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DEGENERATE C-ULTRADISTRIBUTION SEMIGROUPS IN LOCALLY CONVEX SPACES

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A b s t r a c t. The main subject in this paper are degenerate C-ultradistribution semigroups in barreled sequentially complete locally convex spaces. Here, the regularizing operator C is not necessarily injective and the infinitesimal generator of semigroup is a multivalued linear operator. We also consider exponential degenerate C-ultradistribution semigroups.

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1. *Introduction and preliminaries*

This is an expository paper. We collect results which simply follows from the known ones. Because of that, proofs are not given. In [18] the classes of C-distribution and C-ultradistribution semigroups in locally convex spaces (cf. [4]–[8], [10], [12], [14]–[16], [22]–[24], [27]–[30] and references cited therein for the current state of

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theory) are introduced and systematically analyzed. The recent paper [20] motivates us to continue the study of generalized degenerate C-regularized semigroups in locally convex spaces in the case of ultradistribution semigroups. The main aim of this paper is to investigate the degenerate C-ultradistribution semigroups in the setting of barreled sequentially complete locally convex spaces. We refer to [5], [11], [17], [27] and [29] for further information about well-posedness of abstract degenerate differential equations of first order. Here, we consider multivalued linear operators as infinitesimal generators of degenerate C-ultradistribution semigroups (cf. [3], [12], [22], [25]). The organization of the paper is as follows. In Section 1 we expose the basic facts about vector-valued ultradistributions. Our main results are contained in Section 2, in which we analyze various themes concerning degenerate C-ultradistribution semigroups in locally convex spaces and further generalize some of our recent results from [18] and [20].

1.1. Notation

We use the standard notation throughout the paper. Unless specified otherwise, we assume that E is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short. For the sake of brevity and better exposition, our standing assumption henceforth will be that the state space E is barreled. By $L(E)$ we denote the space consisting of all continuous linear mappings from E into E and by the symbol $\circledast_E$ (usually we will denote $\circledast$ if there is no risk for confusion) denotes the fundamental system of seminorms which defines the topology of E. Let X be also an SCLCS, let $\mathcal{B}$ be the family of bounded subsets of E, and let $p_B(T) := \sup_{x \in B} p(Tx)$, $p \in \circledast_X$, $B \in \mathcal{B}$, $T \in L(E, X)$. Then $p_B(\cdot)$ is a seminorm on $L(E, X)$ and the system $(p_B)_{(p,B) \in \circledast_X \times \mathcal{B}}$ induces the Hausdorff locally convex topology on $L(E, X)$. The Hausdorff locally convex topology on E^*, the dual space of E, defines the system $(|\cdot|_B)_{B \in \mathcal{B}}$ of seminorms on E^*, where $|x^*|_B := \sup_{x \in B} |\langle x^*, x\rangle|$, $x^* \in E^*$, $B \in \mathcal{B}$. The bidual of E is denoted by E^{**}. The polars of nonempty sets $M \subseteq E$ and $N \subseteq E^*$ are defined as follows

$$M^\circ := \{y \in E^* : |y(x)| \leq 1 \text{ for all } x \in M\}$$

and

$$N^\circ := \{x \in E : \ |y(x)| \leq 1 \text{ for all } y \in N\}.$$

If A is a linear operator acting on E, then the domain, kernel space and range of A will be denoted by $D(A)$, $N(A)$ and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. Since we have assumed that the state space E is barreled, the spaces $L(E)$ and E^* are sequentially complete ([26]) and any strongly continuous operator family $(S(t))_{t\in[0,\tau)} \subseteq L(E)$, where $0 < \tau \leq \infty$, is locally equicontinuous. The reader may consult [31] and [17] for further information on the Laplace transform of functions with values in SCLCS's; cf. [2] for the Banach space case.

We assume that (M_p) is a sequence of positive real numbers such that $M_0 = 1$ and the following conditions hold:

(M.1): $M_p^2 \leq M_{p+1}M_{p-1},\ \ p \in \mathbb{N},$

(M.2): $M_p \leq AH^p \sup\limits_{0\leq i\leq p} M_i M_{p-i},\ \ p \in \mathbb{N},$ for some $A,\ H > 1,$

(M.3)': $\sum\limits_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$

Every employment of the condition

(M.3): $\sup\limits_{p\in\mathbb{N}} \sum\limits_{q=p+1}^{\infty} \frac{M_{q-1}M_{p+1}}{pM_pM_q} < \infty,$

which is a slightly stronger than (M.3)', will be explicitly emphasized.

Let $s > 1$. Then the Gevrey sequence $(p!^s)$ satisfies the above conditions. The associated function of sequence (M_p) is defined by $M(\rho) := \sup\limits_{p\in\mathbb{N}} \ln \frac{\rho^p}{M_p},\ \rho > 0$; $M(0) := 0,\ M(\lambda) := M(|\lambda|),\ \lambda \in \mathbb{C} \setminus [0, \infty)$.

