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BOGOLJUB STANKOVIĆ: CONTRIBUTIONS TO GENERALIZED ASYMPTOTICS AND MECHANICS

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Dedicated to Professor Bogoljub Stanković (1924–2018)

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A b s t r a c t. As a closest collaborators of Academician Bogoljub Stanković, the authors of this article outline some fields of his scientific interests and contributions

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1. *S. Pilipović: Contributions to generalized asymptotics*

Academician Bogoljub Stanković and Stevan Pilipović (SP) had collaborated on 22 publications four of which are monographs published by the world's leading publishers. Their main joint interests were generalized asymptotics, especially quasi-asymptotics and the S-asymptotics. Note that there are several approaches to generalized asymptotics in spaces of generalized functions. The most developed approaches are those of Vladimirov, Drozhinov and Zavialov [12] (see also [6], [11]), and of Kanwal and Estrada [7], [8]. The first approach is extended in the direction of the

$S-$asymptotics by Stanković an SP [4], [5]. Separately, with the coauthors Arpad Takači and Jasson Vindas, Bogoljub Stanković and SP have published two monographs. The first one [3] deals with the results concerning basic notions of quasi-asymptotics and integral transforms, while the second one [10] published 22 years later, gives a more complete analysis of various types of generalized asymptotics including the S-asymptotics, on various spaces of distributions, ultradistributions and Fourier hyperfunctions with the emphasis on Tauberian type theorems for generalized integral transforms. Of course, it is not possible in this very short article present even a part of those results with all the details. The writer decided to quote here basic definitions and results related to the S-asymptotics. The notion of S-asymptotics was mentioned marginally earlier in the literature without special attention to the structural representations and general theory related to Abelian and Tauberian type results for functions, distributions, ultradistributions and hyperfunctions.

It is necessary to mention that the class of regularly varying functions, so important in the classical asymptotic analysis introduced by Karamata [9] (see also [1]), also obtained very important role in the generalized asymptotic analysis. (The monograph [1]. gives collected results related to the theory and the applications of regular varying functions.)

The shift asymptotics defined by Drozhinov and Zavialov, was later called S-asymptotics by the authors of [3] and [10]. It is said that a tempered distribution T supported by $[0, \infty)$ has the shift asymptotics at infinity related to the regularly varying function $r(t) = t^a L(t)$ if

$$T(t+k)/r(t) \to g(t), k \to \infty \tag{1.1}$$

in the sense of convergence in the space of tempered distributions. We have adapted and made precise the S-asymptotic behavior of generalized functions and studied this type of the asymptotic behavior for distributions, ultradistributions and hyperfunctions giving a structural characterizations of comparison functions and the limit function, cf. [3], [4], [5] .

Conceptually, in this short presentation, the asymptotic behavior is considered within a dual space $\mathcal{F}'$ of a barrelled and Montel locally convex space $\mathcal{F}$. The spaces of distributions $\mathcal{D}'$, tempered distributions $\mathcal{S}'$ non-quasy analytic ultradistributions $\mathcal{D}'^*$ ($*$ is the joint notation for Beurling and Roumieu type ultradistributions which correspond to the Gevrey sequence $(M_p = p!^s), s > 1$), Fourier hyperfunctions $\mathcal{Q}$ are of this kind. In this presentation $\mathcal{F}$ will always denote one of these spaces. As it is written, the length of this article is very limited so the definition of any of these spaces is skipped. The same is done for a very long list of references relevant for this field; for the most completed list of references one should look [10] from 2012.

Moreover, several important papers vere publised (after this book) by J. Vindas and SP.

Asymptotic behavior of generalized functions Let Γ be a convex cone. Let $h_1, h_2 \in \Gamma$. We say that $h_1 \geq h_2$ if and only if $h_1 \in h_2 + \Gamma$; Γ is now partially ordered. For a real-valued function f defined on Γ, we write $\lim_{h\in\Gamma,\, h\to\infty} f(h) = A \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists $h(\varepsilon) \in \Gamma$ such that $f(h) \in (A-\varepsilon, A+\varepsilon)$ when $h \geq h(\varepsilon),\ h \in \Gamma$. The S-asymptotics is defined as:

$$\lim_{h\in\Gamma,\, h\to\infty} \langle T(x+h)/c(h), \varphi(t)\rangle = \langle u, \varphi\rangle, \quad \varphi \in \mathcal{F}. \tag{1.2}$$

In case $n = 1$ limits (1.1) and (1.2) coincide. We list the main properties.

