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SPECIAL CASES OF ORTHOGONAL POLYNOMIALS ON THE SEMICIRCLE AND APPLICATIONS IN NUMERICAL ANALYSIS

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A b s t r a c t. Orthogonal polynomials on the semicircle was introduced by Gautschi and Milovanović in [*Rend. Sem. Mat. Univ. Politec. Torino, Special Issue* (*July* 1985)*,* 179 − 185] *and* [*J. Approx. Theory* **46** (1986)*,* 230 − 250]*. In this paper we give an account of this kind of orthogonality, weighted generalizations mainly oriented to Chebyshev weights of the first and second kind, as well as the corresponding applications in numerical analysis. Moreover, we also present a number of new results including those for Laurent polynomials orthogonal to a semicircle and applications to quasi-singular integrals.*

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1. *Introduction and preliminaries*

In this paper we consider some classes of polynomials, as well as rational functions (precisely the so-called Laurent polynomials), orthogonal on the semicircle in the complex plane $\mathbb{C}$.

First, we start with the well-known definition of the inner product space and orthogonal systems (cf. [19, p. 75]).

Definition 1.1. Let X be a complex linear space of functions with an inner (scalar) product $(f, g) : X^2 \to \mathbb{C}$ such that

$$
\begin{array}{lll}
\text{(a)} & (f+g,h) = (f,h) + (g,h) & \text{(Linearity)}, \\
\text{(b)} & (\alpha f, g) = \alpha (f,g) & \text{(Homogeneity)}, \\
\text{(c)} & (f,g) = \overline{(g,f)} & \text{(Hermitian Symmetry)}, \\
\text{(d)} & (f,f) > 0\,,\ (f,f) = 0 \Leftrightarrow f = 0 & \text{(Positivity)},
\end{array}
$$

where $f, g, h \in X$ and α is a complex scalar. The bar in the above line denotes the complex conjugate. The space X is called an inner product space.

If X is a real linear space of functions, then the inner product $(f, g) : X^2 \to \mathbb{R}$ is such that the condition (c) is replaced by

$$
\text{(c}'\text{)} \quad (f,g) = (g,f) \qquad \text{(Symmetry)}.
$$

Several examples of interesting orthogonal systems are presented in the monograph [19, pp. 79–89]. A standard system of orthogonal polynomials $\{p_k\}$, where

$$
p_k(t) = t^k + \text{terms of lower degree}, \quad k = 0, 1, \ldots\,, \tag{1.1}
$$

and

$$
(p_k, p_m) = 0 \quad (k \neq m), \quad (p_k, p_m) > 0 \quad (k = m)\,,
$$

is called a system of (monic) orthogonal polynomials with respect to the inner product $(\cdot\,,\,\cdot)$. With $\mathcal{P}$ we denote the space of all algebraic polynomials, and with $\mathcal{P}_n$ $(\subset \mathcal{P})$ its subset of polynomials of degree at most n.

In this section we mention only three types of orthogonal polynomials (on the real line, unit circle, and unit semicircle) in the separate subsections. Some considerations on orthogonality on the semicircle with respect to the Gegenbauer weight function is treated in Section 2. Section 3 is devoted to integration of quasi-singular integrals and introducing orthogonal Laurent's polynomials on the semicircle. Some properties of mentioned Laurent's polynomials will be proved elsewhere.

1.1. Orthogonal polynomials on the real line

The most common type of orthogonality is one with respect to the following inner product on $\mathbb{R}$,

$$
(f, g) = \int_{\mathbb{R}} f(t)\overline{g(t)}\,\mathrm{d}\lambda(t)\,, \tag{1.2}
$$

where $\mathrm{d}\lambda(t)$ is a nonnegative measure on the real line $\mathbb{R}$, with finite or unbounded support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k\,\mathrm{d}\lambda(t),\ k = 0, 1, \ldots,$ exist and are finite, and $\mu_0 > 0$. If λ is an absolutely continuous function, then we say that $\lambda'(t) = w(t)$ is a weight function and in that case, the measure $\mathrm{d}\lambda(t)$ can be expressed as $\mathrm{d}\lambda(t) = w(t)\,\mathrm{d}t$. In the general case, the function λ can be written in the form $\lambda = \lambda_{\mathrm{ac}} + \lambda_{\mathrm{s}} + \lambda_{\mathrm{j}}$, where λ_{ac} is absolutely continuous, λ_{s} is singular, and λ_{j} is a jump function.

Since the inner product (1.2) in this case has the following property $(tf, g) = (f, tg)$, the (monic) orthogonal polynomials $p_k(t) \equiv p_k(\mathrm{d}\lambda; t)$ satisfy the fundamental three-term recurrence relation

$$\begin{aligned} &p_{k+1}(t) = (t - a_k)p_k(t) - b_k p_{k-1}(t), \quad k = 0, 1, 2, \ldots, \\ &p_{-1}(t) = 0, \ \ p_0(t) = 1, \end{aligned} \tag{1.3}$$

where the coefficients a_k and b_k are given by

$$a_k = \frac{(tp_k, p_k)}{(p_k, p_k)}, \qquad k = 0, 1, 2, \ldots ,$$

$$b_k = \frac{(p_k, p_k)}{(p_{k-1}, p_{k-1})}, \quad k = 1, 2, \ldots ,$$

and they depend only on the measure $\mathrm{d}\lambda(t)$ (or the weight function w). We note that $b_k > 0,\ k \geq 1$. The coefficient b_0 in (1.3) can be arbitrary, but the definition $b_0 = \mu_0 = \int_{\mathbb{R}} \mathrm{d}\lambda(t)$ is sometimes convenient.

These polynomials $p_n(\mathrm{d}\lambda; t)$, $n \geq 1$, orthogonal with respect to the inner product (1.2), have only real zeros, mutually different and all these zeros are located in the support of the measure $\mathrm{d}\lambda(t)$. Furthermore, the zeros of $p_n(t)$ and $p_{n+1}(t)$ interlace, i.e.,

$$\tau_1^{(n+1)} < \tau_1^{(n)} < \tau_2^{(n+1)} < \tau_2^{(n)} < \cdots < \tau_n^{(n+1)} < \tau_n^{(n)} < \tau_{n+1}^{(n+1)},$$

where $\tau_1^{(n)} < \tau_2^{(n)} < \cdots < \tau_n^{(n)}$ denote the zeros of $p_n(\mathrm{d}\lambda; t)$ in increasing order (for proofs see [19, pp. 99–101]).

The typical examples of these polynomials are *classical orthogonal polynomials* of Jacobi, Laguerre and Hermite (cf. [19, p. 121–146]), for which the recurrence coefficients a_k and b_k in (1.3) are known in the explicit form. There are many important applications of these polynomial in different areas of mathematics (approximation theory, numerical analysis, ...), mathematical physics, as well as in other computational and applied sciences.

The same recursion coefficients a_k and b_k appear in the Jacobi continued fraction associated with the measure $\mathrm{d}\lambda$ (Stieltjes transform of the measure)

$$F(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\lambda(t)}{z - t} \sim \frac{b_0}{z - a_0 -}\,\frac{b_1}{z - a_1 -}\,\cdots$$

The n-th convergent of this continued fraction is a rational function

$$R_n(z) = \frac{b_0}{z - a_0 -} \, \frac{b_1}{z - a_1 -} \cdots \frac{b_{n-1}}{z - a_{n-1}} = \frac{\sigma_n(z)}{p_n(z)},$$

with simple poles at the zeros of $p_n(z)$, $z = \tau_k^{(n)}$, $k = 1, \ldots, n$. The numerators $\sigma_n(z)$ ($\deg \sigma_n = n - 1$) in $R_n(z)$ are the so-called associated polynomials, which satisfy the same three-term recurrence relation, but with the different starting values ($\sigma_0(z) = 0$, $\sigma_{-1} = -1$). Expanding $R_n(z)$ in partial fractions we get

$$R_n(z) = \frac{\sigma_n(z)}{p_n(z)} = \sum_{k=1}^{n} \frac{A_k^{(n)}}{z - \tau_k^{(n)}}. \tag{1.4}$$

On the other side, let $\mathrm{d}\lambda(t)$ be as in (1.2). Then, for each $n \in \mathbb{N}$, there exists the n-point Gauss-Christoffel quadrature formula

$$\int_{\mathbb{R}} f(t) \, \mathrm{d}\lambda(t) = \sum_{k=1}^{n} A_k^{(n)} f(\tau_k^{(n)}) + R_n(f), \tag{1.5}$$

which is exact for all algebraic polynomials of degree at most $2n-1$, i.e., $R_n(f) = 0$ for each $f \in \mathcal{P}_{2n-1}$.

There is a deep connection between (1.4) and the Gauss-Christoffel quadrature formula (1.5). Namely, the coefficients $A_k^{(n)}$ in (1.4) are exactly the weight coefficients (Christoffel numbers) in (1.5) and zeros of the polynomial $\pi_n(t)$ are the nodes of (1.5), so that orthogonal polynomials are main tool in construction of Gaussian quadrature formulas.

