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ON CERTAIN SUMS OVER ORDINATES OF ZETA ZEROS III

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Dedicated to the memory of Professor Bogoljub Stanković (1924–2018)

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A b s t r a c t. The upper bound

$$\int_2^T |G(\tfrac{1}{2} + \mathrm{i}t)|^2 \,\mathrm{d}t \ll T \log^2 T$$

is proved, where initially $G(s) = \sum_{\gamma>0} \gamma^{-s}$*. Here* γ *denotes ordinates of complex zeros of the Riemann zeta-function* $\zeta(s)$*. This coincides with the lower bound for the integral in question.*

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1. *Introduction*

This paper is a continuation of the author's work [5] and the joint work [1]. It deals with a mean square estimate for the function

$$G(s) := \sum_{\gamma>0} \gamma^{-s} \qquad (s = \sigma + \mathrm{i}t; \sigma, t \in \mathbb{R}, \sigma > 1),$$

where γ denotes ordinates of complex zeros of the Riemann zeta-function $\zeta(s)$. Here, as usual, the zeros are counted with their respective multiplicities. For a comprehensive account on $\zeta(s)$, the reader is referred to the monographs of E.C. Titchmarsh [9] and the author [4]. The series for $G(s)$ does not converge for $\operatorname{Re} s \leqslant 1$, but the function itself possesses unconditionally analytic continuation at least to the region $\operatorname{Re} s > -1$. The mean square estimate

$$T\log^2 T \ll \int_0^T \left|G\left(\tfrac12 + \mathrm{i}t\right)\right|^2 \,\mathrm{d}t \ll T\log^2 T\sqrt{\log\log T} \tag{1.1}$$

was proved in [1]. The lower bound in (1.1) is new, and the upper bound improves and rectifies the corresponding result of [5], whose proof was not complete. The Vinogradov symbol $f(x) \ll g(x)$ (same as $f(x) = O(g(x))$) is defined in the usual way: $f(x) \ll g(x)$ means that $|f(x)| \leqslant Cg(x)$ for $x \geqslant x_0$, some constant $C > 0$, provided that $g(x) > 0$ for $x \geqslant x_0$.

The aim of this note is to improve the upper bound in (1.1). Efforts have been made to keep the exposition as complete as possible. We shall prove

Theorem 1.1. *We have*

$$\int_0^T \left|G\left(\tfrac12 + \mathrm{i}t\right)\right|^2 \,\mathrm{d}t \ll T\log^2 T. \tag{1.2}$$

Remark 1.1. The lower bound in (1.1) and the upper bound in (1.2) are both of the form $T\log^2 T$, so it is plausible to conjecture that

$$\int_0^T \left|G\left(\tfrac12 + \mathrm{i}t\right)\right|^2 \,\mathrm{d}t = (C + o(1))T\log^2 T \qquad (T \to \infty) \tag{1.3}$$

for some positive constant C. Proving (1.3), however, is out of reach at present.

2. *Proof of Theorem 1.1*

Instead of (1.2) it is sufficient to prove

$$I(T) := \int_{T/2}^T \left|G\left(\tfrac12 + \mathrm{i}t\right)\right|^2 \,\mathrm{d}t \ll T\log^2 T, \tag{2.1}$$

replace T by $T2^{-j}$ and sum the resulting expressions over $O(\log T)$ values $j = 1, 2, \ldots$.

To start with a workable expression for $G(\frac{1}{2}+\mathrm{i}t)$ we proceed as in [5], using the zero counting function

$$N(T) := \sum_{0<\gamma\leqslant T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + f(T), \tag{2.2}$$

$$f(T) \ll \frac{1}{T}, \quad f'(T) \ll \frac{1}{T^2}, \tag{2.3}$$

$$S(T) = \frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+\mathrm{i}T\right) = \frac{1}{\pi}\operatorname{Im}\left\{\log\zeta\left(\tfrac{1}{2}+\mathrm{i}T\right)\right\} \ll \log T. \tag{2.4}$$

This is known as the Riemann – von Mangoldt formula (see [9] or [4]). Here the argument of $\zeta(\frac{1}{2}+\mathrm{i}T)$ is obtained by continuous variation along the straight lines joining the points 2, $2+\mathrm{i}T$, $\frac{1}{2}+\mathrm{i}T$, starting with the value 0. If T is an ordinate of a zero, then we set $S(T) = S(T+0)$.