Proposition 1.1. *Let Γ be a convex cone and $T(t+h) \overset{s}{\sim} c(h)U(t),\ h \in \Gamma$. Then:* a) *There exists a function d on Γ such that*

$$\lim_{h\in\Gamma,\, \|h\|\to\infty} c(h+h_0)/c(h) = d(h_0) \ \textit{for every} \ \ h_0 \in \Gamma\,.$$

b) *The limit U satisfies the equation $U(\cdot + h) = d(h)U,\ \ h \in \Gamma$.*

c) *Let* $\operatorname{int}\Gamma \neq \emptyset$ ($\operatorname{int}\Gamma$ *is the interior of* Γ) *and* $T(x+h) \overset{s}{\sim} c(h)U(t),\ h \in \Gamma$, *where c is a positive function defined on $\mathbb{R}^n$. Then:*

c_1) *For every $h_0 \in \mathbb{R}^n$ there exists*

$$\lim_{h\in(h_0+\Gamma)\cap\Gamma,\|h\|\to\infty} c(h+h_0)/c(h) = \tilde{d}(h_0)\,.$$

c_2) $\tilde{d}(x) = \exp(\alpha\cdot x),\ x \in \mathbb{R}^n$, *where α is a fixed element of* $\mathbb{R}^n$.

c_3) $U(t+h) = \tilde{d}(h)U(t),\ h \in \mathbb{R}^n$ *and* $U(t) = C\exp(\alpha\cdot t)$ *for some* $C \in \mathbb{R}$.

We only assumed that c is a positive function. But if we know that there exist $T \in \mathcal{F}'$ and $U \neq 0$ such that $T(x+h) \overset{s}{\sim} c(h)U(t),\ h \in \Gamma$, then we can find a function $\tilde{c} \in C^\infty$ and with the property

$$\lim_{h\in\Gamma,\|h\|\to\infty} \tilde{c}(h)/c(h) = 1.$$

This function $\tilde{c}$ can be defined as follows: $\tilde{c}(h) = \langle T(x+h),\ \tilde{\varphi}(x)\rangle/\langle U, \tilde{\varphi}\rangle$, where $\tilde{\varphi}$ is chosen to give $\langle U, \tilde{\varphi}\rangle \neq 0$. In this sense we can sometimes suppose that $c \in C^\infty$, and we do not lose in generality.

Similarly, we have $\lim_{h\in\Gamma',\|h\|\to\infty} c(h)/\tilde{c}(h+x) = \exp(-\alpha\cdot x)$ in $\mathcal{E}$ if $\operatorname{int}\Gamma \neq \emptyset, \Gamma' \subset\subset \Gamma$ ($\Gamma \cap \mathbb{S}^{n-1}$ is compact in $\Gamma \cap \mathbb{S}^{n-1}$).

In the one-dimensional case a cone Γ can be only $\mathbb{R}, \mathbb{R}_+$ or $\mathbb{R}_-$. In all three cases $\operatorname{int}\Gamma \neq \emptyset$. Consequently, $\tilde{d}$ from Proposition 1.2 has the form $\tilde{d}(x) = \exp(\alpha x)$, where $\alpha \in \mathbb{R}$.