The quadrature nodes $\tau_1^{(n)}, \ldots, \tau_n^{(n)}$ in (1.5) are eigenvalues of the symmetric tridiagonal Jacobi matrix

$$J_n(\mathrm{d}\lambda) = \begin{bmatrix} a_0 & \sqrt{b_1} & & & \mathbf{O} \\ \sqrt{b_1} & a_1 & \sqrt{b_2} & & \\ & \sqrt{b_2} & a_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{b_{n-1}} \\ \mathbf{O} & & & \sqrt{b_{n-1}} & a_{n-1} \end{bmatrix},$$

and the weight coefficients are given by $A_k^{(n)} = b_0 v_{k,1}^2$, $k = 1, \ldots, n$, where $v_{k,1}$ is the first component of the normalized eigenvector $\mathbf{v}_k = [v_{k,1} \; \ldots \; v_{k,n}]^{\mathrm{T}}$ such that $\mathbf{v}_k^{\mathrm{T}} \mathbf{v}_k = 1$. The most popular method for solving this eigenvalue problem is the Golub-Welsch procedure [15], obtained by a simplification of the QR algorithm. Thus, the knowledge of the coefficients a_k and b_k in the recurrence relation (1.3) is of exceptional importance, which are unfortunately known in explicit form only for a narrow class of orthogonal polynomials, including classical orthogonal polynomials.

For a wide class of the so-called *strong nonclassical orthogonal polynomials*, these recursive coefficients must be constructed numerically. Such approaches belong to the *constructive theory of orthogonal polynomials* developed by Walter Gautschi in the 1980s (see [7], [8], [22]).

In general, in numerical construction of recursion coefficients an important aspect is the sensitivity of the problem with respect to small perturbation in the input. Recent progress in symbolic computation and variable-precision arithmetic now makes it possible to generate a_k and b_k directly by using the original Chebyshev method of moments, but in sufficiently high precision. Such an approach enables us to overcome the numerical instability! Symbolic/variable-precision software for orthogonal polynomials is available:

- Gautschi's package `SOPQ` in MATLAB (see [9, 10]
- Our MATHEMATICA package `OrthogonalPolynomials` (see [5], [23]).

Both packages are freely downloadable.

1.2. Orthogonal polynomials on the unit circle

The second type of important orthogonality is one on the unit circle, with respect to the inner product

$$(f, g) = \int_0^{2\pi} f\left(\mathrm{e}^{\mathrm{i}\theta}\right) \overline{g\left(\mathrm{e}^{\mathrm{i}\theta}\right)} \,\mathrm{d}\mu(\theta), \quad \mathrm{d}\mu(\theta) \geq 0.$$

The polynomials $\phi_k(z)$ orthogonal with respect to this inner product have been introduced and studied by Szegő [31, 32]. The inner product has not the property $(zf, g) = (f, zg)$ and therefore the three-term recurrence relation does not exist! But, $(zf, zg) = (f, g)$, which is important in proving that all zeros of $\phi_k(z)$ are inside the unit circle $|z| = 1$.

The monic orthogonal polynomials $\{\phi_k\}$ on the unit circle $|z| = 1$, for $k = 0, 1, \ldots,$ satisfy the recurrence relations

$$\phi_{k+1}(z) = z\phi_k(z) + \phi_{k+1}(0)\phi_k^*(z), \quad \phi_{k+1}^*(z) = \phi_k^*(z) + \overline{\phi_{k+1}(0)} z\phi_k^*(z),$$

where $\phi_k^*(z) = z^k \overline{\phi}_k(1/z)$.

For details see Nevai [25], as well as an extensive book in two volumes published by Barry Simon in 2005 [29, 30]. These polynomials have many applications in theory of time series, digital filters, statistics, image processing, scattering theory, control theory, etc.

Similarly, orthogonal polynomials on a rectifiable curve or arc lying in the complex plane can be considered (see for example, [14] and [33]). Also, complex orthogonal polynomials may be constructed with double integrals. For example, by

$$(f, g) = \iint_B f(z)\,\overline{g(z)}\,w(z)\,\mathrm{d}x\,\mathrm{d}y$$

for a suitable positive weight functions $w(z)$, where B is a bounded region lying in the complex plane, a system of orthogonal polynomials can be generated (see Carlemann [3] and Bochner [1]).

1.3. Motivation for introducing orthogonal polynomials on the semicircle

In the eighties of the last century, one new type of orthogonality – orthogonality on the semicircle – was introduced by Gautschi and Milovanović [13] (see also [12]). We give first a motivation for this kind of orthogonality.

One of the attractive problems in the area of Numerical Integration after 1970 was the numerical integration of the Cauchy principal value integrals, e.g.,

$$\text{v.p.} \int_a^b \frac{f(x)w(x)}{x-c}\,\mathrm{d}x, \quad a < c < b \tag{1.6}$$

(see, for example, [6, 16, 17, 18, 24, 26, 27, 28, 34]). Our main idea in computing (1.6) for a holomorphic function $z \mapsto f(z)$ in a domain $D \subset \mathbb{C}$, containing the segment $[a, b]$, was to integrate the complex function

$$z \mapsto F(z) = \frac{f(z)}{z-c} \quad (z \in D)$$

over some contour $\Gamma \in D$, which connects the points $z = a$ and $z = b$, bypassing the point $z = c$ (because of simplicity we put $w(z) = 1$). Namely, taking a contour C and a semicircle γ_ε, with a small radius ε, so that $C = [-1, c-\varepsilon] \cup \gamma_\varepsilon \cup [c+\varepsilon] \cup \Gamma$ is a closed contour (see Figure 1), we can apply Cauchy's theorem. Thus, we have

$$\int_a^{c-\varepsilon} F(z)\,\mathrm{d}z + \int_{\gamma_\varepsilon} F(z)\,\mathrm{d}z + \int_{c+\varepsilon}^b F(z)\,\mathrm{d}z + \int_\Gamma F(z)\,\mathrm{d}z = 0,$$

i.e.,

$$\text{v.p.} \int_a^b \frac{f(x)}{x-c}\,\mathrm{d}x = -\int_\Gamma F(z)\,\mathrm{d}z - \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} \frac{f(z)}{z-c}\,\mathrm{d}z,$$

where the last term on the right side can be expressed in an explicit form over residue of the function $z \mapsto F(z)$ at the point $z = c$, i.e., $\mathrm{i}\pi f(c)$. This means that we have replaced a direct calculation of the Cauchy principal value integrals with the corresponding integrals over the curve Γ.

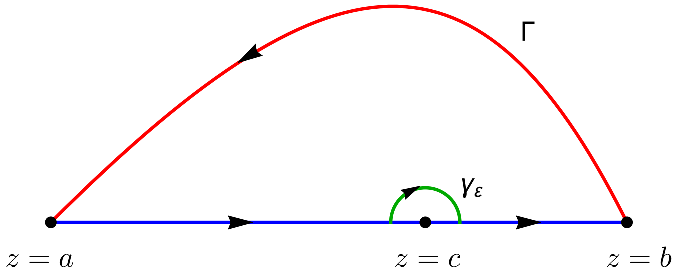

Figure 1: Bypassing the point $z = c$ over the contour Γ

Suppose now that $a = -1$, $b = 1$, $c = 0$, and let the contour Γ be a unit semicircle $\Gamma = \{z = \mathrm{e}^{\mathrm{i}\theta} : 0 \le \theta \le \pi\}$ and $w(z) = 1$. Then the integral in this symmetric case over the semicircle Γ, after putting $z = \mathrm{e}^{\mathrm{i}\theta}$, becomes

$$\int_\Gamma F(z)\,\mathrm{d}z = \int_\Gamma \frac{f(z)}{z}\,\mathrm{d}z = \mathrm{i}\int_0^\pi f\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta. \tag{1.7}$$

Remark 1.1. The Cauchy principal value integral (1.6), for $a = -1$, $b = 1$, $-1 < c < 1$ and the Gegenbauer $w(z) = (1 - z^2)^{\lambda - 1/2}$, $\lambda > -1/2$, using bilinear transformation $t = (x + c)/(cx + 1)$, can be transformed to the symmetric case as follows

$$\text{v.p.}\int_{-1}^1 \frac{f(t)w(t)}{t-c}\,\mathrm{d}t = w(c)\ \text{v.p.}\int_{-1}^1 \frac{g(x;c)w(x)}{x}\,\mathrm{d}x,$$

where

$$g(x;c) = \frac{1}{(cx+1)^{2\lambda}} f\left(\frac{x+c}{cx+1}\right).$$

If we want to calculate the last integral in (1.7), with quadrature formulas of the maximal degree of exactness, i.e., with formulas of Gaussian type, we need orthogonal polynomials with respect to the inner product

$$\langle f, g\rangle = \int_0^\pi f\big(\mathrm{e}^{\mathrm{i}\theta}\big) g\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta, \tag{1.8}$$

or alternatively,

$$\langle f, g\rangle = \int_\Gamma f(z) g(z) (\mathrm{i}z)^{-1}\,\mathrm{d}z\,. \tag{1.9}$$

Note that the product introduced in this way does not satisfy the conditions (c) and (d) in Definition 1.1. Namely, the second factor in (1.8), i.e., (1.9), is not conjugated, so that this product has no Hermitian Symmetry, but it possesses the standard Symmetry property (c').