Let us recall that the spaces of Beurling, respectively, Roumieu ultradifferentiable functions are defined by $\mathcal{D}^{(M_p)} := \mathcal{D}^{(M_p)}(\mathbb{R}) := \text{indlim}_{K\Subset\Subset\mathbb{R}}\mathcal{D}_K^{(M_p)}$, respectively, $\mathcal{D}^{\{M_p\}} := \mathcal{D}^{\{M_p\}}(\mathbb{R}) := \text{indlim}_{K\Subset\Subset\mathbb{R}}\mathcal{D}_K^{\{M_p\}}$, (where K goes through all compact sets in $\mathbb{R}$ where $\mathcal{D}_K^{(M_p)} := \text{projlim}_{h\to\infty}\mathcal{D}_K^{M_p,h}$, respectively, $\mathcal{D}_K^{\{M_p\}} := \text{indlim}_{h\to 0}\mathcal{D}_K^{M_p,h}$,

$$\mathcal{D}_K^{M_p,h} := \bigl\{\phi \in C^\infty(\mathbb{R}) : \text{supp}(\phi) \subseteq K,\ \|\phi\|_{M_p,h,K} < \infty\bigr\},$$

$$\|\phi\|_{M_p,h,K} := \sup\left\{\frac{h^p\bigl|\phi^{(p)}(t)\bigr|}{M_p} : t \in K,\ p \in \mathbb{N}_0\right\}.$$

Spaces of tempered ultradistributions are defined as strong dual of corresponding test spaces:

$$\mathcal{S}^{(M_p)}(\mathbb{R}) := \text{proj}\lim_{k\to\infty} \mathcal{S}^{M_p,k}(\mathbb{R}),\ \text{resp.},\ \ \mathcal{S}^{\{M_p\}}(\mathbb{R}) := \text{ind}\lim_{k\to 0} \mathcal{S}^{M_p,k}(\mathbb{R}),$$

where
$$\mathcal{S}^{M_p,k}(\mathbb{R}) := \Big\{\phi \in C^{\infty}(\mathbb{R}) : ||\phi||_{M_p,k} < \infty\Big\},\ k > 0,$$
$$||\phi||_{M_p,k} := \ \sup\Bigg\{\frac{k^{\alpha+\beta}}{M_\alpha M_\beta}(1+|t|^2)^{\beta/2}|\phi^{(\alpha)}(t)| : t \in \mathbb{R},\ \alpha,\ \beta \in \mathbb{N}_0\Bigg\}.$$
Henceforth the asterisk $*$ stands for both cases.

Let $\emptyset \neq \Omega \subseteq \mathbb{R}$. The spaces $\mathcal{D}'^*(E) := L(\mathcal{D}^*, E)$, $\mathcal{D}^*_\Omega$, $\mathcal{D}^*_0$, $\mathcal{E}'^*_\Omega$, $\mathcal{E}'^*_0$, $\mathcal{D}'^*_\Omega(E)$, $\mathcal{D}'^*_0(E)$ and $\mathcal{S}'^*_0(E)$ are defined as in distribution case. We know that there exists a regularizing sequence in $\mathcal{D}^*$. Regularizing sequence in $\mathcal{D}^*$ is any sequence $(\rho_n)_{n\in\mathbb{N}}$ in $\mathcal{D}^*_0$ for which there exists a function $\rho \in \mathcal{D}^*$ such that $\int_{-\infty}^{\infty} \rho(t)\,dt = 1$, supp$(\rho) \subseteq [0,1]$ and $\rho_n(t) = n\rho(nt)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$. We define the convolution products $\varphi * \psi$ and $\varphi *_0 \psi$ by
$$\varphi * \psi(t) := \int_{-\infty}^{\infty} \varphi(t-s)\psi(s)\,\mathrm{d}s \ \text{ and } \ \varphi *_0 \psi(t) := \int_0^t \varphi(t-s)\psi(s)\,\mathrm{d}s,\ t\ \in \mathbb{R},$$
for $\varphi, \psi : \mathbb{R} \to \mathbb{C}$ locally integrable functions. Notice that $\varphi * \psi = \varphi *_0 \psi$, provided that supp$(\varphi)$ and supp(ψ) are subsets of $[0,\infty)$. Given $\varphi \in \mathcal{D}^*$ and $f \in \mathcal{D}'^*$, or $\varphi \in \mathcal{E}^*$ and $f \in \mathcal{E}'^*$, we define the convolution $f * \varphi$ by $(f * \varphi)(t) := f(\varphi(t-\cdot))$, $t \in \mathbb{R}$. For $f \in \mathcal{D}'^*$, or for $f \in \mathcal{E}'^*$, define $\check{f}$ by $\check{f}(\varphi) := f(\varphi(-\cdot))$, $\varphi \in \mathcal{D}^*$ $(\varphi \in \mathcal{E}^*)$. The convolution of two ultradistributions $f, g \in \mathcal{D}'^*$, denoted by $f * g$, is defined by $(f * g)(\varphi) := g(\check{f} * \varphi)$, $\varphi \in \mathcal{D}^*$.

We recall the definition of a multivalued map (multimap) (cf. [9] by R. Cross, [11] by A. Favini-A. Yagi). Let X and Y be two SCLCSs. Then a multivalued map (multimap) $\mathcal{A} : X \to P(Y)$ is said to be a multivalued linear operator (MLO) iff the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a subspace of X;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

If $X = Y$, then it is also said that $\mathcal{A}$ is an MLO in X. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It is easily seen that $\mathcal{A}^{-1}$ is an MLO in X, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective.

If $\mathcal{A},\ \mathcal{B} : X \to P(Y)$ are two MLOs, then we define its sum $\mathcal{A} + \mathcal{B}$ by
$$D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$$
and $(\mathcal{A}+\mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A}+\mathcal{B})$. It can be simply checked that $\mathcal{A}+\mathcal{B}$ is likewise an MLO.

Let $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : Y \to P(Z)$ be two MLOs, where Z is an SCLCS. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{BA}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{BA}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. Then $\mathcal{BA} : X \to P(Z)$ is an MLO and $(\mathcal{BA})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A} : X \to P(Y)$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$. It is clear that $z\mathcal{A} : X \to P(Y)$ is an MLO and $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, $z,\ \omega \in \mathbb{C}$.