Let $c(x) = L(e^x)\exp(\alpha x),\ x \in \mathbb{R}^n$. We will show that L is a slowly varying function. By Proposition 1.2 a), there exists the limit

$$\lim_{h\in\Gamma,\|h\|\to\infty} L(\exp(h+h_0))/L(\exp(h)) = 1,\ \ h_0 \in \mathbb{R}^n,$$

which implies $\lim_{x\in\mathbb{R}_+,x\to\infty} L(xp)/L(x) = 1,\ \ p \in \mathbb{R}_+$ and this defines a slowly varying function. Thus if $T \in \mathcal{F}'(\mathbb{R})$ and $T(x+h) \overset{s}{\sim} c(h)U(t)$, in $\Gamma \subset \mathbb{R}$, then it follows that c has the form $c(x) = \exp(\alpha x)L(\exp(x)), x \geq a > 0$, where L is a slowly varying function.

Basic properties of the S-asymptotics.

We continue with the collection of main assertions.

Theorem 1.1. *Let* $T \in \mathcal{F}'$. a) *If* $T(t+h) \overset{s}{\sim} c(h)U(t),\ h \in \Gamma$, *then for every* $k \in \mathbb{N}_0^n,\ T^{(k)}(t+h) \overset{s}{\sim} c(h)U^{(k)}(t),\ h \in \mathcal{G}a$. *In particular, we have* $U^{(k)} \neq 0$;

b) *Let* $g \in M_{(\cdot)}$ *(set of multipliers of* $\mathcal{F}'$*; let* c, c_1 *be positive functions. If for every* $\varphi \in \mathcal{F},\ (g(t+h)/c_1(h))\varphi(t)$ *converges to* $G(t)\varphi(t)$ *in* $\mathcal{F}$ *when* $h \in \Gamma,\ \|h\| \to \infty$ *and if* $T(t+h) \overset{s}{\sim} c(h)U(t),\ h \in \Gamma$, *then* $g(t+h)T(t+h) \overset{s}{\sim} c_1(h)c(h)G(t)U(t)$, $h \in \Gamma$.

c) *If* $T \in \mathcal{F}',\ \mathrm{supp}f$ *is a compact set in* $\mathbb{R}^n$, *then* $T(t+h) \overset{s}{\sim} c(h)\cdot 0,\ h \in \gamma$, *for every positive function* c.

d) *Let* $S \in \mathcal{F}',\ \mathrm{supp}S$ *being compact. Suppose that in* $\mathcal{F}'$, *the convolution is defined and is hypocontinuous. If* $T(t+h) \overset{s}{\sim} c(h)U(t),\ h \in \gamma$, *then* $(S*T)(t+h) \overset{s}{\sim} c(h)(S*U)(t),\ h \in \Gamma$;

e) *Let* $T \in \mathcal{F}' = \mathcal{D}'^*$ *and* $T \overset{s}{\sim} c(h)\cdot U,\ \ h \in \Gamma$, *in* $\mathcal{D}'^*$. *Let* $P(D)$ *be an ultradifferential operator of class* $*$. *Then*

$$P(D)T \overset{s}{\sim} c(h)\cdot P(D)U,\ \ h \in \gamma,\ \text{in}\ \mathcal{D}'^*.$$

f) *Let* $f \in \mathcal{F} = Q(\mathcal{D}^n)$. *Let* $P(D)$ *be a local operator and* $f(x+h) \overset{s}{\sim} c(h)\cdot u(x),\ h \in \Gamma,\ \|h\| \to \infty$ *in* $\mathcal{Q}(\mathcal{D}^n)$, *then* $P(D)f(x+h) \overset{s}{\sim} c(h)\cdot P(D)u(x),\ h \in \Gamma,\ \|h\| \to \infty$ *in* $\mathcal{Q}(\mathbb{D}^n)$, *as well.*

Theorem 1.2. 1) *Let* $f, g \in \mathcal{D}'(\mathbb{R})$ *and for some* $m \in \mathbb{N},\ g^{(m)} = f$.

a) *If* $f(x+h) \overset{s}{\sim} h^\nu L(h)\cdot 1,\ \ h \in \mathbb{R}_+$, *where* $\nu > -1$, *then* $g(x+h) \overset{s}{\sim} h^{\nu+m}L(h)\cdot 1,\ h \in \mathbb{R}_+$.