However, the corresponding (monic) orthogonal polynomials with respect to this not-hermitian inner product exist uniquely and satisfy a three-term recurrence relation like (1.3), because of the property $\langle zf, g\rangle = \langle f, zg\rangle$. Otherwise, this kind of orthogonality can be correctly treated in this case, using the approach based on a *complex moment functional* (see Chihara [4, pp. 6–10]) given by

$$\mathcal{L}z^k = \mu_k, \quad \mu_k = \int_0^\pi \mathrm{e}^{\mathrm{i}k\theta}\,\mathrm{d}\theta = \begin{cases} \pi, & k = 0, \\ 2\mathrm{i}/k, & k \text{ odd}, \\ 0, & k \text{ even}, \ k \neq 0. \end{cases} \tag{1.10}$$

After much calculation, using moment determinants, Gautschi and Milovanović [13] obtained the three-term recurrence relation for the monic orthogonal polynomials $\{\pi_k\}$,

$$\pi_{k+1}(z) = (z - \mathrm{i}\alpha_k)\pi_k(z) - \beta_k\pi_{k-1}(z), \quad k = 0, 1, \ldots, \tag{1.11}$$

with starting polynomials $\pi_{-1}(z) = 0$, $\pi_0(z) = 1$, where

$$\alpha_0 = \theta_0, \quad \alpha_k = \theta_k - \theta_{k-1}, \quad \beta_k = \theta_{k-1}^2, \ \ k \geq 1,$$

and

$$\theta_k = \frac{2}{2k+1}\left[\frac{\Gamma((k+2)/2)}{\Gamma((k+1)/2)}\right]^2, \quad k \geq 0. \tag{1.12}$$

The sequence (1.12) can be also expressed in the form

$$\theta_k = \begin{cases} \dfrac{2^{2k+1}}{\pi(2k+1)}\dbinom{k}{k/2}^{-2}, & k \text{ is even}, \\[2ex] \dfrac{\pi(k+1)^2}{(2k+1)2^{2k+1}}\dbinom{k}{(k-1)/2}^{2}, & k \text{ is odd}. \end{cases}$$

Otherwise, this sequence is

$$\{\theta_k\}_{k=0}^{\infty} = \left\{\frac{2}{\pi}, \frac{\pi}{6}, \frac{8}{5\pi}, \frac{9\pi}{56}, \frac{128}{81\pi}, \frac{225\pi}{1408}, \frac{512}{325\pi}, \frac{245\pi}{1536}, \frac{32768}{20825\pi}, \frac{99225\pi}{622592}, \ldots\right\},$$

and polynomials $\pi_k(z)$, $k = 0, 1, \ldots$, are

$$\pi_0(z) = 1, \quad \pi_1(z) = z - \frac{2\mathrm{i}}{\pi}, \quad \pi_2(z) = z^2 - \frac{\mathrm{i}\pi}{6} z - \frac{1}{3},$$

$$\pi_3(z) = z^3 - \frac{8\mathrm{i}}{5\pi} z^2 - \frac{3}{5} z + \frac{8\mathrm{i}}{15\pi},$$

$$\pi_4(z) = z^4 - \frac{9}{56}\mathrm{i}\pi z^3 - \frac{6}{7} z^2 + \frac{27\mathrm{i}\pi}{280} z + \frac{3}{35},$$

$$\pi_5(z) = z^5 - \frac{128\mathrm{i}}{81\pi} z^4 - \frac{10}{9} z^3 + \frac{256\mathrm{i}}{189\pi} z^2 + \frac{5}{21} z - \frac{128\mathrm{i}}{945\pi},$$

$$\pi_6(z) = z^6 - \frac{225\mathrm{i}\pi}{1408} z^5 - \frac{15}{11} z^4 + \frac{125}{704}\mathrm{i}\pi z^3 + \frac{5}{11} z^2 - \frac{375\mathrm{i}\pi}{9856} z - \frac{5}{231}, \quad \text{etc.}$$

It was proved that these polynomials can be expressed in terms of the monic Legendre polynomials $\widehat{P}_k(z)$ as

$$\pi_k(z) = \widehat{P}_k(z) - \mathrm{i}\theta_{k-1}\widehat{P}_{k-1}(z), \quad k \geq 1,$$

where θ_k is given by (1.12).

Also we proved that all zeros of $\pi_k(z)$ are simple, contained in the upper semidisc $D_+ = \{z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im} z > 0\}$ and located symmetrically with respect to the imaginary axis. Zeros of $\pi_k(z)$ for $k = 3$, 6, and 10 are presented in Figure 2. With increasing k the zeros tend to fall to the interval $[-1, 1]$.

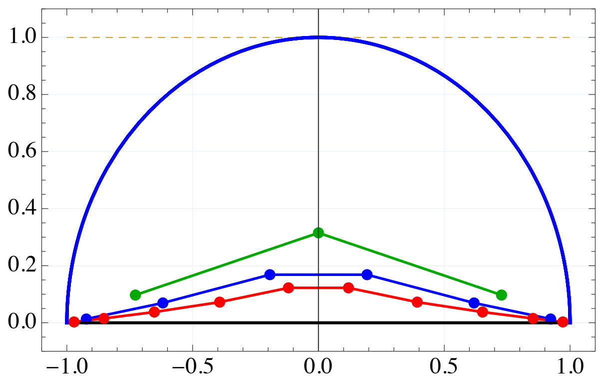

Figure 2: Zeros of $\pi_k(z)$ for $k = 3$ (green), $k = 6$ (blue) and $k = 10$ (red)

2. *Some considerations on orthogonality on the semicircle with respect to the Gegenbauer weight function*

A more general problem with a complex weight function w, which is positive and integrable on the open interval $(-1,1)$, though possibly singular at the endpoints, and which can be extended to a function $w(z)$ holomorphic in the half disc $D_+ = \{z \in \mathbb{C} : |z| < 1,\ \mathrm{Im}\, z > 0\}$ were studied in [11] and [21]. Namely, we considered the following two inner products (see [11]), one on the real line as (1.2) with $\mathrm{d}\lambda(x) = w(x)\,\mathrm{d}x$ on $[-1,1]$, i.e.,

$$(f,g) = \int_{-1}^{1} f(x)\overline{g(x)}w(x)\,\mathrm{d}x, \tag{2.1}$$

and the second one on the semicircle Γ,

$$\langle f,g\rangle = \int_{\Gamma} f(z)g(z)w(z)(\mathrm{i}z)^{-1}\,\mathrm{d}z = \int_0^{\pi} f\big(\mathrm{e}^{\mathrm{i}\theta}\big)g\big(\mathrm{e}^{\mathrm{i}\theta}\big)w\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta, \tag{2.2}$$

and established the existence of orthogonal polynomials $\{\pi_k\}$ on the semicircle with respect to the non-Hermitian product (2.2), assuming only that

$$\mathrm{Re}\langle 1,1\rangle = \mathrm{Re}\int_0^{\pi} w\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta \neq 0. \tag{2.3}$$

The first inner product (2.1) is positive definite and evidently generates a unique system of real orthogonal polynomials $\{p_k\}$, which satisfy the three-term recurrence relation (1.3).

