propagation families for (2.1).

A k-regularized C_1-existence propagation family for (2.1) is said to be an exponentially bounded (resp., bounded), analytic k-regularized C_1-existence propagation family for (2.1), of angle $\alpha \in (0, \pi/2]$, if and only if for each single operator family $(R(t))_{t\geq 0}$ of it, the following holds:

(a) For every $x \in X$, the mapping $t \mapsto R(t)x$, $t > 0$, can be analytically extended to the sector Σ_α; we denote this extension by the same symbol.

(b) For every $x \in X$ and $\beta \in (0, \alpha)$, we have $\lim_{z\to 0, z\in\Sigma_\beta} R(z)x = R(0)x$.

(c) For every $\beta \in (0, \alpha)$, there exists $\omega_\beta \geq \max(0, \text{abs}(k))$ (resp., $\omega_\beta = 0$) such that the family $\{e^{-\omega_\beta z} R(z) : z \in \Sigma_\beta\} \subseteq L(X)$ is bounded.

We similarly introduce the classes of exponentially bounded (resp., bounded), analytic k-regularized C_2-uniqueness propagation families for (2.1) and exponentially bounded (resp., bounded), analytic k-regularized C-resolvent propagation families for (2.1).

Concerning the well-posedness of abstract inhomogenous Cauchy problem (2.4) ($f \equiv 0$), we want only to observe that the assertions (A)-(B) clarified in [15] continue to hold in our new framework without any terminological changes. We leave to the interested readers a problem of transferring the assertions clarified in [15, Theorem 2.6, Remark 2.7] to multivalued linear operators case.

3. *Asymptotically almost periodic and asymptotically almost automorphic solutions of* (2.4)

In this section, it will be always assumed that $\mathcal{B} = I$, $\mathcal{A}_j = A_j \in L(X)$ for $1 \leq j \leq n-1$, and $\mathcal{A}$ is a closed MLO. Then we can profile the class of k-regularized C-resolvent propagation families for (2.1) by means of vector-valued Laplace transform. In order to present the main ideas for applications, in the subsequent three theorems, we will consider only the subcase (SC1) in which $0 \notin \mathcal{D}_i^l \cup S$ ($1 \leq i \leq m_n - 1$, $1 \leq l \leq s_i$).

The following results can be deduced by using the argumentation contained in the proofs of [14, Theorem 3.4.5, Theorem 3.4.6, Theorem 3.4.8].

Theorem 3.1. *Suppose $k(t)$ satisfies* (P1), $\omega \geq \max(0, \text{abs}(k))$, *as well as that for every $i \in \mathbb{N}_{m_n-1}$ and $l \in \mathbb{N}_{s_i}$, $(e^{-\omega t} R_{i,l}(t))_{t\geq 0} \subseteq L(X)$ is a strongly continuous bounded family, as well as that, in the case that $S \neq \emptyset$, $(e^{-\omega t} R_{0,1}(t))_{t\geq 0} \subseteq L(X)$ is a strongly continuous bounded family. Let $0 \notin \mathcal{D}_i^l \cup S$ ($1 \leq i \leq m_n - 1, 1 \leq l \leq s_i$).*

(I) *Let the following two conditions hold:*

(i) $CA_j \subseteq A_j C$, $j \in \mathbb{N}_{n-1}^0$, $A_j \in L(X)$, $j \in \mathbb{N}_{n-1}$, $A_i A_j = A_j A_i$, i, $j \in \mathbb{N}_{n-1}$ *and* $A_j \mathcal{A} \subseteq \mathcal{A} A_j$, $j \in \mathbb{N}_{n-1}$.

(ii) *If $S = \emptyset$, there exist an integer $i \in \mathbb{N}_{m_n-1}$ and an integer $l \in [1, s_i]$ satisfying that the operator*

$$Z_\lambda := \chi_{\mathcal{D}_i^l}(n) \lambda^{m_n} + \sum_{j=1}^{n-1} \chi_{\mathcal{D}_i^l}(j) \lambda^{m_j} A_j$$

is injective for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$.

