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FROM FREDHOLM OPERATORS TO FIXED POINT THEORY

VLADIMIR RAKOČEVIĆ

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A b s t r a c t. *In this paper we give a survey of some of our results on Fredholm operators, generalized inverses, measures of noncompactness and from fixed point theory.*

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1. *Fredholm operators*

A Fredholm operator is an operator that arises in the Fredholm theory of integral equations. It is named in honor of Erik Ivar Fredholm.

1.1. Introduction and preliminaries

Let X and Y be infinite-dimensional Banach spaces, and $B(X,Y)\,(K(X,Y))$ the set of all bounded (compact) linear operators from X into Y. We shall write $B(X)$ instead of $B(X,X)$. For an element T in $B(X,Y)$ let $N(T)$ and $R(T)$ denote, respectively, the null space and the range of T. Recall that the *reduced minimum modulus* of T, $\gamma(T)$, is defined by

$$\gamma(T) = \inf\{\|Tz\|/\mathrm{dist}\,(z,N(T)) : \mathrm{dist}\,(z,N(T)) > 0\}.$$

$R(T)$ is closed if and only if $\gamma(T) > 0$.

For T in $B(X,Y)$ set $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$. Recall that an operator $T \in B(X,Y)$ is *semi-Fredholm* if $R(T)$ is closed and at least one of $\alpha(T)$ and $\beta(T)$ is finite. For such an operator we define an *index* $i(T)$ by $i(T) = \alpha(T) - \beta(T)$. It is well known that the index is a continuous function on the set of semi-Fredholm operators. Let $\Phi_+(X,Y)$ $(\Phi_-(X,Y))$ denote the set of *upper (lower) semi-Fredholm* operators, i.e., the set of semi-Fredholm operators with $\alpha(T) < \infty$, $(\beta(T) < \infty)$. An operator T is *Fredholm* if it is both upper semi-Fredholm and lower semi-Fredholm. Let $\Phi(X,Y)$ denote the set of Fredholm operators, i.e., $\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$. We shall write $\Phi(X)$ instead of $\Phi(X,X)$, and $\Phi_\pm(X)$ instead of $\Phi_\pm(X,X)$.

The *semi-Fredholm radii* of the operator T are

$$r_+(T) = \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_+(X) \quad \text{for} \quad |\lambda| < \epsilon\},$$
$$r_-(T) = \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_-(X) \quad \text{for} \quad |\lambda| < \epsilon\}.$$

The fact that $K(X)$ is a closed two-sided ideal in $B(X)$ enables us to define the *Calkin algebra* over X as the quotient algebra $C(X) = B(X)/K(X)$; $C(X)$ is itself a Banach algebra in the quotient algebra norm

$$\|T\|_e \equiv \|T + K(X)\| = \inf_{K \in K(X)} \|T + K\|.$$

We shall use π to denote the natural homomorphism of $B(X)$ onto $C(X)$; $\pi(T) = T + K(X)$, $T \in B(X)$. Let $r_e(T) = \lim \|\pi(T^n)\|^{1/n}$ be the *essential spectral radius* of T. An operator $T \in B(X)$ is said to be a *Riesz* operator if and only if $r_e(T) = 0$, i.e., if and only if $\pi(T)$ is quasinilpotent in $C(X)$. Let $R(X)$ denote the set of Riesz operators in $B(X)$.

We denote by $\mathcal{A}$ a complex Banach algebra with identity 1. If $a \in \mathcal{A}$, then $\rho(a)$, $\sigma(a)$, $r(a)$ denote the resolvent set, spectrum and the spectral radius of a, respectively; acc $\sigma(a)$ and iso $\sigma(a)$ denote the sets of all accumulation and isolated points of $\sigma(a)$, respectively. The symbols qNil($\mathcal{A}$), Inv ($\mathcal{A}$) and Idem ($\mathcal{A}$) denote the sets of all quasinilpotent, invertible and idempotent elements of $\mathcal{A}$, respectively.

Let us recall that the Fredholm operators $\Phi(X)$ constitute a multiplicative open semigroup in $B(X)$, and by the Atkinson theorem [10, Theorem 3.2.8] we have

$$\Phi(X) = \pi^{-1}(\text{Inv } C(X)).$$

Thus, $T \in \Phi(X)$ if and only if there are $S \in B(X)$ and $K_1, K_2 \in K(X)$ such that

$$TS = I - K_1 \quad \text{and} \quad ST = I - K_2.$$

Such an operator S, if it exists, is called a *Fredholm inverse* of T.