In order to connect these system of polynomials, $\{\pi_k\}$ and $\{p_k\}$, i.e., the inner product (2.1) and not-hermitian product (2.2), when the functions f and g are algebraic polynomials, we take a contour C_ε, $\varepsilon > 0$, with small circular parts of radius ε and centers at ± 1 (see Figure 3), i.e.,

$$C_\varepsilon = [-1+\varepsilon, 1-\varepsilon] \cup \gamma_{\varepsilon,1} \cup \Gamma_\varepsilon \cup \gamma_{\varepsilon,-1},$$

and consider the weighted integral of an arbitrary polynomial $g \in \mathcal{P}$ over C_ε. Then, by Cauchy's theorem, we have $\int_{C_\varepsilon} g(z)w(z)\,\mathrm{d}z = 0$. Supposing that the weight function w is such that integrals over $\gamma_{\varepsilon,\pm 1}$ tend to zero when $\varepsilon \to 0$, we obtain in that case the following connection

$$\int_{\Gamma} g(z)w(z)\,\mathrm{d}z + \int_{-1}^{1} g(z)w(z)\,\mathrm{d}z = 0$$

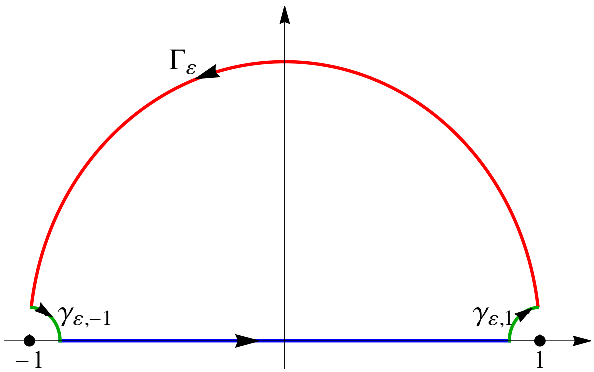

Figure 3: The contour $C_\varepsilon = [-1+\varepsilon, 1-\varepsilon] \cup \gamma_{\varepsilon,1} \cup \Gamma_\varepsilon \cup \gamma_{\varepsilon,-1}$

for each $g \in \mathcal{P}$. It enables us to express the polynomials π_k in terms of the real polynomials $\{p_k\}$,

$$\pi_k(z) = p_k(z) - \mathrm{i}\theta_{k-1} p_{k-1}(z), \quad k = 0, 1, \ldots,$$

where

$$\theta_{k-1} = \frac{\mu_0 p_k(0) + \mathrm{i} q_k(0)}{\mathrm{i}\mu_0 p_{k-1}(0) - q_{k-1}(0)}, \quad k = 0, 1, \ldots, \tag{2.4}$$

and $\{q_k\}$ are the associated polynomials, defined by (cf. [19, pp. 111–114])

$$q_k(z) = \int_{-1}^{1} \frac{p_k(z) - p_k(x)}{z - x} w(x)\,\mathrm{d}x, \quad k = 0, 1, \ldots.$$

The sequence (2.4) can be expressed also in the form $\theta_k = \mathrm{i}a_k + b_k/\theta_{k-1}$, $\theta_{-1} = \mu_0$, where a_k and b_k are the recursion coefficients in the three-term relation (1.3) for the real orthogonal polynomials $\{p_k\}$.

The polynomials π_k satisfy the three-term recurrence relation of the form (1.11), where the coefficients α_k and β_k are given by

$$\alpha_0 = \frac{b_0}{\mu_0}, \quad \alpha_k = -\theta_{k-1} + \frac{b_k}{\theta_{k-1}} \quad (k \geq 1)$$

and

$$\beta_k = \frac{\theta_{k-1}}{\theta_{k-2}}\, b_{k-1} = \theta_{k-1}(\theta_{k-1} - \mathrm{i}a_{k-1}).$$

For details see [11].

Under certain conditions, all zeros of polynomials $\{\pi_k\}$ orthogonal on the semicircle are in D_+.

Some applications of polynomials $\{\pi_k\}$ orthogonal on the semicircle in numerical differentiation and numerical integration were given in [2] and [20].

Several interesting properties of such polynomials $\{\pi_k\}$ were shown in [11] and [21], especially for Gegenbauer weight function $w(z) = w^\lambda(z) = (1-z^2)^{\lambda-1/2}$, with the parameter $\lambda > -1/2$. Since the assumption (2.3) is satisfied ($\mu_0 = \pi \neq 0$) in that case, the corresponding polynomials $\{\pi_k^\lambda\}$ orthogonal on the semicircle Γ can be expressed in terms of monic Gegenbauer polynomials $\widehat{C}_k(z)$ as

$$\pi_k^\lambda(z) = \widehat{C}_k^\lambda(z) - \mathrm{i}\theta_{k-1}\widehat{C}_{k-1}^\lambda(z), \tag{2.5}$$

where the sequence $\{\theta_{k-1}\}$ is given recursively by

$$\theta_0 = \frac{\Gamma\left(\lambda+\dfrac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\lambda+1)}, \quad \theta_k = \frac{k(k+2\lambda-1)}{4(k+\lambda)(k+\lambda-1)}\cdot\frac{1}{\theta_{k-1}}, \quad k=1,2,\ldots,$$

wherefrom we can obtain an explicit form in terms of the gamma function,

$$\theta_k = \frac{1}{\lambda+k}\cdot\frac{\Gamma\left(\dfrac{k+2}{2}\right)\Gamma\left(\lambda+\dfrac{k+1}{2}\right)}{\Gamma\left(\dfrac{k+1}{2}\right)\Gamma\left(\lambda+\dfrac{k}{2}\right)}, \quad k \geq 0. \tag{2.6}$$

These polynomials $\{\pi_k\}$ satisfy the three-term recurrence relation (1.11), where

$$\alpha_0 = \theta_0, \quad \alpha_k = \theta_k - \theta_{k-1}, \quad \beta_k = \theta_{k-1}^2, \quad k \geq 1.$$

Using Stirling's formula in (2.6), we find that

$$\theta_k \to \frac{1}{2}, \ \alpha_k \to 0, \ \beta_k \to \frac{1}{4}, \text{ when } k \to +\infty.$$

Especially, interesting cases are:

(1) $\lambda = 0$: *Chebyshev polynomials of the first kind*

$$T_k(t) = \cos(k\theta), \quad t = \cos\theta,$$

orthogonal on $(-1,1)$, with respect to the weight function $w^0(t) = 1/\sqrt{1-t^2}$.

(2) $\lambda = 1$: *Chebyshev polynomials of the second kind*

$$U_k(t) = \frac{\sin((k+1)\theta}{\sin\theta}, \quad t = \cos\theta,$$

orthogonal on $(-1, 1)$, with respect to the weight function $w^1(t) = \sqrt{1-t^2}$.

Their three-term recurrence relations are the same (cf. [19, p. 10])

$$T_{k+1}(t) = 2tT_k(t) - T_{k-1}(t), \quad U_{k+1}(t) = 2tU_k(t) - U_{k-1}(t),$$

but with different starting polynomials

$$T_0(t) = 1,\ T_1(t) = t \quad \text{and} \quad U_0(t) = 1,\ U_1(t) = 2t.$$

For the corresponding sequences (2.6) we have here (see [21]):

(1) for $\lambda = 0$, $\quad \theta_0 = 1, \theta_k = \frac{1}{2},\ \ k \geq 1;$

(2) for $\lambda = 1$, $\quad \theta_k = \frac{1}{2},\ \ k \geq 0.$

Using these values of θ_k, the relation (2.5), as well as the representations of Chebyshev polynomials in the complex plane [19, §1.1.4] we can prove the following two theorems:

Theorem 2.1. *The monic polynomials orthogonal on the semicircle* Γ *with respect to the complex function* $w(z) = w^0(z) = (1-z^2)^{-1/2}$ *can be expressed in terms of Chebyshev polynomials of the first kind*

$$\pi_0^0(z) = 1, \quad \pi_k^0(z) = \frac{1}{2^{k-1}}\left(T_k(z) - \mathrm{i}T_{k-1}(z)\right),\ \ k \geq 1,$$

or in the explicit form

$$\pi_k^0(z) = \frac{1}{2^k}\left\{\left[1 - \mathrm{i}\left(z - \sqrt{z^2-1}\right)\right]\left(z + \sqrt{z^2-1}\right)^k + \left[1 - \mathrm{i}\left(z + \sqrt{z^2-1}\right)\right]\left(z - \sqrt{z^2-1}\right)^k\right\}. \tag{2.7}$$

Here, $|z + \sqrt{z^2-1}| > 1$, *when* $z \in \mathbb{C} \setminus [-1, 1]$.

A few first polynomials in this sequence are

$$\pi_0^0(z) = 1,\ \ \pi_1^0(z) = z - \mathrm{i},\ \ \pi_2^0(z) = z^2 - \frac{\mathrm{i}}{2}z - \frac{1}{2},$$

$$\pi_3^0(z) = z^3 - \frac{\mathrm{i}}{2}z^2 - \frac{3}{4}z + \frac{\mathrm{i}}{4},\ \ \pi_4^0(z) = z^4 - \frac{\mathrm{i}}{2}z^3 - z^2 + \frac{3\mathrm{i}}{8}z + \frac{1}{8},$$

$$\pi_5^0(z) = z^5 - \frac{\mathrm{i}}{2}z^4 - \frac{5}{4}z^3 + \frac{\mathrm{i}}{2}z^2 + \frac{5}{16}z - \frac{\mathrm{i}}{16},$$

$$\pi_6^0(z) = z^6 - \frac{\mathrm{i}}{2}z^5 - \frac{3}{2}z^4 + \frac{5\mathrm{i}}{8}z^3 + \frac{9}{16}z^2 - \frac{5\mathrm{i}}{32}z - \frac{1}{32},$$

$$\pi_7^0(z) = z^7 - \frac{i}{2}z^6 - \frac{7}{4}z^5 + \frac{3i}{4}z^4 + \frac{7}{8}z^3 - \frac{9i}{32}z^2 - \frac{7}{64}z + \frac{i}{64},$$

$$\pi_8^0(z) = z^8 - \frac{i}{2}z^7 - 2z^6 + \frac{7i}{8}z^5 + \frac{5}{4}z^4 - \frac{7i}{16}z^3 - \frac{1}{4}z^2 + \frac{7i}{128}z + \frac{1}{128}, \quad \text{etc.}$$