(iii) *If* $S = \emptyset$ *and there do not exist integers* $i \in \mathbb{N}_{m_n-1}$ *and* $l \in [1, s_i]$ *such that the operator* Z_λ *is injective for all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *and* $\tilde{k}(\lambda) \neq 0$, *then there exists* $j \in \mathbb{N}_n$ *with* $\alpha_j \in \mathbb{N}$.

If $((R_{i,l}(t))_{t\geq 0})_{1\leq i\leq m_n-1, 1\leq l\leq s_i}$ *if* $S = \emptyset$, *resp.,*

$$((R_{i,l}(t))_{t\geq 0},\ (R_{0,1}(t))_{t\geq 0})_{1\leq i\leq m_n-1, 1\leq l\leq s_i}$$

if $S \neq \emptyset$, *is a* k*-regularized* C*-resolvent propagation family for* (2.1), *then* P_λ *is injective for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *and* $\tilde{k}(\lambda) \neq 0$, *as well as the equalities*

$$P_\lambda \int_0^\infty \mathrm{e}^{-\lambda t} R_{i,l}(t)x\,\mathrm{d}t = \tilde{k}(\lambda)\lambda^{-\alpha-i}\Bigg[\chi_{\mathcal{D}_i^l}(n)\lambda^{m_n}Cx + \sum_{j=1}^{n-1}\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j}A_jCx\Bigg] \tag{3.1}$$

and

$$P_\lambda \int_0^\infty \mathrm{e}^{-\lambda t} R_{0,1}(t)x\,\mathrm{d}t = \tilde{k}(\lambda)\sum_{j=1}^{n}\chi_S(j)A_jCx \tag{3.2}$$

are fulfilled for $\operatorname{Re}\lambda > \omega$ *and* $\tilde{k}(\lambda) \neq 0$.

(II) *Suppose that* P_λ *is injective for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *and* $\tilde{k}(\lambda) \neq 0$ *as well as the equalities* (3.1)–(3.2) *are fulfilled and the condition* (I)(i) *holds. Then* $((R_{i,l}(t))_{t\geq 0})_{1\leq i\leq m_n-1, 1\leq l\leq s_i}$ *if* $S = \emptyset$, *resp.,*

$$((R_{i,l}(t))_{t\geq 0},\ (R_{0,1}(t))_{t\geq 0})_{1\leq i\leq m_n-1, 1\leq l\leq s_i}$$

if $S \neq \emptyset$, *is a* k*-regularized* C*-resolvent propagation family for* (2.1).

Theorem 3.2. *Assume* $k(t)$ *satisfies* (P1), $\omega \geq \max(0, \operatorname{abs}(k))$, $\beta \in (0, \pi/2]$ *and, for every* $i \in \mathbb{N}_{m_n-1}^0$, *the function* $(k * g_i)(t)$ *can be analytically extended to a function* $k_i : \Sigma_\beta \to \mathbb{C}$ *satisfying that, for every* $\gamma \in (0, \beta)$, *the set*

$$\{\mathrm{e}^{-\omega z}k_i(z) : z \in \Sigma_\gamma\}$$

is bounded. Let $0 \notin \mathcal{D}_i^l \cup S$ $(1 \leq i \leq m_n - 1, 1 \leq l \leq s_i)$, *and let the following four conditions hold:*

(i) $CA_j \subseteq A_jC$, $j \in \mathbb{N}^0_{n-1}$, $A_j \in L(X)$, $j \in \mathbb{N}_{n-1}$, $A_iA_j = A_jA_i$, $i,\ j \in \mathbb{N}_{n-1}$ *and* $A_j\mathcal{A} \subseteq \mathcal{A}A_j$, $j \in \mathbb{N}_{n-1}$.

(ii) *The operator* P_λ *is injective for all* $\omega + \Sigma_{\beta+\pi/2}$.