Let X be a parameter, to be chosen later, which satisfies $1 \ll X \leqslant T$. Then we write

$$G(s) = \sum_{\gamma\leqslant X}\gamma^{-s} + R(s),$$

say, where on using (2.2) it follows that

$$\begin{aligned} R(s) &= \sum_{\gamma>X}\gamma^{-s} = \int_X^\infty x^{-s}\,\mathrm{d}N(x) \\ &= \int_X^\infty \frac{x^{-s}}{2\pi}\log\frac{x}{2\pi}\,\mathrm{d}x + \int_X^\infty x^{-s}\,\mathrm{d}\big(S(x)+f(x)\big). \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} R(s) &= \frac{X^{1-s}}{2\pi(s-1)}\log\frac{X}{2\pi} + \frac{X^{1-s}}{2\pi(s-1)^2} - X^{-s}\big(S(X)+f(X)\big) \\ &\quad + s\int_X^\infty x^{-s-1}(S(x)+f(x))\,\mathrm{d}x. \end{aligned} \tag{2.5}$$

Initially (2.5) is valid for for $\sigma > 1$, but it possesses meromorphic continuation for all $\sigma > 0$, since $S(x) \ll \log x$. Henceforth we set $s = \frac{1}{2}+\mathrm{i}t$ and choose

$$X = \frac{T}{\log T}. \tag{2.6}$$

Then the contribution of the terms in the first line of (2.5) to $I(T)$ in (2.1) is

$$\ll X\log^2 X\int_{T/2}^T t^{-2}\,\mathrm{d}t + \frac{T}{X}\log^2 X \ll \log^3 T. \tag{2.7}$$

Now we use Lemma 4 of the author's paper [6], which says that

$$\int_0^T\left|\int_a^b g(x)x^{-s}\,\mathrm{d}x\right|^2\mathrm{d}t \leqslant 2\pi\int_a^b g^2(x)x^{1-2\sigma}\,\mathrm{d}x \quad (s=\sigma+\mathrm{i}t, T>0, a<b),$$

if $g(x)$ is a real-valued, integrable function on $[a,b]$, a subinterval of $[2,\infty)$, which is not necessarily finite. With $s=\frac12+\mathrm{i}t, A=X, b=+\infty$ this gives

$$\begin{aligned}\int_{T/2}^T\left|s\int_X^\infty x^{-s-1}\big(S(x)+f(x)\big)\,\mathrm{d}x\right|^2\mathrm{d}t \\ &\ll T^2\int_X^\infty\big(S^2(x)+f^2(x)\big)x^{-2}\,\mathrm{d}x \\ &\ll T^2X^{-1}\log\log X \ll T\log T\log\log T.\end{aligned} \tag{2.8}$$

Here we used the elementary bound

$$\int_1^X S^2(x)\,\mathrm{d}x \;\ll\; X\log\log X.$$

An elementary calculation shows that

$$\int_{-1}^1(1-|y|)\mathrm{e}^{-2\pi\mathrm{i}xy}\,\mathrm{d}y = \Big(\frac{\sin\pi x}{\pi x}\Big)^2.$$

Therefore on applying the Fourier inversion one has

$$\frac12\int_{-\infty}^\infty \mathrm{e}^{2\pi\mathrm{i}xy}\left(\frac{\sin\pi x}{\pi x}\right)^2\mathrm{d}x = \begin{cases}1-|y|, & \text{if } |y|\leqslant 1,\\ 0, & \text{if } |y|>1.\end{cases} \tag{2.9}$$

To estimate the contribution of $\sum_{0<\gamma\leqslant X}\gamma^{-s}$ to $I(T)$ in (2.1) we use (2.9) and the fact that

$$1 \;\leqslant\; \frac{\pi^2}{4}\left(\frac{\sin\frac{\pi t}{2T}}{\frac{\pi t}{2T}}\right)^2 \qquad (|t|\leqslant T).$$

We obtain

$$\int_{T/2}^{T}\Big|\sum_{0<\gamma\leqslant X}\gamma^{-1/2-\mathrm{i}t}\Big|^2\,\mathrm{d}t \ll \int_{T/2}^{T}\left(\frac{\sin\frac{\pi t}{2T}}{\frac{\pi t}{2T}}\right)^2\Big|\sum_{0<\gamma\leqslant X}\gamma^{-1/2-\mathrm{i}t}\Big|^2\,\mathrm{d}t$$

$$\leqslant \sum_{0<\gamma,\gamma'\leqslant X}(\gamma\gamma')^{-1/2}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\pi t}{2T}}{\frac{\pi t}{2T}}\right)^2 \mathrm{e}^{\mathrm{i}t\log\gamma/\gamma'}\,\mathrm{d}t,$$

where both γ and γ' denote ordinates of zeta-zeros, counted with their respective multiplicities. In the last integral we make the change of variable $t = 2Tx$ and apply (2.9) with