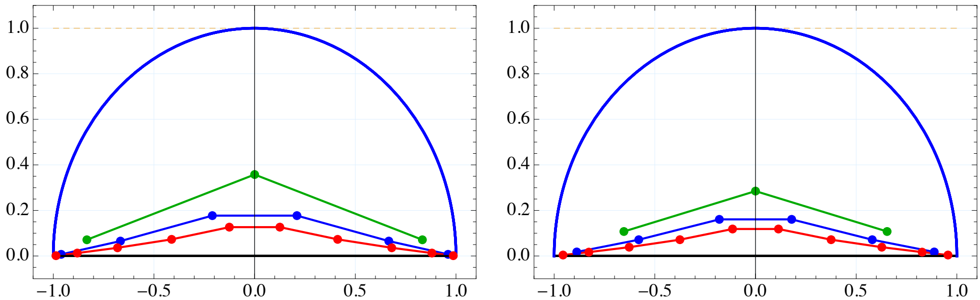

Figure 4: Zeros of $\pi_k^\lambda(z)$ for $k = 3$ (green), $k = 6$ (blue) and $k = 10$ (red) for $\lambda = 0$ (left) and $\lambda = 1$ (right)

Theorem 2.2. *The monic polynomials orthogonal on the semicircle* Γ *with respect to the complex function* $w(z) = w^1(z) = (1-z^2)^{1/2}$ *can be expressed in terms of Chebyshev polynomials of the second kind*

$$\pi_0^1(z) = 1, \quad \pi_k^1(z) = \frac{1}{2^k}\left(U_k(z) - iU_{k-1}(z)\right), \quad k \geq 1,$$

or in the explicit form

$$\pi_k^1(z) = \frac{1}{2^{k+1}\sqrt{z^2-1}}\left\{\left[\left(z+\sqrt{z^2-1}\right) - i\right]\left(z+\sqrt{z^2-1}\right)^k - \left[\left(z-\sqrt{z^2-1}\right) - i\right]\left(z-\sqrt{z^2-1}\right)^k\right\}. \qquad (2.8)$$

Here, $|z+\sqrt{z^2-1}| > 1$*, when* $z \in \mathbb{C} \setminus [-1,1]$.

A few first polynomials in this sequence are

$$\pi_0^1(z) = 1, \quad \pi_1^1(z) = z - \frac{i}{2}, \quad \pi_2^1(z) = z^2 - \frac{i}{2}z - \frac{1}{4},$$

$$\pi_3^1(z) = z^3 - \frac{i}{2}z^2 - \frac{1}{2}z + \frac{i}{8}, \quad \pi_4^1(z) = z^4 - \frac{i}{2}z^3 - \frac{3}{4}z^2 + \frac{i}{4}z + \frac{1}{16},$$

$$\pi_5^1(z) = z^5 - \frac{\mathrm{i}}{2}z^4 - z^3 + \frac{3\mathrm{i}}{8}z^2 + \frac{3}{16}z - \frac{\mathrm{i}}{32},$$

$$\pi_6^1(z) = z^6 - \frac{\mathrm{i}}{2}z^5 - \frac{5}{4}z^4 + \frac{\mathrm{i}}{2}z^3 + \frac{3}{8}z^2 - \frac{3\mathrm{i}}{32}z - \frac{1}{64},$$

$$\pi_7^1(z) = z^7 - \frac{\mathrm{i}}{2}z^6 - \frac{3}{2}z^5 + \frac{5\mathrm{i}}{8}z^4 + \frac{5}{8}z^3 - \frac{3\mathrm{i}}{16}z^2 - \frac{1}{16}z + \frac{\mathrm{i}}{128},$$

$$\pi_8^1(z) = z^8 - \frac{\mathrm{i}}{2}z^7 - \frac{7}{4}z^6 + \frac{3\mathrm{i}}{4}z^5 + \frac{15}{16}z^4 - \frac{5\mathrm{i}}{16}z^3 - \frac{5}{32}z^2 + \frac{\mathrm{i}}{32}z + \frac{1}{256}, \quad \text{etc.}$$

3. *Integration of quasi-singular integrals and orthogonality on the semicircle*

As we mentioned in Section 2, using polynomials $\{\pi_k\}$ we can construct quadrature formulas for Cauchy Principal Value integrals by construction quadrature formulas of maximal degree of exactness over semicircle ([12, 20, 21]). The nodes of such formulas are inside of the half disc $D_+ = \{z \in \mathbb{C} \,:\, |z| < 1,\ \mathrm{Im}\, z > 0\}$ and we can see that with increasing the number of nodes k, these zeros of $\pi_k(z)$ tend to fall to the interval $[-1, 1]$ (see Figures 2 and 4). Because of that such quadrature formulas cannot be applied to calculation of real quasi-singular integrals. As an illustration we consider a simple example with quasi-singular integral

$$\int_{-1}^{1} \frac{f(x)}{(x-c)^2 + d^2} w(x)\,\mathrm{d}x, \quad -1 < c < 1,\ \ |d| \ll 1,$$

where $x \mapsto w(x)$ is a given weight function.

The quasi-singularities in this case are $c \pm \mathrm{i}d$. For $d = 0$ this integral diverges, unless this singularity (a pole of second order) is “killed” by the weight function $z \mapsto w(z)$!

For $d \neq 0$ the integral exists, but for small values of d the convergence of quadrature formulas is very slow and standard methods are unusable! Such integrals are very common in many fields (physics, electrical engineering, telecommunications, mechanics, etc.).

In the sequel we consider a simple case with $f(x) = \cos x$, $w(x) = 1$, and $c = 0$, i.e., the integral

$$I(d) = \int_{-1}^{1} F(x,d)\,\mathrm{d}x = \int_{-1}^{1} \frac{\cos x}{x^2 + d^2}\,\mathrm{d}x, \tag{3.1}$$

when $d = 1/2, 10^{-1}, 10^{-2}$, and $d = 10^{-3}$ (see graphics in Figure 5), and then we apply the standard n-point Gauss-Legendre quadrature formulas,

$$\int_{-1}^{1} F(x)\,\mathrm{d}x = \sum_{\nu=1}^{n} A_k^{(n)} F(x_k^{(n)}) + R_n(F), \tag{3.2}$$

where $R_n(F)$ is the corresponding remainder term. The nodes $x_k^{(n)}$, $k = 1, \ldots, n$ are zeros of the Legendre polynomial of degree n, and $A_k^{(n)}$ are the corresponding weight coefficients (Christoffel numbers). General form of Gauss-Christoffel formulas on the real line (1.5) and their construction, including MATHEMATICA package `OrthogonalPolynomials` (see [5], [23]) are given in Subsection 1.1.

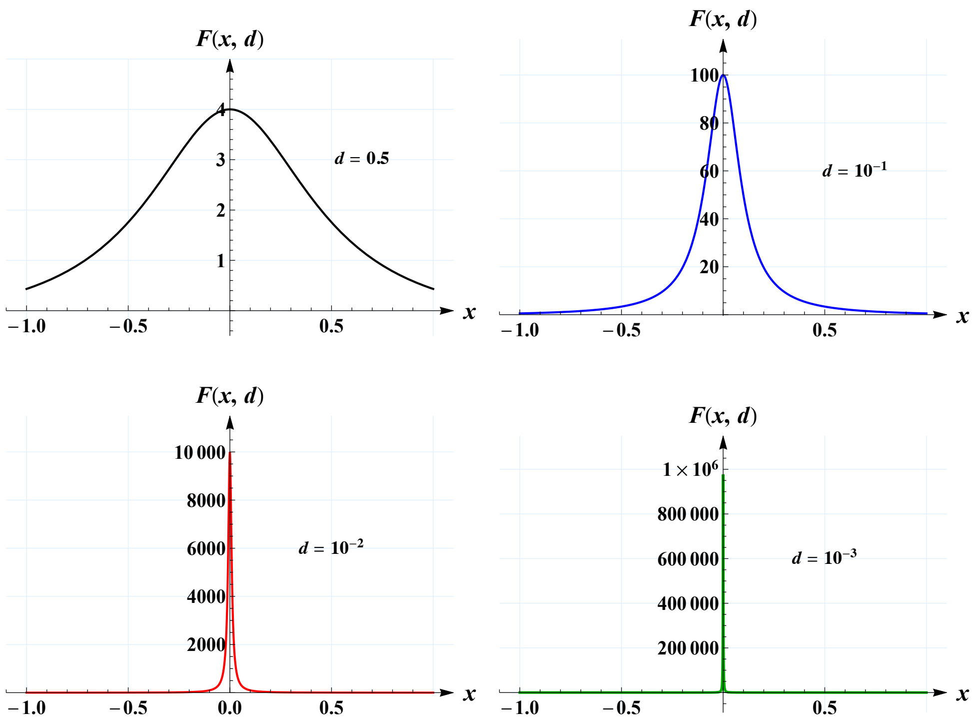

Figure 5: Integrand $x \mapsto F(x,d)$ for $d = 1/2, 10^{-1}, 10^{-2}, 10^{-3}$

Absolute values of the relative errors in the Gauss-Legendre quadrature formulas with $n = 5(5)50$ nodes are presented in Figure 6 in log-scale. As we can see for $d \leq 10^{-2}$, application of the Gauss-Legendre rule for $n \leq 50$ gives quite wrong results, without any exact decimal digits, but for $d = 0.1$ this Gaussian rule with 50 nodes give four exact decimal digits.