The integer powers of an MLO $\mathcal{A} : X \to P(X)$ is defined recursively as follows: $\mathcal{A}^0 =: I$; if $\mathcal{A}^{n-1}$ is defined, set $D(\mathcal{A}^n) := \{x \in D(\mathcal{A}^{n-1}) : D(\mathcal{A}) \cap \mathcal{A}^{n-1}x \neq \emptyset\}$, and $\mathcal{A}^n x := (\mathcal{A}\mathcal{A}^{n-1})x = \bigcup_{y \in D(\mathcal{A}) \cap \mathcal{A}^{n-1}x} \mathcal{A}y,\ \ x \in D(\mathcal{A}^n)$. It is well known that $(\mathcal{A}^n)^{-1} = (\mathcal{A}^{n-1})^{-1}\mathcal{A}^{-1} = (\mathcal{A}^{-1})^n =: \mathcal{A}^{-n}$, $n \in \mathbb{N}$ and $D((\lambda - \mathcal{A})^n) = D(\mathcal{A}^n)$, $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$. Moreover, if $\mathcal{A}$ is single-valued, then the above definitions are consistent with the usual definition of powers of $\mathcal{A}$.

If $\mathcal{A} : X \to P(Y)$ is an MLO, then we define the adjoint $\mathcal{A}^* : Y^* \to P(X^*)$ of $\mathcal{A}$ by its graph

$$\mathcal{A}^* := \Big\{ \big(y^*, x^*\big) \in Y^* \times X^* : \big\langle y^*, y \big\rangle = \big\langle x^*, x \big\rangle \text{ for all pairs } (x, y) \in \mathcal{A} \Big\}.$$

In [17], we have recently considered the C-resolvent sets of MLOs in locally convex spaces (where $C \in L(X)$ is injective, $C\mathcal{A} \subseteq \mathcal{A}C$). The C-resolvent set of an MLO $\mathcal{A}$ in X, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which $R(C) \subseteq R(\lambda - \mathcal{A})$ and $(\lambda - \mathcal{A})^{-1}C$ is a single-valued bounded operator on X. The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$). Here, we analyze the general situation in which the operator $C \in L(X)$ is not necessarily injective. Then the operator $(\lambda - \mathcal{A})^{-1}C$ is no longer single-valued, which additionally hinders our considerations and work.

2. *Properties of degenerate C-ultradistribution semigroups in locally convex spaces*

Throughout this section, we assume that $C \in L(E)$ is not necessarily injective operator. Since E is barreled, the uniform boundedness principle [26, p. 273] implies that each $\mathcal{G} \in \mathcal{D}'^*(L(E))$ is boundedly equicontinuous, i.e., that for every $p \in ⊛$ and for every bounded subset B of $\mathcal{D}^*$, there exist $c > 0$ and $q \in ⊛$ such that $p(\mathcal{G}(\varphi)x) \leq cq(x),\ \varphi \in B,\ x \in E$.

Definition 2.1. Let $\mathcal{G} \in \mathcal{D}_0'^*(L(E))$ satisfy $C\mathcal{G} = \mathcal{G}C$. Then it is said that $\mathcal{G}$ is a pre-(C-UDS) of $*$-class iff the following holds:

$$\mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi), \quad \varphi,\ \psi \in \mathcal{D}^*. \tag{C.S.1}$$

If, additionally,

$$\mathcal{N}(\mathcal{G}) := \bigcap_{\varphi \in \mathcal{D}_0^*} N(\mathcal{G}(\varphi)) = \{0\}, \tag{C.S.2}$$

then $\mathcal{G}$ is called a C-ultradistribution semigroup of $*$-class, (C-UDS) of $*$-class in short. A pre-(C-UDS) $\mathcal{G}$ is called dense iff

$$\mathcal{R}(\mathcal{G}) := \bigcup_{\varphi \in \mathcal{D}_0^*} R(\mathcal{G}(\varphi)) \text{ is dense in } E. \tag{C.S.3}$$

If $C = I$, then we also write pre-(UDS), (UDS), instead of pre-(C-UDS), (C-UDS).

Suppose that $\mathcal{G}$ is a pre-(C-UDS) of $*$-class. Then $\mathcal{G}(\varphi)\mathcal{G}(\psi) = \mathcal{G}(\psi)\mathcal{G}(\varphi)$ for all φ, $\psi \in \mathcal{D}^*$ and $\mathcal{N}(\mathcal{G})$ is a closed subspace of E.

The structural characterization of a pre-(C-UDS) $\mathcal{G}$ of $*$-class on its kernel space $\mathcal{N}(\mathcal{G})$ is described in the following theorem (cf. [15, Proposition 3.1.1] and the proofs of [22, Lemma 2.2], [15, Proposition 3.5.4]).