Recall that if $T \in B(X)$, then $a(T)$ ($d(T)$), the *ascent (descent)* of $T \in B(X)$, is the smallest non-negative integer n such that $N(T^n) = N(T^{n+1})$ $(R(T^n) = R(T^{n+1}))$. If no such n exists, then $a(T) = \infty$ ($d(T) = \infty$).

For T in $B(X)$ set $N^\infty(T) = \cup_n N(T^n),\ R^\infty(T) = \cap_n R(T^n)$ An operator T is called *upper semi-Browder* if $T \in \Phi_+(X)$ and $a(T) < \infty$; T is called *lower semi-Browder* if $T \in \Phi_-(X)$ and $d(T) < \infty$ [71, Definition 7.9.1]. Let $\mathcal{B}_+(X)$ $(\mathcal{B}_-(X))$ denote the set of upper (lower) semi-Browder operators. An operator in a Banach space is called semi-Browder if it is upper semi-Browder or lower semi-Browder. Let $\mathcal{B}(X)$ denote the set of Browder operators, i.e., $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$.

If M and N are two closed subspaces of the Banach space X we set

$$\delta(M, N) = \sup\{\text{dist}\,(u, N) : u \in M, \|u\| = 1\}$$

and

$$\hat{\delta}(M, N) = \max\,[\delta(M, N), \delta(N, M)].$$

$\hat{\delta}$ is called the *gap* (or *opening*) between the M and N [28, 71, 95, 172].

1.2. Semi Browder operators and perturbations

Let us recall that $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$ are open subsets in $B(X)$, but not stable under finite-rank perturbations [10, pp. 13–14]. In [155], among other things, we generalize the well-known Grabiner theorem [67, Theorem 2], and our result [154, Theorem 1] on the perturbations of semi-Fredholm operators with finite ascent or descent. Now our arguments are based on the observation that both [67, Theorem 2] and [154, Theorem 1] have been presented in the global form, i.e., they have been stated for all semi-Fredholm operators with finite ascent or descent, while the perturbation results have been in the local form, i.e., they have depended on the particular choice of semi-Fredholm operator.

Theorem 1.1 ([155])**.** *Suppose that $T,\ S \in B(X)$ and $TS = ST$. Then*

$$T \in \mathcal{B}_+(X) \quad \textit{and} \quad r_e(S) < r_+(T) \quad \Longrightarrow \quad T + S \in \mathcal{B}_+(X),$$

and

$$T \in \mathcal{B}_-(X) \textit{and} \quad r_e(S) < r_-(T) \quad \Longrightarrow \quad T + S \in \mathcal{B}_-(X).$$

Let us remark that the commutativity condition in Theorem 1.1 is essential, even for finite dimensional perturbation S [10, pp. 13–14].

Corollary 1.1. *Suppose that $T \in B(X)$, $S \in R(X)$ and $TS = ST$. Then*

$$T \in \mathcal{B}_+(X) \quad \Longrightarrow \quad T + S \in \mathcal{B}_+(X).$$

and

$$T \in \mathcal{B}_-(X) \quad \Longrightarrow \quad T + S \in \mathcal{B}_-(X).$$

Now as a corollary, we get the main result of S. Grabiner [67, Theorem 2].

Corollary 1.2. *Suppose that* $T \in B(X)$, $S \in K(X)$ *and* $TS = ST$. *Then*

$$T \in \mathcal{B}_+(X) \quad \Longrightarrow \quad T + S \in \mathcal{B}_+(X).$$

and

$$T \in \mathcal{B}_-(X) \quad \Longrightarrow \quad T + S \in \mathcal{B}_-(X).$$

Recall that the *perturbation classes* associated with $\Phi_+(X)$ and $\Phi_-(X)$ are denoted, respectively, by $P(\Phi_+(X))$ and $P(\Phi_-(X))$, i.e.,

$$P(\Phi_+(X)) = \{T \in B(X) : T + S \in \Phi_+(X) \quad \text{for all} \quad S \in \Phi_+(X)\}$$

and

$$P(\Phi_-(X)) = \{T \in B(X) : T + S \in \Phi_-(X) \quad \text{for all} \quad S \in \Phi_-(X)\}.$$

Now as a corollary, we get the main result [154, Theorem 1].

