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A COMMON FIXED POINT RESULT IN STRONG JS-METRIC SPACE

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Dedicated to Professor Bogoljub Stanković (1924–2018)

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A b s t r a c t. The aim of our paper is the prove that the common fixed point result due to Sehgal and Thomas is valied in a class of generalized metric spaces in sence of Jleli and Samet.

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1. *Introduction*

Metric fixed point theory, motivated by Banach fixed point theorem, is one of the most powerful and useful tools in nonlinear functional analysis. The application of this theory is remarkable in wide scale of mathematical, engineering, economic, physical, computer science and other fields of science. Following the Banach contraction principle in many papers the existence of weaker contractive conditions combined with stronger additional assumptions on the mapping or on the space, are investigated.

In [8] Sehgal initiated the study of fixed point for mappings with contractive iterate at a point. This result was extended by many authors (see [2], [3], [4], [5]).

Definition 1.1. [8] In metric space (X, d), the mapping $T : X \to X$ is said to be with contractive iterate at a point $x \in X$ if there is a positive integer $n(x)$ such that for all $y \in X$

$$d(T^{n(x)}x, T^{n(x)}y) \leq qd(x, y), \tag{1.1}$$

where $q \in (0, 1)$.

In [9] V. M. Sehgal and J. W. Thomas proved a common fixed point result for a family of pointwise contractive self-mappings in metric space (X, d).

Definition 1.2. [9] Let (X, d) be a complete metric space and $M \subseteq X$. Let $\mathcal{F}$ be a commutative semigroup of self mappings (not necessarily continuous) of M. The semigroup $\mathcal{F}$ is pointwise contractive in M if for each $x \in M$, there is an $f_x \in \mathcal{F}$ such that

$$d(f_x(y), f_x(x)) \leq \varphi(d(y, x)), \tag{1.2}$$

for all $y \in M$, where φ is some real valued function defined on the nonnegative reals.

Theorem 1.1. [9] *Let M be a closed subset of X and $\mathcal{F}$ a commutative semigroup of self-mappings of M, which is pointwise contractive in M for some $\varphi : [0, \infty) \to [0, \infty)$, where φ id nondecreasing, continuous on the right and satisfies $\varphi(r) < r$ for all $r > 0$. If for some $x_0 \in M$,*

$$\sup\{d(f(x_0), x_0) : f \in \mathcal{F}\} < \infty, \tag{1.3}$$

then, there exists a unique $\xi \in M$ such that $f(\xi) = \xi$ for each $f \in \mathcal{F}$. Moreover, there is a sequence $\{g_n\} \subset \mathcal{F}$ with $g_n(x) \to \xi$ for each $x \in M$.

On the other side, recent research in fixed point theory are focused on generalizing the underlying space. Consideration various generalization of metric spaces (partial metric spaces, fuzzy metric spaces, probability metric spaces, quasy-metric spaces, uniform spaces, ultra metric spaces, $b-$metric spaces, cone metric spaces and so on) leads to opportunity to use distinct advantages by creating topological structure suitable for application in some cases when the classical metric does not give the answer.

Recently, Jleli and Samet [6] introduced a generalization of the notion of a metric spaces which they called a generalized metric space. They noted that some above-mentioned abstract metric spaces may be regarded as particular cases of their general definition. They also stated and proved fixed point theorems for some contractions defined on these spaces.

In this paper we consider Sehgal and Thomas common fixed point result in setting of generalized metric spaces in sence of Jleli and Samet.

2. *Preliminaries*

Let X be a nonempty set and $\mathcal{D} : X \times X \to [0, \infty]$ be a mapping. For every $x \in X$ define a set

$$C(\mathcal{D}, X, x) = \{\{x_n\} \subset X : \lim_{n\to\infty} \mathcal{D}(x_n, x) = 0\}.$$

In this case, we say that $\mathcal{D}$ is a generalized metric in the sense of Jleli and Samet [6] (for short JS-metric) on X if it satisfies the following conditions:

$(\mathcal{D}_1)$ for every $(x, y) \in X \times X,\ \mathcal{D}(x, y) = 0 \Rightarrow x = y$;

$(\mathcal{D}_2)$ for every $(x, y) \in X \times X,\ \mathcal{D}(x, y) = \mathcal{D}(y, x)$;

$(\mathcal{D}_3)$ there exists $C > 0$ such that for every $(x, y) \in X \times X$ and $\{x_n\} \in C(\mathcal{D}, X, x)$,

$$\mathcal{D}(x, y) \leq C \limsup_{n\to\infty} \mathcal{D}(x_n, y).$$

In this case $(X, \mathcal{D})$ is said to be JS-metric space. Note that, if $C(\mathcal{D}, X, x) = \emptyset$ for all $x \in X$, then $(\mathcal{D}_3)$ is trivially hold.

