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GENERALIZED LAPLACE TRANSFORM OF LOCALLY INTEGRABLE FUNCTIONS DEFINED ON $[0, \infty)$

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A b s t r a c t. In [*Bull. Cl. Sci. Math. Nat. Sci. Math.* **40** $(2015),\ 99-113$] *we defined the Laplace transform on a bounded interval* $[0, b]$*, denoted by* $^0\mathcal{L}$*, using some ideas of H. Komatsu* [*J. Fac. Sci. Univ. Tokyo, IA,* **34** (1987), 805–820] *and* [*Structure of solutions of differential equations* (*Katata/Kyoto,* 1995)*, pp.* 227–252*, World Sci. Publishing, River Edge, NJ,* 1996]. *We use this definition to extend it to the space of locally integrable functions defined on* $[0, \infty)$*, which is a wider class then functions* L *used by G. Doetsch* [*Handbuch der Lalace-Transformation I, Basel – Stuttgart,* $1950-1956$*, p.* 32]. *As an application we give solutions of integral equations of the convolution type, defined on a bounded interval, or on the half-axis as well, and of equations with fractional derivatives.*

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1. *Introduction*

To make easier reading this paper we quoted some definitions and propositions from [9] and [1].

Vector space $L[0,b]$, $0 < b < \infty$. The function $f \in L[0,b]$ if there exists $\int_0^b f(\tau)\,\mathrm{d}\tau$ in the sense of Lebesgue; $\int_0^b f(\tau)\,\mathrm{d}\tau = 0 \Leftrightarrow f(t) = 0$ a.e. on $[0,b]$.

Vector space of locally integrable functions $L_{\mathbf{loc}}[a,\infty)$. A function f is said to be locally integrable in the interval $[0,\infty)$ if it is integrable in the sense of Lebesgue in each bounded interval $[0,T], T > 0$. We know that if f and g are locally integrable on $[0,\infty)$, so is their convolution ([6]).

Lemma 1.1 ([6], p. 96)**.** *If the convolution of two locally integrable functions f and g vanishes almost everywhere in the interval $0 \leq t < \infty$, then at least one of these functions vanishes almost everywhere in this interval.*

Vector space $L^{\mathbf{exp}}[0,\infty)$. The function $f \in L^{\exp}[0,\infty)$ if $f \in L_{loc}[0,\infty)$ and such that for an $s_0 > 0$ there exists $\int_0^\infty |\mathrm{e}^{-s_0\tau} f(\tau)|\,\mathrm{d}\tau$. With operation addition and multiplication by $c \in \mathbb{C}$ this is a vector space on $\mathbb{C}$. A function $f \in L^{\exp}[b,\infty)$ if $f \in L^{\exp}[0,\infty)$ and $f(t) = 0, 0 \leq t < b < \infty$. $L^{\exp}[b,\infty)$ is also a vector space. The convolution of f and g which belong to $L^{\exp}[0,\infty)$ is also in $L^{\exp}[0,\infty)$.

In the following we take care of Theorem 2 in [1], I, p. 33: If the Laplace integral in s_0 is absolute integrable, it is absolute integrable in $\mathrm{Re}\, s > s_0$, as well.

Lemma 1.2 ([1], I, p. 123)**.** *If $f \in L^{\exp}[0,\infty)$, and $g \in L^{\exp}[b,\infty)$, then the convolution $f * g \in L^{\exp}[b,\infty)$.*

Vector space $L^{\mathbf{exp}}[0,\infty)/L^{\mathbf{exp}}[b,\infty)$. In $L^{\exp}[0,\infty)$ we define a two elements relation: $f \sim g \underset{\mathrm{def}}{=} f - g \in L^{\exp}[b,\infty),\ b > 0$. Since $L^{\exp}[b,\infty)$ is a vector space, a subspace of $L^{\exp}[0,\infty)$, the relation $\sim$ is an equivalence relation in accordance with the vector space $L^{\exp}[0,\infty)$. The equivalence classes are elements of $L_b = L^{\exp}[0,\infty)/L^{\exp}[b,\infty)$. An element $f_b \in L_b$ is defined by $\bar{f} + L^{\exp}[b,\infty)$, where $\bar{f} \in L^{\exp}[0,\infty)$. In L_b is defined the addition and product by $r \in \mathbb{R}$: $f_b + g_b = \bar{f} + \bar{g} + L^{\exp}[b,\infty)$, and $rf_b = (rf)_b$. With this two operation L_b is also a vector space.