Theorem 2.1. *Let (M_p) satisfy* (M.3)*, let $\mathcal{G}$ be a pre-(C-UDS) of $*$-class, and let the space $\mathcal{N}(\mathcal{G})$ be barreled. Then, with $N = \mathcal{N}(\mathcal{G})$ and G_1 being the restriction of $\mathcal{G}$ to N ($G_1 = \mathcal{G}_{|N}$), we have: There exists a unique set of operators $(T_j)_{j\in\mathbb{N}_0}$ in $L(\mathcal{N}(\mathcal{G}))$ commuting with C so that $G_1 = \sum_{j=0}^{\infty} \delta^{(j)} \otimes T_j$, $T_j C^j = (-1)^j T_0^{j+1}$, $j \in \mathbb{N}$ and the set $\{M_j T_j L^j : j \in \mathbb{N}_0\}$ is bounded in $L(\mathcal{N}(\mathcal{G}))$, for some $L > 0$ in the Beurling case, resp. for every $L > 0$ in the Roumieu case.*

Let $\mathcal{G} \in \mathcal{D}_0'^*(L(E))$, and let $T \in \mathcal{E}_0'^*$, i.e., T is a scalar-valued ultradistribution of $*$-class with compact support contained in $[0, \infty)$. Define

$$G(T) := \Big\{(x, y) \in E \times E : \mathcal{G}(T * \varphi)x = \mathcal{G}(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0^*\Big\}.$$

Then it can be easily seen that $G(T)$ is a closed MLO; furthermore, if $\mathcal{G} \in \mathcal{D}_0'^*(L(E))$ satisfy (C.S.2), then $G(T)$ is a closed linear operator. Assuming that the regularizing operator C is injective, definition of $G(T)$ can be equivalently introduced by replacing the set $\mathcal{D}_0^*$ with the set $\mathcal{D}^*_{[0,\epsilon)}$ for any $\epsilon > 0$. In general case, for every $\psi \in \mathcal{D}^*$, we have $\psi_+ := \psi \mathbf{1}_{[0,\infty)} \in \mathcal{E}_0'^*$, where $\mathbf{1}_{[0,\infty)}$ stands for the characteristic function of $[0, \infty)$, so that the definition of $G(\psi_+)$ is clear. We define the (infinitesimal) generator of a pre-(C-UDS) $\mathcal{G}$ by $\mathcal{A} := G(-\delta')$ (cf. [18] for more details about non-degenerate case, and [3, Definition 3.4] and [12] for some other approaches used in degenerate case). Then $\mathcal{N}(\mathcal{G}) \times \mathcal{N}(\mathcal{G}) \subseteq \mathcal{A}$ and $\mathcal{N}(\mathcal{G}) = \mathcal{A}0$, which simply implies that $\mathcal{A}$ is single-valued iff (C.S.2) holds. If this is the case, then we also have that the operator C must be injective: Suppose that $Cx = 0$ for some $x \in E$. By (C.S.1), we get that $\mathcal{G}(\varphi)\mathcal{G}(\psi)x = 0$, φ, $\psi \in \mathcal{D}$. In particular, $\mathcal{G}(\psi)x \in \mathcal{N}(\mathcal{G}) = \{0\}$ so that $\mathcal{G}(\psi)x = 0$, $\psi \in \mathcal{D}$. Hence, $x \in \mathcal{N}(\mathcal{G}) = \{0\}$ and therefore $x = 0$.

Further on, if $\mathcal{G}$ is a pre-(C-UDS) of $*$-class, $T \in \mathcal{E}_0'^*$ and $\varphi \in \mathcal{D}^*$, then $\mathcal{G}(\varphi)G(T) \subseteq G(T)\mathcal{G}(\varphi)$, $CG(T) \subseteq G(T)C$ and $\mathcal{R}(\mathcal{G}) \subseteq D(G(T))$. If $\mathcal{G}$ is a pre-(C-UDS) of $*$-class and φ, φ, $\psi \in \mathcal{D}^*$, then the assumption $\varphi(t) = \psi(t)$, $t \geq 0$, implies $\mathcal{G}(\varphi) = \mathcal{G}(\psi)$. As in the Banach space case, we can prove the following (cf. [15, Proposition 3.1.3, Lemma 3.1.6]): Suppose that $\mathcal{G}$ is a pre-(C-UDS) of $*$-class. Then $(Cx, \mathcal{G}(\psi)x) \in G(\psi_+)$, $\psi \in \mathcal{D}^*$, $x \in E$ and $\mathcal{A} \subseteq C^{-1}\mathcal{A}C$, while $C^{-1}\mathcal{A}C = \mathcal{A}$ provided that C is injective. The following two propositions holds in degenerate C-ultradistribution case (see [20] for degenerate C-distribution case). Note that the reflexivity of the space E implies that the spaces E^* and $E^{**} = E$ are both barreled and sequentially complete locally convex spaces.

Proposition 2.1. *Let $\mathcal{G}$ be a pre-(C-UDS) of $*$-class, S, $T \in \mathcal{E}_0'^*$, $\varphi \in \mathcal{D}_0^*$, $\psi \in \mathcal{D}^*$ and $x \in E$. Then we have:*

(i) $(\mathcal{G}(\varphi)x, \mathcal{G}(\overbrace{T * \cdots * T}^{m} * \varphi)x) \in G(T)^m$, $m \in \mathbb{N}$.

(ii) $G(S)G(T) \subseteq G(S * T)$ *with* $D(G(S)G(T)) = D(G(S * T)) \cap D(G(T))$, *and* $G(S) + G(T) \subseteq G(S + T)$.

(iii) $(\mathcal{G}(\psi)x, \mathcal{G}(-\psi')x - \psi(0)Cx) \in G(-\delta')$.

(iv) *If $\mathcal{G}$ is dense, then its generator is densely defined.*

The assertions (ii)–(vi) of [15, Proposition 3.1.2] can be reformulated for pre-(C-UDS)'s of $*$-class in locally convex spaces.