The class of JS-metric spaces is larger than many known class of abstract metric spaces. For example, every standard metric space, every b-metric space, every dislocated metric space (in the sense of Hitzler-Seda), and every modular space with the Fatou property is a JS-metric space. For more details see [6].

Given a JS-metric space $(X, \mathcal{D})$ and a point $x \in X$, a sequence $\{x_n\} \subseteq X$ is said to be:

1° $\mathcal{D} - convergent\ to\ x$ if $\{x_n\} \in C(\mathcal{D}, X, x)$, in such a case, we will write

$$\{x_n\} \xrightarrow{\mathcal{D}} x;$$

2° $\mathcal{D} - Cauchy$ if

$$\lim_{n,m\to\infty} \mathcal{D}(x_n, x_m) = 0. \tag{2.1}$$

A JS-metric space $(X, \mathcal{D})$ is *complete* if every $\mathcal{D}$-Cauchy sequence in X is $\mathcal{D}$-convergent.

Jleli and Samet proved that the limit of a $\mathcal{D}$-convergent sequence is unique.

Proposition 2.1. [6] *Let $(X, \mathcal{D})$ be a JS-metric space. Let $\{x_n\}$ be a sequence in X and $(x, y) \in X \times X$. If $\{x_n\}$ $\mathcal{D}$-converges to x and $\{x_n\}$ $\mathcal{D}$-converges to y, then $x = y$.*

They also proved fixed point theorem for q-contraction defined on these spaces as a generalization of Banach contraction principle.

Theorem 2.1. [6] *Suppose that the following conditions hold:*

(i) $(X, \mathcal{D})$ *is a complete JS-metric space,*

(ii) $f : X \to X$ *is a k-contraction for some* $k \in (0,1)$*, that is,*

$$\mathcal{D}(f(x), f(y)) \leq k\mathcal{D}(x,y)$$

for all $(x,y) \in X \times X$.

(iii) *there exists* $x_0 \in X$ *such that* $\sup\{\mathcal{D}(x_0, f^n(x_0) : n \in \mathbb{N}\} < \infty$.

Then $\{f^n(x_0\}$ *converges to* $\omega \in X$*, a fixed point of* f*. Moreover, if* $\omega' \in X$ *is another fixed point of* f *such that* $\mathcal{D}(\omega, \omega') < \infty$*, then* $\omega = \omega'$.

For more fixed point results in JS-metric spaces we refer the reader to [6], [7].

Now we are going to introduce an interesting subclass of JS-metric spaces as a frame for our researches.

Definition 2.1. Let X be a nonempty set and let $\mathcal{D}^* : X \times X \to [0, \infty)$ be a function which satisfies the following conditions:

($\mathcal{D}^*1$) $\mathcal{D}^*(x,y) = 0 \Leftrightarrow x = y$;

($\mathcal{D}^*2$) $\mathcal{D}^*(x,y) = \mathcal{D}^*(y,x)$, for all $x, y \in X$;

($\mathcal{D}^*3$) there exists $C > 0$ such that,
for every $x, y \in X$ and $\{x_n\} \in C(\mathcal{D}^*, X, x), \{y_n\} \in C(\mathcal{D}^*, X, y)$
$\mathcal{D}^*(x,y) \leq C \limsup_{n\to\infty} \mathcal{D}^*(x_n, y_n)$.

Then $\mathcal{D}^*$ is a *strong JS-metric* and the pair $(X, \mathcal{D}^*)$ is a *strong JS-metric space*.

Remark 2.1. Since sequence $x_n = x$, $n \in \mathbf{N}$, $\mathcal{D}^*$-converges to x every strong JS-metric space is JS-metric space.

Example 2.1. Every standard metric space (X, d) is a strong JS-metric space.

Since, we only have to check condition ($\mathcal{D}^*3$), let $x, y \in X$, $\{x_n\} \stackrel{d}{\to} x$, and $\{y_n\} \stackrel{d}{\to} y$. Using triangle inequality twice

$$d(x,y) \leq d(x,x_n) + d(x_n,y_n) + d(y_n,y)$$

for all naturale number n, so

$$d(x,y) \leq \limsup_{n\to\infty} d(x_n, y_n)$$

and ($\mathcal{D}^*3$) is satisfied with $C = 1$.

Example 2.2. Every b-metric space (X, d_b) is a strong metric space.

Let us recall the definition of b-metric space.