The "0" element in L_b is $L^{\exp}[b,\infty)$.

Lemma 1.3 ([9])**.** *Every function $f \in L[0,b]$ can be extended to a function $\bar{f} \in L^{\exp}[0,\infty)$. The vector space L_b is algebraically isomorph to the vector space $L[0,b]$ $(f \leftrightarrow L_b)$.*

In this part we use some theorems from [1] where Riemann integral is used. This is possible because: Every $J-$function in the sense of Riemann is also an $J-$function in the sense of Lebesgue ([1], I, Theorem 2, p. 31).

2. *Generalized Laplace transform of locally integrable function defined on* $[0,\infty)$

Definition 2.1 ([9])**.** For a function $f \in L[0,b]$, $b > 0$ the generalized Laplace transform, denoted by ${}^0\mathcal{L}$, is defined as

$$({}^0\mathcal{L}f)(s) \underset{\text{def}}{=} (\mathcal{L}\bar{f})(s) + (\mathcal{L}L^{\exp}[b,\infty))(s), \quad s > s_0, \tag{2.1}$$

where $\mathcal{L}$ is the classical Laplace transform and $\bar{f}$ is the extension of f, which belongs to $L^{\exp}[0,\infty)$.

Some properties of ${}^0\mathcal{L}$ follows from the properties of Laplace transform $\mathcal{L}$. So the inverse operator of ${}^0\mathcal{L}$, of element $f \in L[0,b]$ denoted by ${}^-({}^0\mathcal{L})$ we can defined in the following way: Let ${}^-\mathcal{L}$ be inverse of $\mathcal{L}$ then

$${}^-({}^0\mathcal{L})({}^0\mathcal{L}f(s)) = {}^-\mathcal{L}(\mathcal{L}\bar{f}(s) + \mathcal{L}L^{\exp}[b,\infty)) = \bar{f}(s) + L^{\exp}[b,\infty) \leftrightarrow f \in L_{[0,b]}.$$

Now it is easy to extend ${}^0\mathcal{L}$ to locally integrable functions on $[0,\infty)$.

Definition 2.2. The generalized Laplace transform ${}^0\mathcal{L}$ of a function $f \in L_{\mathrm{loc}}[0,\infty)$ is defined by the family of functions

$$\{{}^0\mathcal{L}(f|_{[0,b]})(s)_{0<b<\infty}\}. \tag{2.2}$$

The inverse operation of the generalized Laplace transform ${}^-({}^0\mathcal{L})$ defines the function $f \in L_{loc}[a,\infty)$ by the family

$$\{{}^-({}^0\mathcal{L})({}^0\mathcal{L}f|_{[0,b]})(s)), \quad 0 < b < \infty\}. \tag{2.3}$$

As an example of Definition 2.2 we show how works the generalized Laplace transform ${}^0\mathcal{L}$ when we solve ordinary differential equation with constant coefficients:

$$\sum_{i=0}^{n} c_i y^{(i)}(t) = f(t), \quad c_n = 1, \quad f \in L[0,b], \tag{2.4}$$

with initial conditions

$$y^{(i)}(0) = 0, \quad i = 0, \dots, n-1. \tag{2.5}$$

First we introduce the following notations:

1. For $K \in L_{\mathrm{loc}}[0,\infty)$, the notation $K_{[0,b]} = K(t)|_{[0,b]}$ is the projection of $K(t)$ on the interval $[0,b]$. $\bar{K}_{[0,b]}(t)$ is the extension of $K_{[0,b]}(t)$ on the whole $[0,\infty)$ belonging to $L^{\exp}[0,\infty)$.