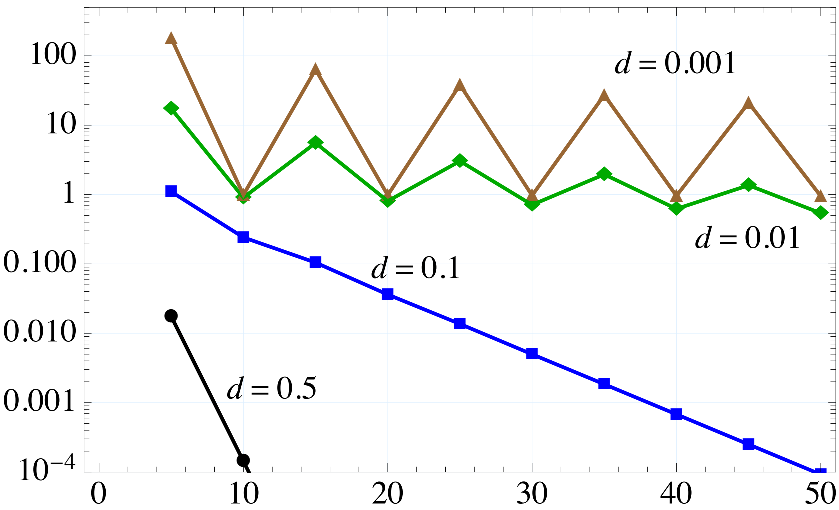

Figure 6: Relative errors of the Gauss-Legendre rules with $n = 5(5)50$ nodes for different values of d

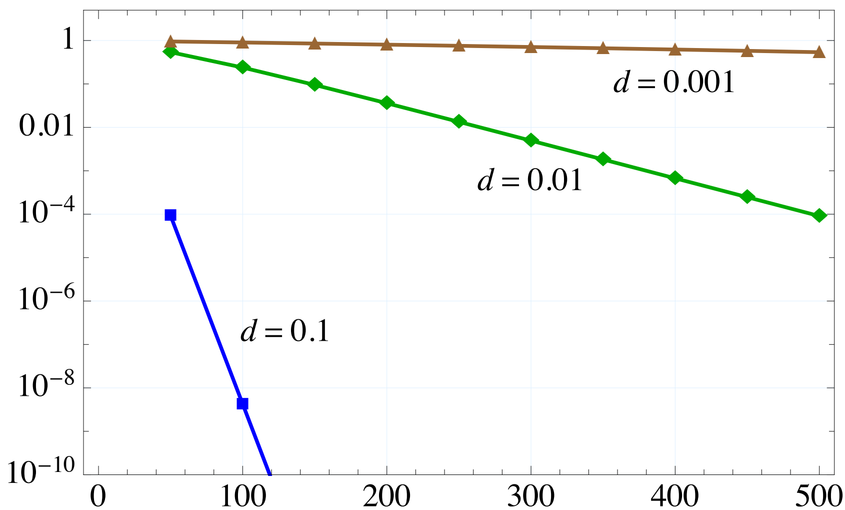

Figure 7: Relative errors of the Gauss-Legendre rules with $n = 100(50)500$ nodes for $d = 0.1,\ 0.01,\ 0.001$

An increasing number of quadrature nodes can also give four exact decimal digits for $d = 0.01$ if we take $n = 500$ nodes (see Figure 7). However, for $d = 10^{-3}$ even 500 nodes do not give any exact digits! The convergence is very slowly!

Absolute values of the relative errors in the Gauss-Legendre quadrature formulas with $n = 100(50)500$ nodes are presented in Figure 7 in log-scale.

In order to apply the idea of integration on the semicircle, we will use here again Cauchy's theorem on residues

$$\int_{-1}^{1} \frac{f(x)w(x)}{(x-c)^2+d^2}\,\mathrm{d}x = \mathrm{Re}\left\{\frac{\pi}{d} f(c+\mathrm{i}d)w(c+\mathrm{i}d) - \int_\Gamma \frac{f(z)w(z)}{(z-c)^2+d^2}\,\mathrm{d}z\right\}.$$

The integral over semicircle Γ can be written in the form

$$\int_\Gamma \frac{f(z)w(z)}{(z-c)^2+d^2}\,\mathrm{d}z = \mathrm{i}\int_0^\pi \frac{\mathrm{e}^{\mathrm{i}\theta} f(\mathrm{e}^{\mathrm{i}\theta})w(\mathrm{e}^{\mathrm{i}\theta})}{(\mathrm{e}^{\mathrm{i}\theta}-c)^2+d^2}\,\mathrm{d}\theta. \tag{3.3}$$

However, an application of the quadrature formula of Gaussian type over the semicircle Γ,

$$\int_0^\pi F(\mathrm{e}^{\mathrm{i}\theta})w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \sum_{\nu=1}^{n} B_\nu^{(n)} F(z_\nu^{(n)}) + R_n^\Gamma(F), \tag{3.4}$$

developed in [12, §7], where the nodes $z_\nu^{(n)}$, $\nu = 1, \ldots, n$, are zeros of the polynomial $\{\pi_n\}$, orthogonal with respect to the non-Hermitian product $\langle f, g\rangle$, defined in (1.9), also are not successful, since the zeros of $\pi_n(z)$ can arbitrarily approach the quasi-singularity $c + \mathrm{i}d$ (see Figure 2).

The basic idea for efficient calculating the last integral (3.3) over the semicircle Γ is to develop a quadrature formula of the form

$$\int_0^\pi F(\mathrm{e}^{\mathrm{i}\theta})w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \sum_{\nu=1}^{n} C_\nu^{(n)} F(\zeta_\nu^{(n)}) + E_n^\Gamma(F), \tag{3.5}$$

such that it be of the maximal degree of exactness on the space of *Laurent*'s polynomials (rational functions). As before, we use the concept of orthogonality with respect to the moment functionals like (1.10),

$$\mathcal{L}z^k = \mu_k, \quad \mu_k = \int_0^\pi \mathrm{e}^{\mathrm{i}k\theta}\,\mathrm{d}\theta = \begin{cases} \pi, & k = 0, \\ 2\mathrm{i}/k, & k \text{ odd}, \\ 0, & k \text{ even},\ k \neq 0. \end{cases} \tag{3.6}$$

including also negative exponents, i.e, $-n+1 \le k \le n$.

Let $\Lambda_{p,q}$ be a linear space of polynomials spanned by the basis

$$\mathcal{B}_{p,q} = \{z^p, z^{p+1}, \dots, z^q\}, \quad p \le q \ \ (p, q \in \mathbb{Z}).$$

For $p = q = 0$, this space reduces to the space of constants

$$\Lambda_{0,0} = \Lambda_0 = \mathcal{P}_0 = \{c \in \mathbb{C} : c \neq 0\}.$$

As before, we use the *non-Hermitian product*, defined by

$$\langle f, g\rangle = \int_\Gamma f(z)g(z)w(z)(\mathrm{i}z)^{-1}\,\mathrm{d}z = \int_0^\pi f\big(\mathrm{e}^{\mathrm{i}\theta}\big)g\big(\mathrm{e}^{\mathrm{i}\theta}\big)w\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta.$$

Such orthogonal systems can be connected with the weighted quadrature formulas of Gaussian type with maximal degree of exactness on $\Lambda_{-n+1,n}$, given by (3.5), where w is a given complex weight function. More details will be given elsewhere, as well as ones for other spaces, e.g., $\Lambda_{-n,n-1}$, $\Lambda_{-n,n}$, etc. Moreover, $\Lambda_{0,n} = \mathcal{P}_n$. In this paper we give only some basis facts in order to show the efficiency of quadrature rules (3.5) for calculating the quasi-singular integrals.

Let $\{R_\nu\}$ be system of orthogonal elements in $\Lambda_{-n+1,n}$ (*the Laurent polynomials*), e.g., generated by the well-known Gram-Schmidt orthogonalization process, starting from the monomials $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$. An alternatively system of Laurent's polynomials $\widetilde{R}_\nu(z)$ can be generated starting from the system of monomials $\{1, z^{-1}, z, z^{-2}, z^2, \dots\}$, and such polynomials can be expressed in terms of the elements $R_\nu(z)$.