Let $d_b : X \times X \to (0, +\infty)$ be a given mapping. We say that d_b is a b-metric on X if it satisfies the following conditions:

$(b1)$ for every $(x, y) \in X \times X$, we have

$$d_b(x, y) = 0 \Leftrightarrow x = y;$$

$(b2)$ for every $(x, y) \in X \times X$,

$$d_b(x, y) = d_b(y, x);$$

$(b3)$ there exists $s \geq 1$ such that, for every $x, y, z \in X$,

$$d_b(x, y) \leq s(d_b(x, z) + d_b(z, y)).$$

In this case, (X, d_b) is said to be a b-metric space.

Obviously, we have to prove that d_b satisfies the property $(\mathcal{D}^*3)$. Let $x, y \in X$ and $\{x_n\} \overset{d_b}{\to} x$, $\{y_n\} \overset{d_b}{\to} y$. By property $(b3)$ we have that

$$\begin{aligned} d_b(x, y) &\leq s(d_b(x, x_n) + d_b(x_n, y)) \\ &\leq s d_b(x, x_n) + s^2 d_b(x_n, y_n) + s^2 d_b(y_n, y) \end{aligned}$$

for every natural number n. Thus, we have

$$d_b(x, y) \leq s^2 \limsup_{n\to\infty} d_b(x_n, y_n)$$

and the property $(\mathcal{D}^*3)$ is satisfied with $c = s^2$.

3. *Main result*

A generalization of the contraction principle can be obtained using different type of a nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$. The most usual additional properties imposed on φ are:

(φ_1) $\varphi(0) = 0$,

(φ_2) φ is right continuous and $\varphi(t) < t$, for all $t > 0$,

(φ_3) $\lim_{i\to\infty} \varphi^i(t) = 0,$ for all $t > 0$,

(φ_4) $\{t_i\} \subset [0, \infty)$ is a sequence such that $t_{i+1} \leq \varphi(t_i)$, then $\lim_{i\to\infty} t_i = 0$.

It is well known that

$$(\varphi_4) \Leftrightarrow (\varphi_3) \Leftrightarrow (\varphi_2) \Rightarrow (\varphi_1),$$

and φ with property (φ_2), is continuous at 0.

Theorem 3.1. *Let B be a closed nonempty subset of complete strong JS-metric space $(X, \mathcal{D}^*)$ and $\mathcal{F}$ a commutative semigroup of self-mapping of B pointwise contractive in B for some $\varphi : [0,\infty) \to [0,\infty)$, where φ is nondecreasing continuous on the right and satisfies $\varphi(t) < t$, for all $t > 0$.*

If for some $x_0 \in B$

$$\sup\{\mathcal{D}^*(x_0, f(x_0)) : f \in \mathcal{F}\} < \infty$$

then, there exists a unique $u \in B$ such that $f(u) = u$ for each $f \in \mathcal{F}$. Moreover, there is a sequence $\{g_n\} \subset \mathcal{F}$ with $\{g_n(x)\} \overset{\mathcal{D}^*}{\to} u$, for each $x \in B$.

PROOF. If $d_0 = \sup\{\mathcal{D}^*(x_0, f(x_0)) : f \in \mathcal{F}\}$, then $\varphi^n(d_0) \to 0$, $n \to \infty$. Let $f_0 = f_{x_0}$ and inductively $f_n = f_{x_n}$, where $x_{n+1} = f_n(x_n)$. Then, for a fixed $k \geq 0$

$$\sup_{n\geq k} \mathcal{D}^*(x_{n+1}, x_{k+1}) = \sup_{n\geq k} \mathcal{D}^*(f_n \circ f_{n-1} \circ \cdots \circ f_k(x_k), f_k(x_k)).$$

Let $h_n = f_n \circ f_{n-1} \circ \cdots \circ f_{k+1}$. It follows that

$$\begin{aligned} \sup_{n\geq k} \mathcal{D}^*(x_{n+1}, x_{k+1}) &= \sup_{n\geq k} \mathcal{D}^*(f_k(h_n(x_k)), f_k(x_k)) \\ &\leq \sup_{n\geq k} \varphi(\mathcal{D}^*(h_n(x_k), x_k)) \\ &\leq \sup_{n\geq k} \varphi^{k+1}(\mathcal{D}^*(h_n(x_0), x_0)) \to 0, \end{aligned}$$

when $k \to \infty$. Thus, the sequence $\{x_n\}$ is $\mathcal{D}^*$-Cauchy. Let $\{x_n\} \overset{\mathcal{D}^*}{\to} u \in B$. By hypotheses, there is a $f_u \in \mathcal{F}$ such that