b) *If* $f(x+h) \overset{s}{\sim} \exp(\alpha h)L(\exp h)\exp(\alpha x),\ h \in \mathbb{R}_+, \alpha \in \mathbb{R}$, *and*

$$\int_0^x \exp(\alpha h)L(\exp h)dh \to \infty,$$

when $x \to \infty$, *then*

$$g(x+h) \overset{s}{\sim} \int_0^h \int_0^{h_{m-1}} \cdots \int_0^{h_1} \exp(\alpha t) L(\exp t) dt dh_1 \dots dh_{m-1} \exp(\alpha x), h \in \mathbb{R}_+.$$

2) *Let* $\phi_0 \in C_0^\infty(\mathbb{R})$ *such that* $\int \phi_0(t)dt = 1$ *and let* $f \in \mathcal{D}'(\mathbb{R})$. *If*

$$\lim_{h\to\infty} \left\langle \frac{f^{(i)}(x+h)}{\exp(\alpha h)L(\exp h)}, \phi_0(x) \right\rangle = \alpha^i \langle \exp(\alpha x), \phi_0(x) \rangle, \ i = 0, 1, \dots, m-1,$$

$$f^{(m)}(x+h) \overset{s}{\sim} \exp(\alpha h) L(\exp h) \alpha^m \exp(\alpha x), \ h \in \mathbb{R}_+,$$

then

$$f(x+h) \overset{s}{\sim} \exp(\alpha h) L(\exp h) \exp(\alpha x), \ h \in \mathbb{R}_+.$$

3) *Suppose that* $T \in \mathcal{D}'$, $\Gamma = \{x \in \mathbb{R}^n; \ x = (0, \dots, x_k, 0, \dots, 0)\}$ *and* $T = (\partial/\partial x_k)S$. *If* $T(t+h) \overset{s}{\sim} c(h)U(t), \ h \in \Gamma$ *and* $c(h)$ *is locally integrable in* h_k *such that*

$$c_1(h_k) = \int_{h_k^0}^{h_k} c(v) dv_k \to \infty \quad \textit{as} \ \ h_k \to \infty, \ h_k^0 \geq 0,$$

then $S(t+h) \overset{s}{\sim} c_1(h)U(t), \ h \in \Gamma$.

4) *Suppose that* $S \in \mathcal{D}'$ *and that for an* $m \in \{1, 2, \dots, n\}$,

$$(D_{t_m} S)(x+h) \overset{s}{\sim} c(h) \cdot U(x), \quad h \in \Gamma.$$

Let $V \in \mathcal{D}'$, $D_{t_m} V = U$ *and* $\phi_0 \in \mathcal{D}(\mathbb{R})$, $\int_{\mathbb{R}} \phi_0(\tau) d\tau = 1$. *Let*

$$\lim_{h\in\Gamma, \|h\|\to\infty} \langle S(x+h)/c(h), \phi_0(x_m)\lambda_m(\tilde{x}) \rangle = \langle V, \phi_0 \lambda_m \rangle,$$

where $\tilde{x} = (x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_n)$ *and*

$$\lambda_m(\tilde{x}) = \int_{\mathbb{R}} \psi(x_1, \dots, x_m, \dots, x_n) dx_m, \quad \psi \in \mathcal{D}.$$

Then $S(x+h) \overset{s}{\sim} c(h)V(x), h \in \Gamma$.

If a distribution has the S-asymptotics in $\mathcal{D}'$, it has the same S-asymptotics in the space of ultradistributions$\mathcal{D}'^*$. The converse does not hold.

Classical and generalized S-asymptotics

Proposition 1.2. *Let a regular generalized function $\tilde{f} \in \mathcal{D}'(\mathbb{R})$, be defined by a function f, which has one of the four properties for $\alpha > 1,\ \beta > 0,\ x \geq x_0,\ h > 0, M > 0$ and $N > 0$:*

a) $f(x+h) \geq M \exp(\beta h^\alpha) f(x) \geq 0,$

a') $-f(x+h) \geq -M \exp(\beta h^\alpha) f(x) \geq 0,$

b) $0 \leq f(x+h) \leq N \exp(-\beta h^\alpha) f(x)$, v

b') $0 \leq -f(x+h) \leq -N \exp(-\beta h^\alpha) f(x).$

Then $\tilde{f}$ cannot have the S-asymptotics with the limit $U \neq 0$. But the function f can have the asymptotics.