Proposition 2.2. *Let $\mathcal{G}$ be a pre-(C-UDS) of $*$-class. Then the following holds:*

(i) $C(\overline{\langle \mathcal{R}(\mathcal{G}) \rangle}) \subseteq \overline{\mathcal{R}(\mathcal{G})}$, *where $\langle \mathcal{R}(\mathcal{G}) \rangle$ denotes the linear span of $\mathcal{R}(\mathcal{G})$.*

(ii) *Assume $\mathcal{G}$ is not dense and $\overline{C\mathcal{R}(\mathcal{G})} = \overline{\mathcal{R}(\mathcal{G})}$. Put $R := \overline{\mathcal{R}(\mathcal{G})}$ and $H := \mathcal{G}_{|R}$. Then H is a dense pre-(C_1-UDS) of $*$-class on R with $C_1 = C_{|R}$.*

(iii) *The dual $\mathcal{G}(\cdot)^*$ is a pre-(C^*-UDS) of $*$-class on E^* and $\mathcal{N}(\mathcal{G}^*) = \overline{\mathcal{R}(\mathcal{G})}^{\circ}$.*

(iv) *If E is reflexive, then $\mathcal{N}(\mathcal{G}) = \overline{\mathcal{R}(\mathcal{G}^*)}^{\circ}$.*

(v) *The $\mathcal{G}^*$ is a (C^*-UDS) of $*$-class in E^* iff $\mathcal{G}$ is a dense pre-(C-UDS) of $*$-class. If E is reflexive, then $\mathcal{G}^*$ is a dense pre-(C^*-UDS) of $*$-class in E^* iff $\mathcal{G}$ is a (C-UDS) of $*$-class.*

The following proposition has been recently proved in [18] in the case that the operator C is injective (cf. [12, Proposition 2]). By the proof of the statement in [18], it is clear that the injectivity of C is superfluous.

Proposition 2.3. *Suppose that $\mathcal{G} \in \mathcal{D}_0'^{*}(L(E))$ and $\mathcal{G}(\varphi)C = C\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}^*$. Then $\mathcal{G}$ is a pre-(C-UDS) of $*$-class if and only if*

$$\mathcal{G}\big(\varphi'\big)\mathcal{G}(\psi) - \mathcal{G}(\varphi)\mathcal{G}\big(\psi'\big) = \psi(0)\mathcal{G}(\varphi)C - \varphi(0)\mathcal{G}(\psi)C, \quad \varphi,\ \psi \in \mathcal{D}^*.$$

In [18], we have recently proved that every (C-UDS) of $*$-class in locally convex space is uniquely determined by its generator. Contrary to the single-valued case, different pre-(C-UDS)'s of $*$-class can have the same generator.

Remark 2.1. Suppose that $\mathcal{G} \in \mathcal{D}_0'^{*}(L(E))$, $\mathcal{G}(\varphi)C = C\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}^*$ and $\mathcal{A}$ is a closed MLO on E satisfying that $\mathcal{G}(\varphi)\mathcal{A} \subseteq \mathcal{A}\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}^*$ and

$$\mathcal{G}\big(-\varphi'\big)x - \varphi(0)Cx \in \mathcal{A}\mathcal{G}(\varphi)x, \quad x \in E,\ \varphi \in \mathcal{D}^*. \tag{2.1}$$

The following statements hold (see [18]):

(i) If $\mathcal{A} = A$ is single-valued, then $\mathcal{G}$ satisfies (C.S.1).

(ii) If $\mathcal{G}$ satisfies (C.S.2) holds, C is injective and $\mathcal{A} = A$ is single-valued, then $\mathcal{G}$ is a (C-UDS) of $*$-class generated by $C^{-1}AC$.

As we have already seen, the conclusion from (ii) immediately implies that $\mathcal{A} = A$ must be single-valued and that the operator C must be injective.

Concerning the assertion (i), its validity is not true in multivalued case: Let $C = I$, let $\mathcal{A} \equiv E \times E$, and let $\mathcal{G} \in \mathcal{D}_0'^{*}(L(E))$ be arbitrarily chosen. Then $\mathcal{G}$ commutes with $\mathcal{A}$ and (2.1) holds but $\mathcal{G}$ need not satisfy (C.S.1).

Next, we give the definition of an $(q-)$exponential pre-(C-UDS) of $*$-class.

Definition 2.2. Let $\mathcal{G}$ be a pre-(C-UDS) of $*$-class. Then $\mathcal{G}$ is said to be an exponential pre-(C-UDS) of $*$-class iff there exists $\omega \in \mathbb{R}$ such that $e^{-\omega t}\mathcal{G} \in \mathcal{S}'^{*}(L(E))$. We use the shorthand pre-(C-EUDS) of $*$-class to denote an exponential pre-(C-UDS) of $*$-class.

Definition 2.3. Let $\mathcal{G}$ be a pre-C-ultradistribution semigroup (pre-C-distribution semigroup). Then $\mathcal{G}$ is said to be a quasi-equicontinuous exponential (short, (q)-exponential) pre-C-ultradistribution semigroup (pre-C-distribution semigroup) if for every $p \in \circledast$ and bounded subset $B \in E$ there exist $M_p \geq 1$, $\omega_p \geq 0$ and q_p seminorm on $\mathcal{S}^*(\mathbb{R})$ ($\mathcal{S}(\mathbb{R})$) such that

$$\sup_{x \in B} p(\mathcal{G}(\varphi)x) \leq M_p e^{\omega_p \cdot} q_p(\varphi),$$

for all $\varphi \in \mathcal{S}_0^*(\mathbb{R})$ ($\varphi \in \mathcal{S}_0(\mathbb{R})$). We use the shorthand pre-q-(C-EUDS) (pre-q-(C-EDS)).