$$\mathcal{D}^*(f_u(x_n), f_u(u)) \leq \varphi(\mathcal{D}^*(x_n, u)) \to 0, \quad n \to \infty,$$

so that

$$\{f_u(x_n)\} \overset{\mathcal{D}^*}{\to} f_u(u).$$

On the other side $\mathcal{D}^*(x_k, f_u(x_k)) \leq \varphi^k(\mathcal{D}^*(x_0, f_u(x_0)) \to 0,\ k \to \infty$, so that

$$\mathcal{D}^*(u, f_u(u)) \leq C \cdot \limsup_{n\to\infty} \mathcal{D}^*(x_k, f_u(x_k)) \to 0, \quad k \to \infty,$$

and consequently $f_u(u) = u$.

Using (1.2) one can prove that u is a unique fixed point of f_u on B. Furthermore, since $\mathcal{F}$ is commutative, for any $f \in \mathcal{F}$

$$f(u) = f(f_u(u)) = f_u(f(u))$$

and therefore $f(u) = u$, for all $f \in \mathcal{F}$.

For each nonnegative integer n, set $g_n = f_u^n$. Obviously $g_n \in \mathcal{F}$.

For any fixed $x \in B$

$$\mathcal{D}^*(g_n(x), u) = \mathcal{D}^*(f_u^n(x), f_u(u)) \leq \varphi^n(\mathcal{D}^*(x, u)) \to 0,\ n \to \infty,$$

so $g_n(x) \stackrel{\mathcal{D}^*}{\to} u$, and proof is completed.

Corollary 3.1. *Let B be a closed subset of complete strong JS-metric space $(X, \mathcal{D}^*)$ and f a self-mapping of B. If f satisfies conditions:*

1° *there exists $x_0 \in B$ such that* $\sup\{\mathcal{D}^*(x_0, f^k(x_0)) : k \in \mathbb{N}\} < \infty$*;*

2° *for each $x \in B$, there exists an integer $n(x) \geq 1$ such that for all $y \in B$*

$$\mathcal{D}^*(f^{n(x)}(x), f^{n(x)}(y)) \leq \varphi(\mathcal{D}^*(x, y)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is nondecreasing right continuous function and $\varphi(t) < t$, for $t > 0$, then there exists a unique $u \in B$ such that $f(u) = u$ and $\{f^k(x)\} \stackrel{\mathcal{D}^*}{\to} u$, for any $x \in B$.

PROOF. Family $\mathcal{F} = \{f^k : k \in \mathbb{N}\}$ is a commutative semigroup of pointwise contractive mappings in B so by Theorem 3.1, there exists a unique fixed point u of f and there is $\{g_k\} \subset \mathcal{F}$ such that $\{g_k(x)\} \stackrel{\mathcal{D}^*}{\to} u$, for each $x \in B$.

Let us prove that $\{f^k(x)\} \stackrel{\mathcal{D}^*}{\to} u$, for any $x \in B$. For k sufficiently large we have $k = r \cdot n(u) + s$, with $r > 0$ and $0 \leq s < n(u)$ and therefore for any $x \in B$

$$\begin{aligned} \mathcal{D}^*(f^k(x), u) &= \mathcal{D}^*(f^{r \cdot n(u)+s}(x), f^{n(u)}(u)) \\ &\leq \varphi^r(\mathcal{D}^*(f^s(x), u)) \\ &\leq \varphi^r(d_0) \end{aligned}$$

for $d_0 = \max\{\mathcal{D}^*(f^s(x), u) : 0 \leq s \leq n(u) - 1\}$. Since $\varphi^r(d_0) \to 0$, $r \to \infty$, it follows that $\{f^k(x)\} \stackrel{\mathcal{D}^*}{\to} u$, for all $x \in B$, so proof is completed.

The next corollary is a generalization of Guseman fixed point result from [5].

Corollary 3.2. *Let B be a closed subset of complete strong JS-metric space $(X, \mathcal{D}^*)$ and f a self-mapping of B. If f is a mapping with contractive iterate at any point $x \in B$ for some $0 < q < 1$ and there exists $x_0 \in B$ such that $\sup\{\mathcal{D}^*(x_0, f^k(x_0)) : k \in \mathbb{N}\} < \infty$, then there exists a unique $u \in B$ such that $f(u) = u$ and $\{f^k(x)\} \overset{\mathcal{D}^*}{\to} u$ for any $x \in B$.*

PROOF. Take, $\varphi(t) = qt$, $t \in [0, \infty)$, and use Corollary 3.1.

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