The construction of the Laurent polynomials from the sequence of monomials $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$ show that an arbitrary orthogonal element (Laurent's polynomial) $R_m \in \Lambda_{-[m/2],[(m+1)/2]}$ $(m \in \mathbb{N}_0)$ can ne expressed as a linear combination

$$R_m(z) = \sum_{k=0}^{m} c_k^{(m)} z^{k-[m/2]}, \quad m \in \mathbb{N}_0, \tag{3.7}$$

for some constants $c_k^{(m)}$, $k = 0, 1, \dots, m$. Moreover, they can give in the following rational form

$$R_m(z) = \frac{Q_m(z)}{z^{[m/2]}}, \tag{3.8}$$

where

$$Q_{2k}(z) = z^k R_{2k}(z) \quad \text{and} \quad Q_{2k+1}(z) = z^k R_{2k+1}(z), \quad k \in \mathbb{N}_0.$$

The coefficients $c_m^{(m)}$ and $c_0^{(m)}$ in (3.7) we are called the *leading* and *trailing* coefficient of $R_m(z)$, respectively.

We always take the leading coefficient to be $c_m^{(m)} = 1$.

The Laurent polynomials $R_m(z)$, as well as its numerator polynomials $\{Q_m(z)\}$, satisfy several interesting properties and such properties will be proven elsewhere. Here we mention some of them.

Theorem 3.1. *The Laurent polynomials $R_k(z)$, orthogonal with respect to the moment functional $\mathcal{L}$ in $\Lambda_{-n+1,n}$, satisfy following two three-term recurrence relations*

$$R_{2k+1}(z) = (z - a_{2k})R_{2k}(z) + b_{2k}R_{2k-1}(z),$$

$$R_{2k+2}(z) = \left(1 - \frac{a_{2k+1}}{z}\right) R_{2k+1}(z) + b_{2k+1}R_{2k}(z)$$

where $R_0(z) = 1$ and $R_{-1}(z) = 0$, and $\{a_k\}$ and $\{b_k\}$ are sequences of complex numbers depending only on the weight function $w(z)$.

For coefficients we can prove also the following formulas of Darboux-type

$$a_{2k} = \frac{(zR_{2k}, R_{2k})}{(R_{2k}, R_{2k})}, \qquad b_{2k} = -\frac{(zR_{2k}, R_{2k-1})}{(R_{2k-1}, R_{2k-1})},$$

$$a_{2k+1} = \frac{(R_{2k+1}, R_{2k+1})}{(z^{-1}R_{2k+1}, R_{2k+1})}, \quad b_{2k+1} = a_{2k+1}\frac{(z^{-1}R_{2k+1}, R_{2k})}{(R_{2k}, R_{2k})}.$$

Theorem 3.2. *Monic polynomials $\{Q_k(z)\}$ satisfy the following three-term recurrence relation of the form*

$$Q_{k+1}(z) = (z - a_k)Q_k(z) + b_k z\, Q_{k-1}(z), \quad k = 0, 1, \ldots,$$

$$Q_0(z) = 1, \ \ Q_{-1}(z) = 0,$$

where $\{a_k\}$ and $\{b_k\}$ are the same sequences of coefficients as in Theorem 3.1.

Theorem 3.3. *The polynomials $Q_n(z)$ can be characterized by the following relations*

$$\int_0^\pi \mathrm{e}^{-\mathrm{i}k\theta} Q_n(\mathrm{e}^{\mathrm{i}\theta}) w(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta = 0, \quad k = 0, 1, \ldots, n-1.$$

For the weight function $w(z) = 1$, the coefficients a_k are:

$$a_0 = \frac{2\mathrm{i}}{\pi},$$

$$a_1 = \frac{2\mathrm{i}\pi}{\pi^2 - 4},$$

$$a_2 = \frac{\mathrm{i}\left(16 - \pi^2\right)\left(\pi^2 - 4\right)}{6\pi\left(\pi^2 - 8\right)},$$

$$a_3 = \frac{12\mathrm{i}\pi\left(\pi^2 - 8\right)\left(32 - 3\pi^2\right)}{4096 - 2048\pi^2 + 256\pi^4 - 9\pi^6},$$

$$a_4 = \frac{\mathrm{i}\left(256 - 112\pi^2 + 9\pi^4\right)\left(16384 - 2448\pi^2 + 81\pi^4\right)}{20\pi\left(180224 - 60672\pi^2 + 6696\pi^4 - 243\pi^6\right)},$$

$$a_5 = \frac{10\,\mathrm{i}\pi\left(88 - 9\pi^2\right)\left(9\pi^2 - 64\right)\left(1048576 - 198144\pi^2 + 9315\pi^4\right)}{\left(16384 - 2448\pi^2 + 81\pi^4\right)\left(4194304 - 2515968\pi^2 + 391716\pi^4 - 18225\pi^6\right)},$$

etc., while the coefficients b_k are

$$b_1 = \frac{2\mathrm{i}}{\pi},$$

$$b_2 = \frac{\mathrm{i}\pi\left(16 - \pi^2\right)}{6\left(\pi^2 - 4\right)},$$

$$b_3 = \frac{4\mathrm{i}\left(\pi^2 - 4\right)\left(3\pi^2 - 32\right)}{3\pi\left(\pi^2 - 16\right)\left(\pi^2 - 8\right)},$$

$$b_4 = \frac{\mathrm{i}\pi\left(\pi^2 - 8\right)\left(16384 - 2448\pi^2 + 81\pi^4\right)}{20\left(32 - 3\pi^2\right)\left(256 - 112\pi^2 + 9\pi^4\right)},$$

$$b_5 = \frac{2\mathrm{i}\left(256 - 112\pi^2 + 9\pi^4\right)\left(1048576 - 198144\pi^2 + 9315\pi^4\right)}{5\pi\left(9\pi^2 - 88\right)\left(9\pi^2 - 64\right)\left(16384 - 2448\pi^2 + 81\pi^4\right)}, \ \ldots$$

If we put $a_k = \mathrm{i}\alpha_k$ and $b_k = \mathrm{i}\beta_k$. Then,

$$\lim_{k\to\infty} \alpha_k = 1, \quad \lim_{k\to\infty} \beta_k = \frac{1}{2}.$$

These convergence properties we can see from Table 1. The exact digits in the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ are shown in bold.

Table 1: The coefficients α_k and β_k, $k \le 17$

k	α_k	β_k
0	0.63661977236758	
1	**1.0**7046146176622	0.63661977236758
2	**1.0**2104877276530	**0.5**4686268616792
3	**1.00**754385210227	**0.5**1972424516012
4	**1.00**328553025512	**0.5**1017104738859
5	**1.00**169153484126	**0.50**612004856595
6	**1.000**98589542046	**0.50**408672459805
7	**1.000**62713848922	**0.50**292614522532
8	**1.000**42471990174	**0.50**220008310515
9	**1.000**30139516589	**0.50**171492617644
10	**1.000**22178414612	**0.50**137443170299
11	**1.000**16802220006	**0.50**112619730380
12	**1.000**13037600341	**0.500**93962699110
13	**1.000**10321476497	**0.500**79585268455
14	**1.0000**8311950419	**0.500**68271506772
15	**1.0000**6792948381	**0.500**59208830190
16	**1.0000**5623226900	**0.500**51837338324
17	**1.0000**4707746447	**0.500**45761013324

A few first orthogonal Laurent polynomials for the Legendre weight function $w(z) = 1$ are:

$$R_0(z) = 1, \qquad R_1(z) = z - \frac{2\mathrm{i}}{\pi},$$

$$R_2(z) = z - \frac{4}{(\pi^2 - 4)\, z} - \frac{2\mathrm{i}\pi}{\pi^2 - 4},$$

$$R_3(z) = z^2 - \frac{2\mathrm{i}\left(3\pi^2 - 16\right) z}{3\pi\,(\pi^2 - 8)} - \frac{2\mathrm{i}\left(\pi^2 - 16\right)}{3\pi\,(\pi^2 - 8)\, z} - \frac{8}{3\,(\pi^2 - 8)},$$

$$R_4(z) = z^2 - \frac{8\left(3\pi^2 - 32\right)}{(256 - 112\pi^2 + 9\pi^4)\, z^2} - \frac{2\mathrm{i}\left(9\pi^3 - 80\pi\right) z}{256 - 112\pi^2 + 9\pi^4} - \frac{2\mathrm{i}\left(9\pi^3 - 112\pi\right)}{3\,(256 - 112\pi^2 + 9\pi^4)\, z} + \frac{512 - 72\pi^2}{3\,(256 - 112\pi^2 + 9\pi^4)},$$

$$R_5(z) = z^3 - \frac{2\mathrm{i}\left(16384 - 5616\pi^2 + 405\pi^4\right) z^2}{5\pi\left(5632 - 1368\pi^2 + 81\pi^4\right)} - \frac{2\mathrm{i}\left(16384 - 2448\pi^2 + 81\pi^4\right)}{5\pi\left(5632 - 1368\pi^2 + 81\pi^4\right) z^2} - \frac{36\left(27\pi^2 - 256\right) z}{5\left(5632 - 1368\pi^2 + 81\pi^4\right)} - \frac{12\left(63\pi^2 - 640\right)}{5\left(5632 - 1368\pi^2 + 81\pi^4\right) z} - \frac{2\mathrm{i}\left(45\pi^2 - 512\right)}{15\pi\left(9\pi^2 - 88\right)},$$

etc.