Concerning degenerate C-ultradistribution semigroups, exponential degenerate C-ultradistribution semigroups and degenerate (q-)exponential C-ultradistribution semigroups, we can give the following theorems (see [19]).

Theorem 2.2. (i) *Suppose that there exist $l > 0$, $\beta > 0$ and $k > 0$, in the Beurling case, resp., for every $l > 0$ there exists $\beta_l > 0$, in the Roumieu case, such that $\Omega^{(M_p)}_{l,\beta} := \{\lambda \in \mathbb{C} : \Re\lambda \geq M(l|\lambda|) + \beta\} \subseteq \rho_C(A)$, resp. $\Omega^{\{M_p\}}_{l,\beta_l} := \{\lambda \in \mathbb{C} : \Re\lambda \geq M(l|\lambda|) + \beta_l\} \subseteq \rho_C(A)$, the mapping $\lambda \to (\lambda - A)^{-1}Cx$, $\lambda \in \Omega^{(M_p)}_{l,\beta}$, resp. $\lambda \in \Omega^{\{M_p\}}_{l,\beta_l}$, is continuous for every fixed element $x \in E$, and the operator family $\{\mathrm{e}^{-M(kl|\lambda|)}(\lambda - A)^{-1}C : \lambda \in \Omega^{(M_p)}_{l,\beta}\} \subseteq L(E)$, resp. $\{\mathrm{e}^{-M(l|\lambda|)}(\lambda - A)^{-1}C : \lambda \in \Omega^{\{M_p\}}_{l,\beta_l}\} \subseteq L(E)$, is equicontinuous. Denote by Γ, resp. Γ_l, the upwards oriented boundary of $\Omega^{(M_p)}_{l,\beta}$, resp. $\Omega^{\{M_p\}}_{l,\beta_l}$. Define, for every $x \in E$ and $\varphi \in \mathcal{D}^*$, the element $\mathcal{G}(\varphi)x$ with*

$$\mathcal{G}(\varphi)x := (-\mathrm{i}) \int_\Gamma \hat{\varphi}(\lambda)(\lambda - A)^{-1}Cx \,\mathrm{d}\lambda, \quad x \in E, \ \varphi \in \mathcal{D}, \tag{2.2}$$

in the Beurling case; in the Roumieu case, for every number $k > 0$ and for every function $\varphi \in \mathcal{D}^{\{M_p\}}_{[-k,k]}$, we define the element $\mathcal{G}(\varphi)x$ in the same way as above, with the contour Γ replaced by $\Gamma_{l(k)}$. Then $\mathcal{G} \in \mathcal{D}_0'^*(L(E))$ is boundedly equicontinuous, $\mathcal{G}(\varphi)C = C\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}^*$, $\mathcal{G}(\varphi)A \subseteq A\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}^*$ and $A\mathcal{G}(\varphi)x = \mathcal{G}(-\varphi')x - \varphi(0)Cx$, $x \in E$, $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{D}^*$). Then, $\mathcal{G}$ is a pre-(C-UDS) of $*$-class.

(ii) *Suppose that A is a closed linear operator on E satisfying that there exist $a \geq 0$ such that $\{\lambda \in \mathbb{C} : \Re\lambda > a\} \subseteq \rho_C(A)$ and the mapping $\lambda \mapsto (\lambda - A)^{-1}Cx$, $\Re\lambda > a$ is continuous for every fixed element $x \in E$. Suppose that there exists a number $k > 0$, in the Beurling case, resp., for every number $k > 0$, in the Roumieu case, such that the operator family $\{\mathrm{e}^{-M(k|\lambda|)}(\lambda - A)^{-1}C : \Re\lambda > a\} \subseteq L(E)$ is equicontinuous. Set*

$$\mathcal{G}(\varphi)x = (-\mathrm{i}) \int\limits_{\bar{a}-\mathrm{i}\infty}^{\bar{a}+\mathrm{i}\infty} \hat{\varphi}(\lambda)\big(\lambda - A\big)^{-1}Cx \ \mathrm{d}\lambda, \quad x \in E, \ \varphi \in \mathcal{D}^*.$$

Then $\mathcal{G} \in \mathcal{D}_0'^*(L(E))$ is boundedly equicontinuous, $\mathrm{e}^{-\omega t}G \in \mathcal{S}'^*(L(E))$ for all $\omega > a$, $\mathcal{G}(\varphi)C = C\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}^*$, $\mathcal{G}(\varphi)A \subseteq A\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}^*$ and

$A\mathcal{G}(\varphi)x = \mathcal{G}\big(-\varphi'\big)x - \varphi(0)Cx, \quad x \in E,\ \varphi \in \mathcal{D}\ \ (\varphi \in \mathcal{D}^*)$. *Then, $\mathcal{G}$ is a pre-(C-EUDS) of $*$-class.*

Remark 2.2. Following J. Chazarain [6], we define (M_p)-ultralogarithmic region $\Lambda_{\alpha,\beta,l}$ of type l as

$$\Lambda_{\alpha,\beta,l} = \{\lambda \in \mathbb{C}\ :\ \Re\lambda \geq \alpha M(l|\Im\lambda|) + \beta\},$$

for $\alpha, \beta > 0$, $l \in \mathbb{R}$. The first part of the Theorem 2.2 can be reformulated with the region $\Omega^{(M_p)}_{l,\beta}$ replaced by $\Lambda_{\alpha,\beta,l}$.