2. $\bar{K}_{[0,b]}(s) = \mathcal{L}(\bar{K}_{[0,b]}(t))(s)$, where $\mathcal{L}$ is the classical Laplace transform.

To equation (2.4) it corresponds by Theorem 3.2 in [9], the family of equations

$$\sum_{i=0}^{n} c_i(\bar{y}_{[0,b]})^{(i)}(t) - \bar{f}_{[0,b]}(t) + L^{\exp}[b,\infty) = L^{\exp}[b,\infty), \quad b > 0. \tag{2.6}$$

We apply now to (2.6) the classical Laplace transform, the theorem which gives the Laplace transform of the derivative of a function ([1], I , p. 100), with (2.5):

$$\sum_{i=0}^{n} c_i s^i(\bar{y}_{[0,b]})(s) - \bar{f}_{[0,b]}(t) + \mathcal{L}(L^{\exp}[b,\infty))(t) = \mathcal{L}L^{\exp}[b,\infty)(s), \tag{2.7}$$

for every $b > 0$, and $\operatorname{Re} s > s_0$. The equality (2.7) can be correct if and only if

$$\left(\sum_{i=0}^{n} c_i s^i\right)(\bar{y}_{[0,b]})(s) - \bar{f}_{[0,b]}(s) = 0, \quad \text{for every} \quad b > 0, \quad \operatorname{Re} s > s_0 \tag{2.8}$$

(because "0" element in L_b is $L^{\exp}[b,\infty)$).

Let us denote by $p(s) = \sum_{i=0}^{n} c_i s^i$. If we know to calculate

$$u(t) = (^{-}\mathcal{L})(\frac{1}{p(s)}) \in L_{\text{loc}}[0,\infty)$$

(see [1], II and [2]), then

$$\bar{y}_{[0,b]}(s) = \frac{\bar{f}_{[0,b]}(s)}{p(s)}, \quad \text{for every} \quad b > 0, \quad \operatorname{Re} s > s_0,$$

and

$$\bar{y}_{[0,b]}(t) = \bar{f}_{[0,b]}(t) * \bar{u}_{[0,b]}(t), \quad \text{for every} \quad b > 0.$$

The solution to equation (2.4) is

$$y(t) \leftrightarrow \bar{f}_{[0,b]}(t) * \bar{u}_{[0,b]}(t) + L^{\exp}[b,\infty), \quad 0 \le t \le b, \quad \text{for every} \quad b > 0, \tag{2.9}$$

which belongs to $L_{\text{loc}}[0,\infty)$.

3. *Application ${}^{0}\mathcal{L}$ to solve integral equations*

We consider the integral equation

$$\int\limits_{0}^{t} K(t-\tau)X(\tau)d\tau = G(t). \tag{3.1}$$

on the interval $[0,b]$ or on the half axis.

Equations (3.1) are equations of the convolution type. They are also Volterra's equations of the first kind (enclosed singular once, as well).

We know that if K and X in equation belong to $L_{\text{loc}}[0,\infty)$, then G belongs to $L_{\text{loc}}[0,\infty)$, as well. ([6]). In this part we look for conditions on functions K and G such that equation (3.1) has a solution X belonging to $L_{\text{loc}}[0,\infty)$. The space $L_{\text{loc}}[0,\infty)$ is a wider class of functions then the class to which one can apply the classical Laplace transform.

In mathematical literature there is only some results for special cases of (3.1) For example see [1] , III, p. 151–163, [2], [6], p. 13–20, and the well-known Abel's equation. We will treat also some special cases, but after all more general, using the generalized Laplace transform ${}^{0}\mathcal{L}$.

3.1. *Unique solution to integral equation* (3.2).