(iii) *For every integers* $i \in \mathbb{N}_{m_n-1}$ *and* $l \in [1, s_i]$, *there exist an operator* $D_{i,l} \in L(X)$ *and a strongly analytic mapping* $q_{i,l} : \omega + \Sigma_{\frac{\pi}{2}+\beta} \to L(X)$ *satisfying the following:*

$$q_{i,l}(\lambda)x = \widetilde{k}_i(\lambda)P_\lambda^{-1}\Bigg[\chi_{\mathcal{D}_i^l}(n)\lambda^{m_n-\alpha}Cx + \sum_{j=1}^{n-1}\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-\alpha}A_jCx\Bigg],$$

for any $x \in X,\ \mathrm{Re}\,\lambda > \omega$, *the family*

$$\big\{(\lambda-\omega)q_{i,l}(\lambda) : \lambda \in \omega + \Sigma_{\frac{\pi}{2}+\gamma}\big\} \text{ is bounded for all } \gamma \in (0,\beta),$$

and, in the case $\overline{D(\mathcal{A})} \neq X$,

$$\lim_{\lambda\to+\infty} \lambda q_{i,l}(\lambda)x = D_{i,l}x, \quad x \notin \overline{D(\mathcal{A})}.$$

(iv) *If* $S = \emptyset$, *then there exist an operator* $D \in L(X)$ *and a strongly analytic mapping* $q : \omega + \Sigma_{\frac{\pi}{2}+\beta} \to L(X)$ *satisfying the following:*

$$q(\lambda)x = \tilde{k}(\lambda)P_\lambda^{-1}\sum_{j=1}^{n}\chi_S(j)A_jCx, \quad x \in X,\ \mathrm{Re}\,\lambda > \omega$$

the family $\big\{(\lambda-\omega)q(\lambda) : \lambda \in \omega + \Sigma_{\frac{\pi}{2}+\gamma}\big\}$ *is bounded for all* $\gamma \in (0,\beta)$,

and, in the case $\overline{D(\mathcal{A})} \neq X$,

$$\lim_{\lambda\to+\infty} \lambda q(\lambda)x = Dx, \quad x \notin \overline{D(\mathcal{A})}.$$

Then there exists an exponentially bounded, analytic k*-regularized* C*-resolvent propagation family*

$$((R_{i,l}(t))_{t\geq 0})_{1\leq i\leq m_n-1, 1\leq l\leq s_i}$$

if $S = \emptyset$, *resp.,*

$$((R_{i,l}(t))_{t\geq 0},\ (R_{0,1}(t))_{t\geq 0})_{1\leq i\leq m_n-1, 1\leq l\leq s_i}$$

if $S \neq \emptyset$, *for* (2.1), *of angle* β. *Furthermore, the family* $\{\mathrm{e}^{-\omega z}R_{\cdot,\cdot}(z) : z \in \Sigma_\gamma\}$ *is bounded for all* $\gamma \in (0,\beta)$, (2.6)–(2.7) *and* $R_{\cdot,\cdot}(z)\mathcal{A}_j \subseteq \mathcal{A}_jR_{\cdot,\cdot}(z)$, $z \in \Sigma_\beta$, $j \in \mathbb{N}^0_{n-1}$ *are valid for any single operator family* $R_{\cdot,\cdot}(\cdot)$.

Observe that the notion from Definition 2.1 can be modified and introduced for single operator families ([10]). The former two theorems can be simply reformulated in this context, which will be important in the sequel.

For our investigation of generalized asymptotically almost periodic and generalized asymptotically almost automorphic solutions of (2.4), the following analogue of [14, Theorem 3.4.10] is crucial to be stated. The proof is very similar to that of afore-mentioned theorem and we will sketch the main details of it, only:

Theorem 3.3. *Suppose that* $c_j \geq 0$, $\mathcal{B} = I$ *and* $A_j = c_j I$ *for* $1 \leq j \leq n-1$, $\zeta' \geq 0$, $\mathcal{A} : X \to P(X)$ *is a closed* MLO, $C \in L(X)$ *is injective,* $C\mathcal{A} \subseteq \mathcal{A}C$ *and the following condition holds:*