Let $\bar{\alpha} > \alpha$. By Γ_l ($\Gamma_{\bar{\alpha}}$) we donte the upwards oriented boundary of the ultra-logarithmic region $\Lambda_{\alpha,\beta,l}$ (the right line connecting the points $\bar{\alpha} - \mathrm{i}\infty$ and $\bar{\alpha} + \mathrm{i}\infty$) and let

$$\mathcal{G}(\varphi)x := (-\mathrm{i}) \int_{\Gamma_l\ (\Gamma_{\bar{\alpha}})} \hat{\varphi}(\lambda)(\lambda - A)^{-1}x \,\mathrm{d}\lambda, \quad x \in E, \quad \varphi \in \mathcal{D}^{(M_p)}. \tag{2.3}$$

The abstract Beurling space of (M_p) class associated to a closed linear operator A is defined as in [7]. Following [7], we put $E^{(M_p)}(A)$:=projlim$_{h\to+\infty}E^{(M_p)}_h(A)$, where

$$E^{(M_p)}_h(A) := \Bigg\{ x \in D_\infty(A) : \|x\|^{(M_p)}_{h,q} = \sup_{p\in\mathbb{N}_0} \frac{h^p q\big(A^p x\big)}{M_p} < \infty \quad \text{for all } h > 0 \text{ and } q \in \circledast \Bigg\}.$$

Then, for each number $h > 0$ the calibration $(\|\cdot\|^{(M_p)}_{h,q})_{q\in\circledast}$ induces a Hausdorff sequentially complete locally convex space on $E^{(M_p)}_h(A)$, $E^{(M_p)}_{h'}(A) \subseteq E^{(M_p)}_h(A)$ provided $0 < h < h' < \infty$, and the spaces $E^{(M_p)}_h(A)$ and $E^{(M_p)}(A)$ are continuously embedded in E.

Theorem 2.3. *Let A be a closed linear operator A and there exist constants $l \geq 1$, $\alpha > 0$, $\beta > 0$ and $k > 0$ such that $\Lambda_{\alpha,\beta,l} \subseteq \rho(A)$ $(RHP_\alpha \equiv \{\lambda \in \mathbb{C} : \Re\lambda > \alpha\} \subseteq \rho(A))$. Let for each seminorm $q \in \circledast$ there exist a number $c_q > 0$ and a seminorm $r \in \circledast$ such that*

$$q\Big(\big(\lambda - A\big)^{-1}x\Big) \leq c_q \mathrm{e}^{M(kl|\lambda|)} r(x), \quad x \in E,\ \lambda \in \Lambda_{\alpha,\beta,l}\ \big(RHP_\alpha\big). \tag{2.4}$$

Moreover, assume that $\mathcal{G}$, defined through (2.3)*, is a (UDS) ((EUDS)) of Beurling class generated by A (i.e., that $\mathcal{G}$ satisfies* (C.S.2)*), and that (M_p) satisfies* (M.1), (M.2) *and* (M.3). *Then the abstract Cauchy problem (ACP) has a unique solution $u(t)$ for all $x \in E^{(M_p)}(A)$.*

Remark 2.3. We would like to observe that Theorem 2.2 and Theorem 2.3 cannot be formulated for multivalued linear operators.

Now we will reconsider some conditions (originally introduced by J. L. Lions [24], for the definition of dense distribution semigroups and for ultradistribution case the conditions in [18]) in our new framework. Suppose that $\mathcal{G} \in \mathcal{D}_0'^{*}(L(E))$ and $\mathcal{G}$ commutes with C. Like in the case of degenerate C-distribution semigroups (see [20]), we analyze the following conditions for $\mathcal{G}$:

(d_1) $\mathcal{G}(\varphi * \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi)$, φ, $\psi \in \mathcal{D}_0^*$,

(d_3) $\mathcal{R}(\mathcal{G})$ is dense in E,

(d_4) for every $x \in \mathcal{R}(\mathcal{G})$, there exists a function $u_x \in C([0,\infty) : E)$ so that $u_x(0) = Cx$ and

$$\mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)u_x(t)\,\mathrm{d}t, \quad \varphi \in \mathcal{D}^*,$$

(d_5) $(Cx, \mathcal{G}(\psi)x) \in G(\psi_+)$, $\psi \in \mathcal{D}^*$, $x \in E$.

We will discuss the connections of the previously given conditions, (d_1), (d_2), (d_3), (d_4) and (d_5). Let $\mathcal{G} \in \mathcal{D}_0'^{*}(L(E))$ be a pre-(C-UDS) of $*$-class. Then $\mathcal{G}$ satisfies (d_1) and from previously $\mathcal{G}$ satisfies (d_5). Also, by the proof of [15, Proposition 3.1.24], we have that $\mathcal{G}$ also satisfies (d_4). On the other hand, it is well known that (d_1), (d_4) and (C.S.2) taken together do not imply (C.S.1), even in the case that $C = I$; see e.g. [15, Remark 3.1.20]. Furthermore, if (d_1), (d_3) and (d_4) hold then (d_5) holds, as well. To prove this, fix $x \in \mathcal{R}(\mathcal{G})$ and $\varphi \in \mathcal{D}^*$. Then it suffices to show that $(Cx, \mathcal{G}(\varphi)x) \in G(\varphi_+)$. Suppose that (ρ_n) is a regularizing sequence and $u_x(t)$ is a function appearing in the formulation of the property (d_4). From the proof of [15, Proposition 3.1.19], for every $\eta \in \mathcal{D}_0^*$, we have