We consider first the unique solutions of the equation

$$\int\limits_{0}^{t} K(t-\tau)X(\tau)d\tau = G(t), \quad t>0. \tag{3.2}$$

Theorem 3.1. *Suppose that:*

1. $K \in L_{\text{loc}}[0,\infty)$;
2. *Equation* (3.2) *has a solution belonging to* $L_{\text{loc}}[0,\infty)$*, then this solution is unique in* $L_{\text{loc}}[0,\infty)$.

PROOF. Let us suppose that there exist two solutions X_1 and X_2 to equation (3.2) belonging to $L_{\text{loc}}[0,\infty)$, Then

$$(K*(X_1-X_2))(t) = 0, \quad t \geq 0. \tag{3.3}$$

(The symbol $*$ marks the operation convolution).

By Lemma 1.1 it follows that $(X_1-X_2)(t)$ vanishes everywhere in the interval $[0,\infty)$.

Theorem 3.2. *Suppose that:*

1. $K \in L[0,b]$ *and non exists number* c, $b > c > 0$, *such that* $K(t) = 0$, $0 \leq t \leq c$*;*

2. *There is a solution to equation*

$$(K * X)(t) = G(t), \quad 0 \leq t \leq b \tag{3.4}$$

belonging to $L[0,b]$.

Then this solution is unique to equation (3.4) *which belongs to* $L[0,b]$.

PROOF. Suppose that there exist two solutions to equation (3.4) X_1 and X_2 belonging to $L[0,b]$, for a given $G(t)$, then

$$(K * (X_1 - X_2))(t) = 0, \quad 0 \leq t \leq b. \tag{3.5}$$

The proof follows when we apply Theorem 12 in [1], I , p.131, which says: If $f * g = 0$ for one interval $(0,b]$, then $f = 0$ in interval $[0,a]$ and $g = 0$ in interval $[0,b]$, where $a + c = b$. In our case $X_1(t) = X_2(t)$ for $t \in [0,b]$, because of supposition 1.

3.2. *Solution of integral equation* (3.1)

To solve equation (3.1) using the generalized Laplace transform we apply already exploited notations:

1. For $K \in L_{\text{loc}}[0,\infty)$ the notation $K_{[0,b]} = K(t)|_{[0,b]}$ is the projection of $K(t)$ on the interval $[0,b]$, and $\bar{K}_{[0,b]}(t)$ is the extension of $K_{[0,b]}(t)$ on the whole $[0,\infty)$ belonging to $L^{\exp}[0,\infty)$.

2. $\bar{K}_{[0,b]}(s) = \mathcal{L}(\bar{K}_{[0,b]}(t))(s)$, where $\mathcal{L}$ is the classical Laplace transform.

Theorem 3.3. *Suppose that in equation* (3.1)*: the functions* K *and* G *belong to* $L_{\text{loc}}[0,\infty)$ *and the number* n *have the following properties:*

1) $n \geq 2$,

2) *There exists the function* $H(t)$ *such that* $H^{(n-1)}(t) \in L_{\text{loc}}[0,\infty)$,

$$({}^{0}\mathcal{L}\bar{H}_{[0,b]}(t))(s) = (s^n \bar{K}_{[0,b]}(s))^{-1}, \quad \operatorname{Re} s > s_0 \quad \textit{for every} \quad b > 0$$

and $H_{[0,b]}(t)$ *is bounded on every interval* $[0,b]$, $\; b > 0$.

3) *$G^{(n)}(t)$ exists for every $t > 0$ and is bounded on every interval $[0,b]$, $b > 0$.*

Then there exists $X(t) \in L_{\rm loc}[0,\infty)$, the unique solution to equation (3.1) belonging to $L_{\rm loc}[0,\infty)$ and

$$X(t) \leftrightarrow \bar{X}_{[0,b]}(t) + L^{\exp}[b,\infty) = (\bar{H}_{[0,b]}(\tau) * \bar{G}_{[0,b]}(\tau))^{(n)}(t) + L^{\exp}[b,\infty)$$
$$= (\bar{H}_{[0,b]}(\tau) * \bar{G}^{(n)}_{[0,b]}(\tau))(t) + \bar{G}_{[0,b]}(0)\bar{H}^{(n-1)}_{[0,b]}(t)$$
$$+ \cdots + \bar{G}^{(n-1)}_{[0,b]}(0)\bar{H}_{[0,b]}(t) + L^{\exp}[b,\infty)$$

for every $b > 0$.