(H): *There exist finite constants* $c < 0$, $M > 0$, $0 < \theta < \pi$ *and* $\beta \in (0, 1]$ *such that*

$$\overline{c + \Sigma_{\pi-\theta}} \subseteq \rho_C(\mathcal{A})$$

and

$$\big\|(\lambda - \mathcal{A})^{-1}C\big\| \leq \frac{M}{|\lambda - c|^{\beta}}, \quad \lambda \in \overline{c + \Sigma_{\pi-\theta}}.$$

Assume that the mapping $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$, $\lambda \in \overline{c + \Sigma_{\pi-\theta}}$, *is strongly continuous. Let the integers* $i \in \mathbb{N}_{m_n - 1}$ *and* $l \in [1, s_i]$ *be fixed, and let* $0 \notin \mathcal{D}_i^l$. *Set*

$$r := \max\big\{s \in \mathbb{N}_n : s \in \mathcal{D}_i^l\big\}.$$

Assume also that

$$r - \alpha - i - \zeta' - \big(\alpha_n - \alpha\big)\beta \leq 0, \tag{3.3}$$

and

$$\nu' := \frac{\pi - \theta}{\alpha_n - \alpha} - \frac{\pi}{2} > 0. \tag{3.4}$$

Set $\zeta := \zeta'$ *if* $\mathcal{A}$ *is densely defined,* $\zeta > \zeta'$ *otherwise, and* $k_i(\cdot) := g_{\zeta+1}(\cdot)$. *Then there exists an exponentially bounded, analytic* k_i*-regularized* C*-propagation family* $(R_{i,l}(t))_{t\geq 0}$ *for* (2.1), *of angle* $\nu := \min(\nu', \pi/2)$. *Moreover,* (2.6)–(2.7) *hold and there exists a finite constant* $M' > 0$ *such that*

$$\big\|R_i(t)\big\| \leq M' \left[t^{\alpha+\zeta+i-m_n} \mathcal{D}_i^l(n) + \sum_{j=1}^{n-1} t^{\alpha+\zeta+i-m_j} \mathcal{D}_i^l(j) \right], \quad t > 0. \tag{3.5}$$

Sketch of proof. As in [10], the mapping $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$, $\lambda \in \overline{c + \Sigma_{\pi-\theta}}$ is strongly continuous and its restriction to $c + \Sigma_{\pi-\theta}$ is strongly analytic. Since $c_j \geq 0$ for $1 \leq j \leq n-1$ and the estimates (3.3)-(3.4) are valid, we can easily show that the conditions of Theorem 3.2 hold (for single operator families, see the short discussion above), with $\omega > 0$ sufficiently large and $k_i(\cdot)$. Hence, $\mathcal{A}$ is a subgenerator of an exponentially bounded, analytic ζ-times integrated C-propagation family $(R_{i,l}(t))_{t\geq 0}$ for (2.1), of angle $\nu = \min(\nu', \pi/2)$, as claimed. The estimate (3.5) can be deduced as in the proof of [14, Theorem 3.4.10], with appealing to some estimates contained in the proof of [2, Theorem 2.6.1].

Remark 3.1. The assertion of Theorem 3.3 can be also formulated, with minor modifications, for exponentially bounded, analytic C-regularized solution operator families whose Laplace transform can be computed as

$$\int_0^{\infty} \mathrm{e}^{-\lambda t} R(t)x \, \mathrm{d}t = \lambda^{-\zeta-1} P_\lambda^{-1} C \left[\lambda^{a_s} + \sum_{j=1}^{s} c_j \lambda^{a_j} \right] x, \quad x \in X, \ \operatorname{Re}\lambda > \omega,$$

where $\omega > 0$, $s \in \mathbb{N}$, $0 \leq a_1 < a_2 < \cdots < a_s$, $\zeta \geq 0$ and $c_j \geq 0$, $j \in \mathbb{N}_s$.