Corollary 1.3. *Suppose that* $T,\ K \in B(X)$ *and* $TK = KT$. *Then*

$$T \in \mathcal{B}_+(X) \quad \textit{and} \quad K \in P(\Phi_+(X)) \quad \Longrightarrow \quad T + K \in \mathcal{B}_+(X).$$

and

$$T \in \mathcal{B}_-(X) \quad \textit{and} \quad K \in P(\Phi_-(X)) \quad \Longrightarrow \quad T + K \in \mathcal{B}_-(X).$$

The sets of upper (lower) semi-Browder operators and Browder operators define, respectively, the corresponding spectra, i.e., for $T \in B(X)$ set

$$\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}_+(X)\},$$

$$\sigma_{db}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}_-(X)\}.$$

$$\sigma_{eb}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}(\mathcal{X})\}.$$

It is clear that $\sigma_{eb}(T) = \sigma_{ab}(T) \cup \sigma_{db}(T)$. $\sigma_{eb}(T)$ is the well known *Browder´s essential spectrum* of T [122, 171]. $\sigma_{ab}(T)$ and $\sigma_{db}(T)$ are non-empty compact subsets of the set of complex plane $\mathbb{C}$, respectively, called the *Browder´s essential approximate point spectrum* of T and *Browder´s essential defect spectrum* of T [144, 154]; Let $\sigma(T), \sigma_a(T)$ and $\sigma_d(T)$ denote, respectively, the *spectrum, approximate point spectrum* and *approximate defect spectrum* of an element T of $B(X)$; recall that

$$\sigma_a(T) = \left\{\lambda \in \mathbb{C} : \inf_{||x||=1} ||(T - \lambda I)x|| = 0\right\}$$

and

$$\sigma_d(T) = \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not onto}\}.$$

It is well known that

$$\sigma_{eb}(T) = \bigcap_{TK=KT,\ K\in K(X)} \sigma(T+K).$$

Recall that [144, 154]

$$\sigma_{ab}(T) = \bigcap_{TK=KT,\ K\in K(X)} \sigma_a(T+K)$$

and

$$\sigma_{db}(T) = \bigcap_{TK=KT,\ K\in K(X)} \sigma_d(T+K).$$

Let $\mathcal{S}$ be a subset of $B(X)$. A subset $\triangle$ of $\sigma(T)$ is said to *remain invariant* under perturbations of T by operators in $\mathcal{S}$ if $\triangle \subset \cap_{S\in\mathcal{S}}\sigma(T+S)$.

Theorem 1.2 ([155]). *Suppose that $T \in B(X)$. Then $\sigma_{ab}(T)$ $(\sigma_{db}(T))$ is the largest subset of the approximate point (defect) spectrum of T which remains invariant under perturbations of T by Riesz operators R which commute with T, i.e.,*

$$\sigma_{ab}(T) = \bigcap_{TS=ST,\ S\in R(X)} \sigma_a(T+S)$$

and

$$\sigma_{db}(T) = \bigcap_{TS=ST,\ S\in R(X)} \sigma_d(T+S).$$

Now as a corollary we get the well known theorem of D. Lay [122, Theorem 4] or M. A. Kaashoek and D. C. Lay [93, Theorem 4.1] for bounded operators.

Corollary 1.4. *Suppose that $T \in B(X)$. Then $\sigma_{eb}(T)$ is the largest subset of the spectrum of T which remains invariant under perturbations of T by Riesz operators R which commute with T.*

Finally, as a corollary we get a theorem of M. Schechter [171, Theorem 2.6].

Corollary 1.5. *An operator $S \in B(X)$ satisfies*

$$\sigma_{eb}(T+S) = \sigma_{eb}(T),$$

for all $T \in B(X)$ which commute with S if and only if $S \in R(X)$.