The function $X(t) \in L_{\rm loc}[0,\infty)$.

PROOF. The equation (3.1) we write in the form which permits to apply the generalized Laplace transform

$$(K(\tau) * X(\tau))(t) = G(t), \quad 0 \le t \le b, \quad b > 0 \tag{3.6}$$

To (3.6) it corresponds the family of equations in L_b, $b > 0$,

$$(\bar{K}_{[0,b]}(\tau) * \bar{X}_{[0,b]}(\tau))(t) - \bar{G}_{[0,b]}(t) + L^{\exp}[b,\infty) = L^{\exp}[b,\infty), \quad b > 0 \tag{3.7}$$

(cf. [9], Lemma 3.2).

The Laplace transform of equation (3.7) is

$$\bar{K}_{[0,b]}(s)\bar{X}_{[0,b]}(s) - \bar{G}_{[0,b]}(s)) + (\mathcal{L}L^{\exp}[b,\infty))(s) = (\mathcal{L}L^{\exp}[b,\infty))(s), \tag{3.8}$$

$b > 0, \quad \mathrm{Re}\, s > s_0$.

This equation (3.8) will be satisfied if:

$$\bar{K}_{[0,b]}(s)\bar{X}_{[0,b]}(s) - \bar{G}_{[0,b]}(s)) = 0, \quad \text{for every} \quad b > 0, \quad s > s_0,$$

(the "0" element in L_b) or

$$s^n\bar{K}_{[0,b]}(s)s^{-n}\bar{X}_{[0,b]}(s) - \bar{G}_{[0,b]}(s))(t) = 0, \quad \text{for every} \quad b > 0, \quad s > s_0. \tag{3.9}$$

Equation (3.9) can be written as

$$s^{-n}\bar{X}_{[0,b]}(s)) = (s^{-n}\bar{K}_{[0,b]}(s))^{-1}\bar{G}_{[0,b]}(s)), \quad \text{for every} \quad b > 0, \quad \mathrm{Re}\, s > s_0.$$

By the supposition 2),

$$(s^n\bar{K}_{[0,b]}(s))^{-1}\bar{G}_{[0,b]}(s)) = \bar{H}_{[0,b]}(s)\bar{G}_{[0,b]}(s), \quad b > 0, \quad \mathrm{Re}\, s > s_0,$$

and

$$\left(\int_0^t \mathrm{d}\tau\right)^{(n)} \bar{X}_{[0,b]}(t) = \left(\bar{H}_{[0,b]}(\tau) * \bar{G}_{[0,b]}(\tau)\right)(t), \quad \text{for every} \quad b > 0.$$

With supposition 3) and Theorem 13 and 10 in [1], I, p. 119 and 117, which give the condition for the derivative of convolution, there exists $(\bar{H}_{[0,b]}(\tau)\bar{G}_{[0,b]}(\tau))^{(n)}(t)$ so that

$$\begin{aligned} \bar{X}_{[0,b]}(t) &= (\bar{H}_{[0,b]}(\tau) * \bar{G}_{[0,b]}(\tau))^{(n)}(t) \\ &= \int_0^t \bar{H}_{[0,b]}(t-\tau)\bar{G}^{(n)}_{[0,b]}(\tau)\,\mathrm{d}\tau \\ &\quad + \bar{G}_{[0,b]}(0)\bar{H}^{(n-1)}_{[0,b]}(t) + \cdots + \bar{G}^{(n-1)}_{[0,b]}(0)\bar{H}_{[0,b]}(t), \quad t \geq 0, \end{aligned} \tag{3.10}$$

for every $b > 0$.

Because of suppositions 2) and 3) the solution $X(t)$ belongs to $L_{\mathrm{loc}}[0,\infty)$. This proves Theorem 3.3 in the interval $[0,b]$, for every $b > 0$.