It is easy to show that for some classes of real functions f on $\mathbb{R}$ the asymptotic behavior at infinity implies the S-asymptotics of $\tilde{f}$.

Proposition 1.3. a) *Let c be a positive function and $\tilde{T}$ be a regular distribution defined by $T \in L^1_{loc}(\mathbb{R}^n)$. Suppose that there exist locally integrable functions $U(t)$ and $V(t), t \in \mathbb{R}^n$, such that for every compact set $K \subset \mathbb{R}^n$ we have in $L^1_{loc}(\mathbb{R}^n)$*

$$|T(t+h)/c(h)| \leq V(t),\ t \in K,\ \|h\| > r_K,$$

$$\lim_{h\in\Gamma, \|h\|\to\infty} T(t+h)/c(h) = U(t),\ \ t \in K.$$

Then, $\tilde{T}(t+h) \overset{s}{\sim} c(h)U(t),\ h \in \Gamma$ in $\mathcal{F}'$.

b) *Let $T \in L_{loc}(\mathbb{R})$ and $T(x) \sim \exp(\alpha x) L(\exp x),\ x \to \infty, \alpha \in \mathbb{R}$, where L is a slowly varying function. Then, $\tilde{T}(x+h) \overset{s}{\sim} \exp(\alpha h) L(\exp h) \exp(\alpha x)$, $h \in \mathbb{R}_+$ in $\mathcal{F}'(\mathbb{R})$.*

A more general assertion is the following one.

Proposition 1.4. *Let Γ be a cone and $\Omega \subset \mathbb{R}^n$ be an open set such that for every $r > 0$ there exists a β_r such that $B(0,r) \subset \{\Omega - h;\ h \in \Gamma,\ \|h\| \geq \beta_r\}$.*

Suppose that $G \in L^1_{loc}(\Omega)$ and has the following properties: There exist locally integrable functions U and V in $\mathbf{R}^n$ such that for every $r > 0$ we have in $L^1_{loc}(\Omega)$

$$|G(x+h)/c(h)| \leq U(x),\ x \in B(0,r),\ h \in \Gamma,\ \|h\| \geq \beta_r;$$

$$\lim_{h\in\Gamma, \|h\|\to\infty} G(x+h)/c(h) = V(x),\ \ x \in B(0,r).$$

If $G_0 \in \mathcal{F}'$ is equal to $\tilde{G}$ defined by G on Ω, then

$$G_0(x+h) \overset{s}{\sim} c(h)V(x),\ \ h \in \Gamma.$$

The following proposition gives the sufficient condition under which the S-asymptotics of an $\tilde{F} \in \mathcal{D}'(\mathbb{R})$ defined by $f \in L^1_{loc}(\mathbb{R})$ implies the asymptotic behavior of f.

Proposition 1.5. *Let $f \in L^1_{loc}(\mathbb{R}),\ c(h) = h^\beta L(h)$, where $\beta > -1$ and L be a slowly varying function. If for some $m \in \mathbb{N}, x^m f(x), x > 0$, is monotonous and $\tilde{f}(x+h) \overset{s}{\sim} c(h) \cdot 1, h \in \mathbb{R}_+$, then $\lim_{h\to\infty} f(h)/c(h) = 1$. If we suppose that L is monotonous, then we can omit the supposition that $\beta > -1$.*

General S-asymptotic expansion

Let Γ be a convex cone with the vertex at zero belonging to $\mathbb{R}^n$ and $\Sigma(\Gamma)$ the set of all real-valued and positive functions $c(h), h \in \Gamma$. We shall consider the asymptotic expansion when $\|h\| \to \infty,\ h \in \Gamma$.