$$y = \frac{T}{\pi}\log\frac{\gamma}{\gamma'}$$

to obtain

$$\int_{T/2}^{T}\Big|\sum_{0<\gamma\leqslant X}\gamma^{-1/2-\mathrm{i}t}\Big|^2\,\mathrm{d}t \ll T\sum_{0<\gamma,\gamma'\leqslant X,|\frac{T}{\pi}\log\frac{\gamma}{\gamma'}|\leqslant 1}(\gamma\gamma')^{-1/2} = T\sum(T), \quad (2.10)$$

say. By symmetry, the portions of $\sum(T)$ in which $\gamma > \gamma'$ and $\gamma < \gamma'$ are equal. Thus we have to distinguish only the cases $\gamma' > \gamma$ and $\gamma' = \gamma$. In the latter case we have a contribution which is

$$\sum_{0<\gamma\leqslant X}\frac{m(\beta+\mathrm{i}\gamma)}{\gamma}, \quad (2.11)$$

where $m(\rho)$ denotes the multiplicity of the zeta-zero $\rho = \beta + \mathrm{i}\gamma$. Let

$$N^*(T) := \sum_{0<\gamma\leqslant T} m(\beta+\mathrm{i}\gamma).$$

Then if we can show that

$$N^*(T) \ll N(T), \quad (2.12)$$

by partial summation and (2.12) it easily follows that the sum in (2.11) is $\ll \log^2 T$, which suffices for (2.1). But A. Fujii (Theorem 3 of [2]) has shown that

$$N_j(T) \leqslant CN(T)\mathrm{e}^{-Aj} \qquad (A, C > 0, j \geqslant j_0), \quad (2.13)$$

where

$$N_j(T) := \sum_{0<\gamma\leqslant T, m(\beta+\mathrm{i}\gamma)=j} 1.$$

M.A. Korolev [7] later found explicit values of A and C in (2.13). Using (2.13) one has

$$N^*(T) = O(N(T)) + \sum_{j=j_0}^{O(\log T)} jN_j(T) \ll N(T) + N(T)\sum_{j=1}^{\infty} j\mathrm{e}^{-Aj} \ll N(T),$$

since the above series is clearly convergent.

It remains to deal with the case when $\gamma' > \gamma$ in $\sum(T)$ in (2.10). If $\gamma' > \gamma$, then the condition

$$\frac{T}{\pi}\log\frac{\gamma'}{\gamma} \leqslant 1$$

implies, for $T \geqslant T_0$,

$$\gamma < \gamma' \leqslant \mathrm{e}^{\pi/T}\gamma \leqslant \Big(1+\frac{2\pi}{T}\Big)\gamma \leqslant \gamma + \frac{2\pi}{\log T},$$

in view of (2.6). It transpires that $\gamma' \sim \gamma$ and using (2.2)–(2.4)we have

$$\begin{aligned}\sum(T) &\ll \sum_{0<\gamma\leqslant X,\gamma<\gamma'\leqslant\gamma+(2\pi)/\log T}\frac{1}{\gamma} \\ &= \sum_{0<\gamma\leqslant X}\frac{1}{\gamma}\Big(N(\gamma+\frac{2\pi}{\log T}) - N(\gamma)\Big) \\ &\ll \sum_{0<\gamma\leqslant X}\frac{1}{\gamma}\Big(1+S(\gamma+\frac{2\pi}{\log T}) - S(\gamma)\Big).\end{aligned} \tag{2.14}$$

To bound the last sum in (2.14) we invoke the estimate

$$\sum_{0<\gamma\leqslant Q,\gamma+a>0} S(\gamma+a) \ll Q\log Q \qquad (0\leqslant |a| \leqslant Q, a\in\mathbb{R}) \tag{2.15}$$

of A. Fujii [3]. Hence, by partial summation, (2.15) yields

$$\sum(T) \ll \log^2 X + \frac{1}{X}X\log X + \int_1^X \frac{x\log x}{x^2}\,\mathrm{d}x \ll \log^2 T. \tag{2.16}$$

Inserting (2.16) in (2.10) we complete the proof of Theorem 1.1.

Concerning (2.15) Fujii even conjectures that, for any given $\alpha > 0$ and $T\to\infty$ one has

$$\sum_{0<\gamma\leqslant T} S\left(\gamma - \frac{2\pi\alpha}{\log T/(2\pi)}\right) = \frac{T}{2\pi}\left\{\int_0^\alpha\Big(\frac{\sin\pi t}{\pi t}\Big)^2\mathrm{d}t + o(1)\right\}.$$

This is closely related to H.L. Montgomery's pair correlation conjecture [8] for the distribution of the zeros of $\zeta(s)$.

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