1.3. $\mathcal{V}_0(X), \mathcal{V}(X)$ and corresponding spectra

For $A \in B(X)$ set

$$k(A) = \dim \frac{N(A)}{N(A) \bigcap R^{\infty}(A)}.$$

$$\mathcal{V}(X) = \{A \in B(X) : R(A) \text{ is closed subspace and } k(A) < \infty\},$$

and

$$\mathcal{V}_0(X) = \{A \in \mathcal{V}(X) : k(A) = 0\}.$$

Let us remark that $k(T) = n < \infty$ precisely when T has Kaashoek's property $P(I, n)$ (see [92, pp. 452–453]) or when T has *almost uniform descent* [68, Definition 1.3]. In particular $k(T) = 0$ if and only if $N(T) \subset R^{\infty}(T)$, or when T is *hyperexact* (cf. [72, 73, 74]).

It is well known that $\Phi_+(X) \cup \Phi_-(X) \subset \mathcal{V}(X)$; $\mathcal{V}_0(X)$ and $\mathcal{V}(X)$ are neither semigroups nor open or closed subsets of $B(X)$. An operator $T \in \mathcal{V}_0(X)$ $(\mathcal{V}(X))$ is called *semi-regular, s-regular, Kato regular, Kato non-singular, ...* (*essential semi regular, essential s-regular, ...*). The semi-Fredholm and semi-Browder operators are closely related to semi-regular and essentially semi-regular operators which (under various names) were intensively studied. From a number of equivalent properties, at the beginning we point out the following Kato–type decomposition theorem [135, 136, 151] for operators in $\mathcal{V}(X)$ which is related to Kato's theorem for semi-Fredholm operators [95].

Let $T_{|M}$ denote the restriction of T to the subspace M of X.

Theorem 1.3 (Kato decomposition [135, 136, 151])**.** *We have $T \in \mathcal{V}(X)$ if and only if $R(T)$ is closed and there exist closed subspaces $X_1, X_2 \subset X$ invariant with respect to T such that $X = X_1 \oplus X_2$, $\dim X_1 < \infty$, $T|_{X_1}$ is nilpotent and $T|_{X_2} \in \mathcal{V}_0(X_2)$.*

Let us remark that if $T \in B(X)$ is a lower semi-Browder operator then the space X_2 in the Kato decomposition is determined uniquely and $X_2 = R^{\infty}(T)$. Thus $T|_{X_2}$ is onto.

If $M \subset X$, then $\overline{M}$ denotes the closure of M in X.

Now, set

$$\sigma_g(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{V}_0(X)\} \quad \text{and} \quad \sigma_{gb}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{V}(X)\}.$$

Now we recall some results for $\sigma_g(T)$ and $\sigma_{gb}(T)$.

Theorem 1.4 ([135, 136, 151])**.** *Let $T \in B(X)$ and f be an analytic function defined on a neighbourhood of the spectrum of T. Then*

$$\sigma_g(f(T)) = f\{\sigma_g(T)\} \quad \textit{and} \quad \sigma_{gb}(f(T)) = f\{\sigma_{gb}(T)\}.$$

Theorem 1.5 ([151]). *Suppose that $T \in B(X)$. Then*

$$\sigma_{gb}(T) = \bigcap_{TK=KT,\, K\in K(X)} \sigma_g(T+K) = \bigcap_{TK=KT,\, K\in F(X)} \sigma_g(T+K).$$

Let

$$\sigma_k(A) = \left\{\lambda \in \mathbb{C} : A - \lambda I \notin \Phi_+(X) \bigcup \Phi_-(X)\right\},$$

be the Kato spectrum of $A \in B(X)$ [96].

Corollary 1.6 ([151]). *Suppose that $T \in B(X)$. Then:*

(1) $\lambda \in \sigma_g(T) \backslash \sigma_{gb}(T)$ *if and only if* λ *is an isolated point of* $\sigma_g(T)$, $0 < k(T-\lambda) < \infty$ *and* $R(T-\lambda)$ *is closed,*

(2) $\sigma_{gb}(T) \subset \sigma_{ek}(T)$,

(3) $\partial\sigma_{ek}(T) \subset \partial\sigma_{gb}(T)$ *and* $\sigma_{gb}(T)$ *is nonempty,*

(4) $\sigma_{gb}(T) = \sigma_{gb}(T^*)$.