If the equation (3.1) is given in the interval $[0,b]$, in the proof of this theorem we have only to leave out that the proof is valid "*for every* $b > 0$" ([9]). This means that (3.10) is the unique solution to equation (3.1) in the interval $[0,b]$.

As an example of Theorem 3.3 we solve Abels integral equation

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}X(\tau)\,\mathrm{d}\tau = G(t), \quad t > 0.$$

Here $\bar{K}_{[0,b]}(t) = t^{-\alpha}/\Gamma(1-\alpha)$ for every $b > 0$ and $t^{-\alpha}/\Gamma(1-\alpha) \in L_{\mathrm{loc}}[0,\infty)$. $\bar{K}_{[0,b]}(s) = s^{\alpha-1}$, for every $b > 0$, and $\mathrm{Re}\, s > s_0$. The functions $\bar{H}_{[0,b]}(s) = s^{-\alpha}$, for every $b > 0$ and

$$X(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}G^{(1)}(\tau)\,\mathrm{d}\tau + G(0)\frac{(t)^{\alpha-1}}{\Gamma(\alpha)}.$$

This is the unique solution in the space $L_{\mathrm{loc}}[0,\infty)$ (cf. for example [7] and [3]).

We apply Theorem 3.3 to solve equation with fractional derivatives. First we show the way how we can transform an equation with fractional derivatives in such a way that we can apply Theorem 3.3 . Let us mention that we can apply operator ${}^0\mathcal{L}$ direct to an equation with fractional derivatives ([9], Part 4).

Consider the equation

$$(D^{\alpha}y)(t) + \lambda^2 y(t) = G(t), \quad t > 0, \quad y^{(i)}(0) = 0, \quad i = 0, 1, \dots, m-2 \tag{3.11}$$

where: D^{α} is the Riemann-Liouville derivative , $\alpha = [\alpha] + \nu$, $[\alpha]$ is the greatest integer part of α, $0 < \nu < 1$, and $m = [\alpha] + 1$, $\alpha \geq 2$.

The fractional derivative $D^{\alpha}y$ is

$$D^{\alpha}y(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\nu} y(\tau)\,\mathrm{d}\tau, \quad t > 0.$$

The function $y(t)$ can be written:

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t y(\tau)\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m-1} \int_0^t (t-\tau)^{m-1} y(\tau)\,\mathrm{d}\tau, \quad t > 0.$$

Supposing for the function $y(t)$: The function $y^{(m-1)}(t)$ exists for $t > 0$, is bounded on every interval $[0, T]$, $0 < T < \infty$ and $y(0) = 0, \dots, y^{(m-2)}(0) = 0$. The same supposition we take for the function $G(t)$. Then we can write equation (3.11) as

$$\frac{1}{\Gamma(1-\nu)} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \int_0^t (t-\tau)^{-\nu} y^{(m-1)}(\tau)\,\mathrm{d}\tau + \lambda^2 \Gamma(1-\nu) \int_0^t (t-\tau)^{m-1} y^{(m-1)}(\tau)\,\mathrm{d}\tau \right.$$
$$\left. - \Gamma(1-\nu) \int_0^t (t-\tau)^{m-1} G^{(m-1)}(\tau)\,\mathrm{d}\tau \right\} = 0, \quad t > 0,$$

(cf. [1], I, Theorem 13, p. 119–120, Theorem which gives derivative of the convolution of two functions, or

$$\begin{aligned} &\int_0^t \left[(t-\tau)^{-\nu} + \lambda^2 \Gamma(1-\nu)(t-\tau)^{m-1}\right] y^{(m-1)}(\tau)\,\mathrm{d}\tau \\ &\qquad = \Gamma(1-\nu) \int_0^t (t-\tau)^{m-1} G^{(m-1)}(\tau) d\tau + c, \quad t > 0. \end{aligned} \tag{3.12}$$