Remark 3.1. In the Chebyshev cases, when $w(z) = (1-z^2)^{\pm 1/2}$, the expressions for recurrence coefficients, as well as ones for the Laurent polynomials are much simpler than the previous in the Legendre case.

Without proof we mention here one of main results:

Theorem 3.4. *The quadrature formula* (3.5) *is exact for each* $F \in \Lambda_{-n+1,n}$ *if and only if its nodes* $\zeta_\nu = \zeta_\nu^{(n)}$ *are zeros of the polynomial* $Q_n(z)$, *and the coefficients* $C_\nu^{(n)}$ *are given by*

$$C_\nu^{(n)} = \frac{1}{Q_n'(\zeta_\nu)} \int_0^\pi \frac{Q_n\left(\mathrm{e}^{\mathrm{i}\theta}\right) w\left(\mathrm{e}^{\mathrm{i}\theta}\right)}{\mathrm{e}^{\mathrm{i}\theta} - \zeta_\nu} \,\mathrm{d}\theta, \quad \nu = 1, \ldots, n. \tag{3.9}$$

In the Legendre case ($w(z) = 1$) for $n = 5$ we give in Table 2 the quadrature parameters (nodes and weight coefficients)

(1) for the standard Gauss-Legendre quadrature formula on $[-1, 1]$, generated by our MATHEMATICA package `OrthogonalPolynomials` (see [5], [23]) by the command

```
{xk,Ak}=aGaussianNodesWeights[5,{aLegendre},...];
```

(2) for the Gautschi-Milovanović quadrature on the semicircle by the same package using the command

```
{zk,Bk}=aGaussianNodesWeights[5,{aGautschiMilovanovic,1/2},...];
```

(3) for new quadrature formulas of Gaussian type given in Theorem 3.4. The nodes ζ_k are zeros of the nominator polynomial $Q_5(z)$ in (3.8), in our case given by

$$Q_5(z) = z^5 - \frac{2\mathrm{i}\left(16384 - 5616\pi^2 + 405\pi^4\right)}{5\pi\left(9\pi^2 - 88\right)\left(9\pi^2 - 64\right)} z^4 - \frac{36\left(27\pi^2 - 256\right)}{5\left(9\pi^2 - 88\right)\left(9\pi^2 - 64\right)} z^3 - \frac{2\mathrm{i}\left(45\pi^2 - 512\right)}{15\pi\left(9\pi^2 - 88\right)} z^2 - \frac{12\left(63\pi^2 - 640\right)}{5\left(9\pi^2 - 88\right)\left(9\pi^2 - 64\right)} z - \frac{2\mathrm{i}\left(16384 - 2448\pi^2 + 81\pi^4\right)}{5\pi\left(9\pi^2 - 88\right)\left(9\pi^2 - 64\right)},$$

and weight coefficients given by (3.9).

Table 2: Nodes and weight coefficients for $n = 5$ in the standard Gauss-Legendre formula (3.2), as well as ones in Gaussian formulas on the semicircle (3.4) and (3.5) for $w(z) = 1$

k	$k = 3$	$k = 2$ and $k = 4$	$k = 1$ and $k = 5$
$x_k^{(5)}$	0.	± 0.5384693101056831	± 0.9061798459386640
$A_k^{(5)}$	0.5688888888888889	0.4786286704993665	0.2369268850561891
$z_k^{(5)}$	$\mathrm{i}\,0.2221614120619286$	∓ 0.4802650814481394 $+\mathrm{i}\,0.1179279409749741$	∓ 0.8905272718373425 $+\mathrm{i}\,0.0224954605960690$
$B_k^{(5)}$	1.991380659532854	0.5027034456938210 $\mp\mathrm{i}\,0.9261893208883687$	0.0724025513346485 $\mp\mathrm{i}\,0.3066364594922929$
$\zeta_k^{(5)}$	$\mathrm{i}\,0.8922797389775098$	∓ 0.6465025063224080 $+\mathrm{i}\,0.6459772999484531$	∓ 0.9563699220720102 $+\mathrm{i}\,0.1706736496489362$
$C_k^{(5)}$	0.8100409076045721	0.7346361111536882 $\mp\mathrm{i}\,0.0470837665077862$	0.4311397618389223 $\mp\mathrm{i}\,0.0627646610077391$

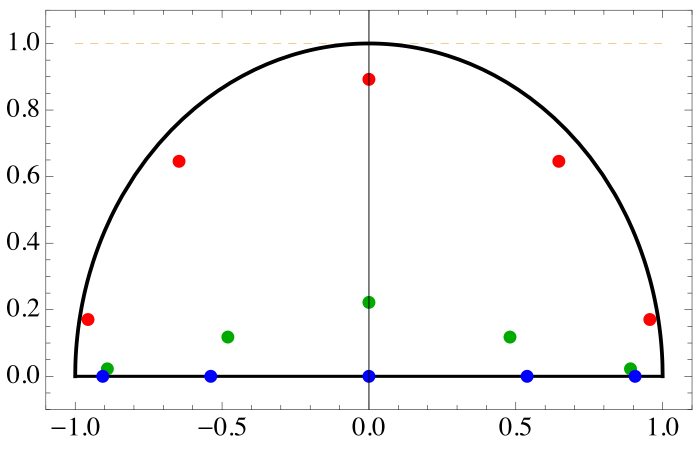

Figure 8: Zeros $x_k^{(5)}$ (blue), $z_k^{(5)}$ (green), and $\zeta_k^{(5)}$ (red), $k = 1, \ldots, 5$, of polynomials $p_5(z)$, $\pi_5(z)$, and $Q_5(z)$, respectively

Nodes of three different Gaussian quadratures with 5 nodes, given in Table 2, $x_k^{(5)}$, $z_k^{(5)}$, and $\zeta_k^{(5)}$, $k = 1, \ldots, 5$, are shown in different colors in Figure 8, in blue, green, and red, respectively.

Finally, we reconsider the example with the integral (3.1), using its reduction to the integral over a semicircle like (3.3), as well as an application of the new quadrature rule (3.5).

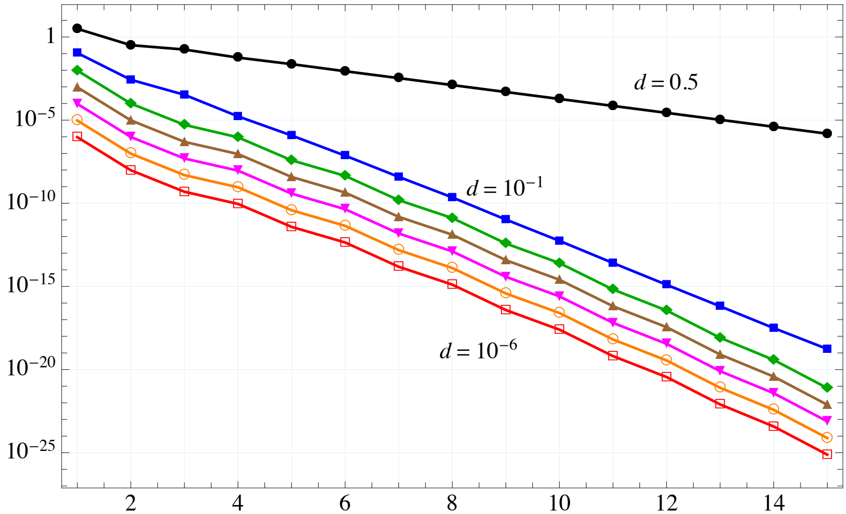

Figure 9: Relative errors of the quadrature rules (3.5) with $n = 1(1)15$ nodes for different values of d

The quadrature sums of this new quadrature rule (3.5) converge very fast. In Figure 9 in log-scale we present relative errors for quadrature rules with a small number of nodes $n \leq 15$, and $d = 0.5$, as well as for very small $d = 10^{-m}$, $m = 1, 2, \ldots, 6$. As we can see, the relative errors are smaller when d tends to zero. For example, for $d = 10^{-6}$ the quadrature formula with only $n = 5$ nodes gives result with about 12 exact decimal digits (the corresponding relative error is 3.00×10^{-12}), while for $n = 10$ quadrature nodes we get result with 18 exact decimal digits (the relative error is 8.85×10^{-19}).

We can conclude that this new quadrature formula is very efficient for calculating real quasi-singular integrals on the finite intervals.

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