A distribution $T \in \mathcal{F}'$ has the S-asymptotic expansion related to the asymptotic sequence $c_n(h) \subset \Sigma(\Gamma)$, if for every $\varphi \in \mathcal{F}$

$$\langle T(t+h), \varphi(t)\rangle \sim \sum_{n=1}^{\infty} \langle U_n(t,h), \varphi(t)\rangle | \{c_n(h)\}, \|h\| \to \infty,\ h \in \Gamma,$$

where $U_n(t,h) \in \mathcal{F}'$ for $n \in \mathbb{N}$ and $h \in \Gamma$. We write for short:

$$T(t+h) \overset{s}{\sim} \sum_{n=1}^{\infty} U_n(t,h) | \{c_n(h)\}, \|h\| \to \infty,\ h \in \Gamma.$$

In the special case $U_n(t,h) = u_n(t)c_n(h),\ u_n \in \mathcal{F}',\ n \in \mathbf{N}$, we write

$$T(t+h) \overset{s}{\sim} \sum_{n=1}^{\infty} u_n(t)c_n(h), \quad \|h\| \to \infty,\ h \in \Gamma.$$

In this case the given S-asymptotic expansion is unique.

Suppose that $f \in \mathcal{F}'(\mathbb{R})$. It is said that $f(x)e^{ixt}$ has the asymptotic expansion related to the asymptotic sequence $\psi_n(t)$ if for every $\varphi \in \mathcal{F}(\mathbf{R})$

$$\langle f(x)e^{ixt}, \varphi(x)\rangle \sim \sum_{n=1}^{\infty} \langle C_n(x,t), \varphi(x)\rangle | \{\psi_n(t)\},\ t \to \infty,$$

where $C_n(x,t) \in \mathcal{F}'(\mathbf{R}),\ n \in \mathbf{N},\ t \geq t_0$.

We write in short

$$fe^{i\cdot t} \sim \sum_{n=1}^{\infty} C_n(\cdot, t) | \{\psi_n(t)\},\ t \to \infty.$$

Concluding remark. This short overview just illustrates a part of the work of Academician Bogoljub Stanković. He published many papers related to this topic. As it is written, a very rich bibliography in [3] and [10] contains the list of all his papers related to generalized asymptotics and applications.

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2. *T. Atanacković: Contributions to Mechanics*

The work of Prof. Stanković in the field of Mechanics belongs to Viscoelasticity of fractional derivative type and to elastic rod theory with discontinues. He was studying problems of existence of solution to the equations of motion as well as regularity and stability of solutions for viscoelastic bodies in the case when constitutive equations contain fractional derivatives. Also, equations arising from variational principles of Mechanics, for the case when Lagrangian density contains fractional derivatives, are studied in several works. He also obtained results in the static theory of rods for the case when, either properties of the material, or geometry of the rod is described by discontinuous functions [11], [16]. The results of Prof. Stanković in the field of Mechanics are published in references [1]–[24]. Most of the results are also presented in books [25]–[26].

In models with fractional partial differential equation, after the separation of variables, the type of ordinary fractional differential equation that was studied, is of the form

$$\sum_{i=0}^{m} A_i \, {}_0D_t^{\alpha_i} y(t) = f(t), \; 0 < t \le b, \tag{2.1}$$

where $\alpha_i = n_i - 1 + \gamma_i, \; i = 0, \dots, m, \; n_i \in \mathbb{N}, \; \gamma_i \in (0,1]; \; 0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_m \le n_m$; if $\alpha_0 = 0$, then $n_0 = 0$ and $\gamma_0 = 1$ and if $\alpha_m = n_m, \; \gamma_m = 1$. Here we use the following notation for the left Riemann-Liouville fractional derivative: Let $\alpha = n - 1 + \gamma, \; n \in \mathbb{N}, \; \gamma \in (0,1]$. Then for $n - 1 < \alpha \le n$ the left Riemann-Liouville fractional derivative of order α is defined as

$$\begin{aligned} {}_0D_t^{\alpha} \, y(t) &= \frac{1}{\Gamma(1-\gamma)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(\tau)d\tau}{(t-\tau)^{\gamma}}, \\ {}_0D_t^{\alpha} \, y(t) &= D^n \, y(t), \; \alpha = n. \end{aligned}$$