Let $\mathcal{F} \in \{AP_T([0,\infty) : \mathbb{C}), AAP([0,\infty) : \mathbb{C}), AAA([0,\infty) : \mathbb{C})\}$, where the symbol $AP_T([0,\infty) : \mathbb{C})$ denotes the space of scalar-valued asymptotically T-periodic functions ($T > 0$), $AAP([0,\infty) : \mathbb{C})$ denotes the space of scalar-valued asymptotically almost periodic functions and $AAA([0,\infty) : \mathbb{C})$ denotes the space of scalar-valued asymptotically almost automorphic functions defined as in [14]. Let the function $k_i(\cdot)$ be defined as above, let $f \in \mathcal{F}$, and let $(R_i(t))_{t\geq 0}$ be the k_i-regularized C-propagation family for (2.1), constructed with the help of Theorem 3.3. Then it can be easily verified that $((R_i * f)(t))_{t\geq 0}$ is a k_i-regularized C-propagation family for (2.1), satisfying additionally (2.6), where $\mathrm{k}_i(\cdot) = (g_{\zeta+1} * f)(\cdot)$. Assume that

$$\big(\alpha + \zeta + i - m_n\big)\mathcal{D}_i^l(n) + \big(\alpha + \zeta + i - m_j\big)\mathcal{D}_i^l(j) < -1. \tag{3.6}$$

Applying Theorem 3.3, some known assertions concerning inheritance of asymptotical periodicity, almost asymptotical almost periodicity and asymptotical almost automorphy under the action of finite convolution products ([14]), we can establish the following result (the condition (3.6) yields the uniform integrability of $(R_{i,l}(t))_{t\geq 0}$, i.e., we have $\int_0^\infty \|R_{i,l}(t)\| \, \mathrm{d}t < \infty$, while the uniqueness of solutions is a simple consequence of the fact that [15, Theorem 2.5] holds in our framework):

Corollary 3.1. *Let the requirements of Theorem* 3.3 *hold, let* $f \in \mathcal{F}$, *and let* $\mathrm{k}_i(\cdot) = (g_{\zeta+1} * f)(\cdot)$. *Assume that* (3.6) *holds. Define* $u_x(t) := (R_{i,l} * f)(t)x$, $t \geq 0$,

$x \in X$. Then, for every $x \in X$, $u_x(\cdot) \in \mathcal{F}$ is a unique mild solution of the abstract Cauchy inclusion

$$\begin{aligned}&\big[u(t) - \chi_{\mathcal{D}_i^l}(n)\big(f * g_{\alpha_n+i-m_n+\zeta+1}\big)(t)Cx\big]\\&\quad+\sum_{j=1}^{n-1} c_j\Big[g_{\alpha_n-\alpha_j} * \Big(u(\cdot)x - \chi_{\mathcal{D}_i^l}(j)\big(f * g_{\alpha_j+i-m_j+\zeta+1}\big)(\cdot)Cx\Big)\Big](t)\\&\quad\in \mathcal{A}\Big[g_{\alpha_n-\alpha} * \Big(u(\cdot)x - \chi_{\mathcal{D}_i^l}(0)\big(f * g_{\alpha+i-m+\zeta+1}\big)(\cdot)Cx\Big)\Big](t), \quad t \geq 0.\end{aligned}$$

Furthermore, $u_x(\cdot)$ is a strong solution of the above inclusion provided that $x \in D(\mathcal{A})$.

Concerning generalized asymptotically almost periodic functions and generalized asymptotically almost automorphic functions (Stepanov and Weyl classes, primarily; see [6], [9] and [14]), some additional conditions on the vanishing part of function $f(\cdot)$ must be imposed in our striving for solution $u_x(\cdot)$ to belong the same class of functions as $f(\cdot)$ (in vector-valued sense). It should be noted that Corollary 3.1 is applicable in the study of certain type of abstract degenerate integral Cauchy problems involving the Poisson heat operator ([8], [14]). Semilinear Cauchy integral inclusions can be also examined, with the help of already established results and theorems from the fixed point theory.