$$\begin{aligned}\mathcal{G}(\rho_n)\mathcal{G}(\varphi_+ * \eta)x &= \mathcal{G}((\varphi_+ * \rho_n) * \eta)Cx = \mathcal{G}(\eta)\mathcal{G}(\varphi_+ * \rho_n)x \\ &= \mathcal{G}(\eta)\int_0^\infty (\varphi_+ * \rho_n)(t)u_x(t)\,\mathrm{d}t \\ &\to \mathcal{G}(\eta)\int_0^\infty \varphi(t)u_x(t)\,\mathrm{d}t = \mathcal{G}(\eta)\mathcal{G}(\varphi)x,\ n \to \infty; \\ \mathcal{G}(\rho_n)\mathcal{G}(\varphi_+ * \eta)x &= \mathcal{G}(\varphi_+ * \eta * \rho_n)Cx \to \mathcal{G}(\varphi_+ * \eta)Cx,\ n \to \infty.\end{aligned}$$

Hence, $\mathcal{G}(\varphi_+ * \eta)Cx = \mathcal{G}(\eta)\mathcal{G}(\varphi)x$ and (d_5) holds, as claimed. On the other hand, (d_1) is a very simple consequence of (d_5). To see this, observe that for each $\varphi \in \mathcal{D}_0^*$

and $\psi \in \mathcal{D}^*$ we have $\psi_+ * \varphi = \psi *_0 \varphi = \varphi *_0 \psi$, so that (d_5) is equivalent to say that $\mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi)$ ($\varphi \in \mathcal{D}_0^*$, $\psi \in \mathcal{D}^*$). In particular,

$$\mathcal{G}(\varphi)\mathcal{G}(\psi) = \mathcal{G}(\psi)\mathcal{G}(\varphi), \quad \varphi \in \mathcal{D}_0^*,\ \psi \in \mathcal{D}^*. \tag{2.5}$$

Now, let (d_5) holds, $\varphi \in \mathcal{D}_0^*$ and $\psi,\ \eta \in \mathcal{D}^*$. Note that $\psi_+ * \eta_+ * \varphi = (\psi *_0 \eta)_+ * \varphi$. Then (cf. also [22, Remark 3.13]):

$$\begin{aligned}
\mathcal{G}(\varphi)\mathcal{G}(\eta)\mathcal{G}(\psi) &= C\mathcal{G}(\eta_+ * \varphi)\mathcal{G}(\psi) \\
&= C\mathcal{G}(\psi_+ * \eta_+ * \varphi) \\
&= C\mathcal{G}\big((\psi *_0 \eta)_+ * \varphi\big)C \\
&= C\mathcal{G}(\varphi)\mathcal{G}(\psi *_0 \eta) \\
&= \mathcal{G}(\varphi)\mathcal{G}(\psi *_0 \eta)C.
\end{aligned} \tag{2.6}$$

By (2.5)–(2.6), we get

$$\mathcal{G}(\eta)\mathcal{G}(\psi)\mathcal{G}(\varphi) = \mathcal{G}(\psi *_0 \eta)C\mathcal{G}(\varphi). \tag{2.7}$$

By (2.5)–(2.7), we have the following conclusions:

(i) (d_5) and (d_3) together imply (C.S.1); in particular, (d_1), (d_3) and (d_4) together imply (C.S.1). This is an extension of [15, Proposition 3.1.19].

(ii) (d_5) and (d_2) together imply that $\mathcal{G}$ is a (C-UDS) of $*$-class; in particular, $\mathcal{A} = A$ must be single-valued and C must be injective.

On the other hand, (d_5) does not imply (C.S.1) even in the case that $C = I$. A simple counterexample is $\mathcal{G} \in \mathcal{D}_0'^*(L(E))$ given by $\mathcal{G}(\varphi)x := \varphi(0)x$, $x \in E$, $\varphi \in \mathcal{D}^*$.

The exponential region $E(a,b)$ has been defined for the first time by W. Arendt, O. El–Mennaoui and V. Keyantuo in [1]:

$$E(a,b) := \Big\{\lambda \in \mathbb{C} : \Re\lambda \geq b,\ |\Im\lambda| \leq e^{a\Re\lambda}\Big\} \quad (a,\ b > 0).$$

Remark 2.4. Suppose that there exist $l > 0$, $\beta > 0$ and $k > 0$, in the Beurling case, resp., for every $l > 0$ there exists $\beta_l > 0$, in the Roumieu case, such that the assumptions of [20, Theorem 4.15] hold with the exponential region $E(a,b)$ replaced with the region

$$\Omega_{l,\beta}^{(M_p)} := \{\lambda \in \mathbb{C} : \Re\lambda \geq M(l|\lambda|) + \beta\},$$

resp.

$$\Omega_{l,\beta_l}^{\{M_p\}} := \{\lambda \in \mathbb{C} : \Re\lambda \geq M(l|\lambda|) + \beta_l\}.$$

Define $\mathcal{G}$ similarly as above. Then $\mathcal{G} \in \mathcal{D}_0'^*(L(E))$, $\mathcal{G}$ commutes with C and $\mathcal{A}$, and (2.1) holds. But, in the present situation, we do not know whether $\mathcal{G}$ has to satisfy (C.S.1) in degenerate case. This is an open problem we would like to address to our readers.

An example of exponential degenerate ultradistribution semigroup of Beurling class can be given by using the consideration from [21, Example 3.25]. By Proposition 2.2(iii), the duals of non-dense (C-UDS)'s of $*$-class serve as examples of pre-(C*-UDS)'s of $*$-class, as well.

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