The fractional differential equations of the type (2.1) appear in many problems of Mechanics. For example, such equations arise in problems describing transversal vibrations of viscoelastic rod. The second type of equations that was studied by Prof. Stanković corresponds to wave propagation in nonlocal elasticity. The corresponding partial differential equation is

$$\frac{\partial^2 u(t,x)}{\partial t^2} + A({}_0D_t^{\beta} u(t,x)) = B \frac{1}{2} \frac{\partial}{\partial x} \Big({}_{-\infty}D_x^{\alpha} u(x,t) - {}_xD_{\infty}^{\alpha} u(x,t) \Big),$$

$$t \ge 0, \;\; x \in \mathbb{R}, \;\; u(0,x) = C_1(x), \; \frac{\partial u(t,x)}{\partial t}\Big|_{t=0} = C_2(x), \tag{2.2}$$

where $0 < \alpha < 1, \; 0 < \beta < 1$ and $A, B \in \mathbb{R}$. Here

$${}_{-\infty}D_x^{\alpha} F(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{F'(u)du}{(x-u)^{\alpha}} du, \;\; {}_xD_{\infty}^{\alpha} F(x) = \frac{-1}{\Gamma(1-\alpha)} \int_{x}^{\infty} \frac{F'(u)du}{(u-x)^{\alpha}}.$$

Both types of equations (2.1) and (2.2) are analyzed in a number of publications, see [1]–[24].

Next we mention another group of results for a special type of fractional differential equations arising in fractional variational principles of Hamilton type. In the

Euler-Lagrange equations for such problems both left and right Riemann-Liouville fractional derivatives appear. Thus a generic equation is of the form [15]

$$
{}_tD_b^{\alpha}\left[{}_0D_t^{\alpha}y\left(x\right)\right]+\left(A_1+A_2\right)\left[{}_0D_t^{\alpha}y\left(x\right)+\ {}_tD_b^{\alpha}y\left(x\right)\right]
$$
$$
+B\left(x\right)y\left(x\right)=C\left(x\right),\quad 0<x<b<\infty,
$$

where A_1, A_2 are given constants and B is given function.

Finally, we present expansion formula for fractional derivatives that was obtained in [8] and [18]. This formula gives the possibility to transform fractional differential equation to the system of ordinary differential equations of integer order. In several publications numerical aspects of such a procedure were examined. The main result of these investigations is the expansion formula, that we state as: Suppose that

Given N and suppose that $0<\alpha<1$. Then the fractional derivative of a function $f\in C^1\left[0,T\right]$ may be approximated as

$$
\begin{aligned}
{}_0D_t^{\alpha}f(t)\quad &\approx\quad \frac{f(t)}{t^{\alpha}}+\frac{1}{\Gamma\left(2-\alpha\right)}\left\{f^{(1)}\left(t\right)\left[1+\sum_{p=1}^{N}\frac{\Gamma\left(p-1+\alpha\right)}{\Gamma\left(\alpha-1\right)p!}\right]t^{1-\alpha}\right.\\
&\left.-\sum_{p=2}^{N}\frac{\Gamma\left(p-1+\alpha\right)}{\Gamma\left(\alpha-1\right)\left(p-1\right)!}\left(\frac{f\left(t\right)}{t^{\alpha}}+\frac{V_p\left(t\right)}{t^{p-1+\alpha}}\right)\right\}, \qquad (2.3)
\end{aligned}
$$

for $t\in\left[0,T\right]$. Here

$$
V_{p-2}=\left(p-1\right)\int_0^t\tau^{p-2}f\left(\tau\right)d\tau,
$$

are moments of the function f. The expansion formula (2.3) was used in solving many concrete problems in Mechanics in which linear and nonlinear fractional differential equations appear. Some of the results of Prof. Stanković presented here are extended and generalized in works of his collaborators. For example, the expansion formula (2.3) was generalized to include fractional derivatives of complex order.

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