Let $\mathcal{F}'$ and $\mathcal{F}''$ be the spaces of generalized asymptotically almost periodic (automorphic) functions defined in [14, Section 3.4]. The following analogue of [14, Proposition 3.4.13] holds in our framework:

Proposition 3.1. *Suppose that $k(t)$ satisfies* (P1)*, $i \in \mathbb{N}_{m_n-1}$, $1 \leq l \leq s_i$, $((R_{i,l}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ if $S = \emptyset$, resp.,*

$$((R_{i,l}(t))_{t\in[0,\tau)},\ (R_{0,1}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$$

if $S \neq \emptyset$, is a strongly Laplace transformable k-regularized C-resolvent propagation family for (2.1).

(i) *For every $\lambda \in \mathbb{C}$, there exists a function $f_\lambda^{i,l}(\cdot)$ satisfying* (P1)-$L(X)$ *and*

$$\begin{aligned}f_\lambda^{i,l}(t) := \mathcal{L}^{-1}\Bigg\{&\Bigg[\Big(1 - \frac{\lambda}{z^{\alpha_n-\alpha}}\Big)I + \sum_{j=1}^{n-1}\frac{A_j}{z^{\alpha_n-\alpha_j}}\Bigg]^{-1}\\&\times\Bigg(\chi_{\mathcal{D}_i^l}(n)\frac{\tilde{k}(z)C}{z^{\alpha_n+i-m_n}} + \sum_{j=1}^{n-1}\chi_{\mathcal{D}_i^l}(j)A_j\frac{\tilde{k}(z)C}{z^{\alpha_n+i-m_j}} - \chi_{\mathcal{D}_i^l}(0)\frac{\lambda\tilde{k}(z)C}{z^{\alpha_n+i-m}}\Bigg)\Bigg\}(t),\end{aligned}$$

for any $t \geq 0$, *and a function* $f_{\lambda}^{0,1}(\cdot)$ *satisfying* (P1)-$L(X)$ *and*

$$f_{\lambda}^{0,1}(t) := \mathcal{L}^{-1}\Biggl\{\Biggl[\Biggl(1 - \frac{\lambda}{z^{\alpha_n - \alpha}}\Biggr)I + \sum_{j=1}^{n-1} \frac{A_j}{z^{\alpha_n - \alpha_j}}\Biggr]^{-1}$$
$$\times \Biggl(\chi_S(n)\tilde{k}(z) + \sum_{j=1}^{n-1} \chi_S(j) A_j \tilde{k}(z) z^{\alpha_j - \alpha_n} - \chi_S(0)\tilde{k}(z) z^{\alpha - \alpha_n}\Biggr)\Biggr\}(t),$$

for any $t \geq 0$.

(ii) *Denote by* D *the set consisting of all eigenvectors* x *of operator* $\mathcal{A}$ *which corresponds to eigenvalues* $\lambda \in \mathbb{C}$ *of operator* $\mathcal{A}$ *(*$\lambda x \in \mathcal{A}x$*) for which the mapping*

$$f_{\lambda,x}^{i,l}(t) := f_{\lambda}^{i,l}(t)x, \quad t \geq 0, \ resp., \ f_{\lambda,x}^{0,1}(t) := f_{\lambda}^{0,1}(t)x, \quad t \geq 0$$

belongs to the space $\mathcal{F}'$. *Then the mapping* $t \mapsto R_{i,l}(t)x$, $t \geq 0$, *resp.,* $t \mapsto R_{0,1}(t)x$, $t \geq 0$, *belongs to the space* $\mathcal{F}'$ *for all* $x \in span(D)$; *furthermore, the mapping* $t \mapsto R_{i,l}(t)x$, $t \geq 0$, *resp.,* $t \mapsto R_{0,1}(t)x$, $t \geq 0$, *belongs to the space* $\mathcal{F}''$ *for all* $x \in \overline{\mathrm{span}(D)}$ *provided additionally that* $(R_i(t))_{t \geq 0}$ *is bounded.*

The assertion of [14, Theorem 3.4.15] can be also rephrased for abstract multi-term fractional differential inclusions with Riemann-Liouville derivatives. Details can be left to the interested readers.

Acknowledgement. The work of the author was supported by the Ministry of Education, Science and Technological Development of Republic of Serbia, grant No. 174024.

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