Now, we introduce a new function

$$G_1(t) = \Gamma(1-\nu) \int_0^t (t-\tau)^{m-1} G^{(m-1)}(\tau) d\tau + c,$$

so that equation (3.12) has the form (under supposed conditions)

$$\int_0^t \left[(t-\tau)^{-\nu} + \lambda^2\Gamma(1-\nu)(t-\tau)^{m-1}\right] y^{(m-1)}(\tau)\,\mathrm{d}\tau = G_1(t), \quad t>0. \quad (3.13)$$

Equation (3.13) is of the form (3.1), with

$$K(t) = t^{-\nu} + \lambda^2\Gamma(1-\nu)t^{m-1}, \quad X(t) = y^{(m-1)}(t), \quad G(t) = G_1(t).$$

We apply Theorem 3.3 if $m=2$, and $n=3$. Then $K(s) = s^{\nu-1} + \lambda^2\Gamma(1-\nu)s^{-2}$ and

$$(s^3K(s))^{-1} = \frac{1}{\Gamma(1-\nu)}(s^{3+\nu-1}+\lambda^2 s)^{-1} = \frac{1}{\Gamma(1-\nu)}\left(\frac{1}{s}\right)\left(s^{1+\nu}+\lambda^2\right)^{-1}.$$

We introduce a new parameter $\beta = (1+\nu)/2,\ 1/2 < \beta < 1$. Then

$$\begin{aligned}(s^3K(s))^{-1} &= \frac{1}{\Gamma(1-2\beta+1)}\frac{1}{s}\left((s^\beta)^2+\lambda^2\right)^{-1} \\ &= \frac{1}{\Gamma(1-2\beta+1)}\left(\frac{\lambda}{s^\beta}\right)s^{\beta-1}((s^\beta)^2+\lambda^2)^{-1}.\end{aligned} \quad (3.14)$$

Now we can use the following relation for the Laplace transform: If $(\mathcal{L}f)(s) = \phi(s)$, then

$$s^{\beta-1}\phi(s^\beta) = \mathcal{L}\left[\int_0^\infty F_\beta\left(\frac{t}{\tau^{1/\beta}}\right) f(\tau)\frac{\mathrm{d}\tau}{\beta\tau}\right](s),$$

([10], p. 119), where $F_\beta(t) = \Phi(0;1/(t^{-\beta}))$ and

$$\Phi(0,-\beta;z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+1)\Gamma(-\beta)}$$

is the Wright‘s function ([10] and [3]). In our case $\phi(s)$ will be

$$\phi(s) = \lambda(s^2+\lambda^2)^{-1} = \mathcal{L}(\sin\lambda t)(s) = (\mathcal{L}f)(s)$$

([2], p. 137) and

$$\left[\mathcal{L}\int_0^\infty F_\beta(\frac{t}{\tau^{1/\beta}})\sin\lambda\tau\frac{\mathrm{d}\tau}{\beta\tau}\right](s) = s^{\beta-1}\phi(s^\beta).$$

Going back to (3.14)

$$(s^3K(s))^{-1} = \frac{1}{\Gamma(2-2\beta)}\frac{\lambda}{s^\beta}\left[\mathcal{L}\int_0^\infty F_\beta\left(\frac{t}{\tau^{\frac{1}{\beta}}}\right)\sin\lambda\tau\frac{\mathrm{d}\tau}{\beta\tau}\right](s).$$

Since

$$H(t) = (^{-}\mathcal{L})(s^3 K(s))^{-1} = \frac{1}{\Gamma(2-2\beta)} \lambda \frac{t^{\beta-1}}{\Gamma(\beta)} * \int_0^\infty F_\beta \left(\frac{t}{\tau^{\frac{1}{\beta}}} \right) \sin \lambda\tau \frac{\mathrm{d}\tau}{\beta\tau}.$$

Now we can use (3.10) in which $X(t) = y^{(1)}(t)$. Then this gives $y^{(1)}(t)$ and $y(t)$ as a solution of (3.11) belonging to $L_{\mathrm{loc}}[0, \infty)$.

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