Corollary 1.7 ([151]). *Let $T \in \mathcal{V}(X)$. Then the following statements are equivalent:*

(1) $T = V + F$, *where* $\alpha(V) = 0$, F *is finite rank and* $VF = FV$*;*

(2) *there exists a finite rank projection* $P, PT = TP$ *and* $\alpha(T_{|N(P)}) = 0$*;*

(3) *there exists* $\epsilon > 0$ *such that* $\alpha(T+\lambda) = 0$ *for* $0 < |\lambda| < \epsilon$*;*

(4) $a(T) < \infty$.

Let us mention that the mappings $A \to \sigma_g(A)$ and $A \to \sigma_{gb}(A)$ are not upper semi-continuous at A (in general, see [151, Remark 4.4]).

Theorem 1.6 ([151]). *Let $T, T_n \in B(X)$ and $TT_n = T_nT$ for each positive integer n. Then*

$$\limsup \sigma_g(T_n) \subset \sigma_g(T) \quad \text{and} \quad \limsup \sigma_{gb}(T_n) \subset \sigma_{gb}(T).$$

Remark 1.1. If $T \in B(X)$, then $\mathbb{C} \setminus \sigma_{gb}(T)$ is an open set in $\mathbb{C}$. Further, let U be a connected component of $\mathbb{C}\setminus\sigma_{gb}(T)$ and $G = \{\lambda \in \mathbb{C}\setminus\sigma_{gb}(T) : k(T-\lambda) \neq 0\}$. A complex number $\lambda \in G \cap U$ is called a *jumping point in U*. If $\lambda \in U$ is a jumping point, then by Theorem 1.3, there is an T–invariant finite dimensional subspace N_λ in X such that $T - \lambda$ is nilpotent on it. Consistent with the matrix case we define the (algebraic) multiplicity of the jumping point λ to be $\dim N_\lambda$.

Theorem 1.7 ([151]). *Let $T \in B(X)$ and let U and G be as above. Then the functions*

$$\lambda \mapsto N^{\infty}(T-\lambda) + R^{\infty}(T-\lambda) \quad \textit{and} \quad \lambda \mapsto N^{\infty}(T-\lambda) \cap R^{\infty}(T-\lambda)$$

are constant on U, while the functions

$$\lambda \mapsto R^{\infty}(T-\lambda) \quad \textit{and} \quad \lambda \mapsto \overline{N^{\infty}(T-\lambda)}$$

are constant on $U \setminus G$.

Now, suppose that the connected component U contains zero. Then the points in $G \cap U$ can be ordered in such a way that

$$|\lambda_1(T)| \leq |\lambda_2(T)| \leq \cdots < v(T),$$

where each jump appears consecutively according to its multiplicity. If there are only $p\,(= 0, 1, 2, \ldots)$ such jumps, we put $|\lambda_{p+1}(T)| = |\lambda_{p+2}(T)| = v(T)$.

Let S denote the closed unit ball of X. Let $q(T) = \sup\{\epsilon \geq 0 : TS \supset \epsilon S\}$ be the *surjection modulus* of T. For each $r = 1, 2, \ldots$, set

$$q_r(T) = \sup\{q(T+F) : \text{rank} F < r\}.$$

Theorem 1.8 ([151]). *Let $T \in B(X)$, $0 \in U$, and let U, G, and $W \equiv N^{\infty}(T-\lambda) + R^{\infty}(T-\lambda)$, $\lambda \in U$ be as above. Then for each jumping point $\lambda_r(T)$, $r = 1, 2, \ldots$, we have*

$$|\lambda_r(T)| = \lim_k q_r((T_{|W})^k)^{1/k}.$$

Corollary 1.8 ([151]). *If $T \in \mathcal{V}(X)$, then $v_0(T) = \lim_k \gamma((T_{|W})^k)^{1/k}$.*

Theorem 1.9 ([117]). *Let T be an operator on a Banach space X. Then the following conditions are equivalent:*

(1) *$T \in \mathcal{V}(X)$,*

(2) *there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Fredholm and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Fredholm,*

(3) *there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Browder and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Browder,*

(4) *there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is surjective and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Browder,*

(5) *there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Browder and the induced operator $\tilde{T} : X/M \mapsto X/M$ is bounded below.*

It is well known that if $T \in \mathcal{V}(X)$ and K is a compact operator commuting with T then $T + K \in \mathcal{V}(X)$.

Theorem 1.10 ([117])**.** *Let $T, S \in B(X)$, $TS = ST$ and let $T \in \mathcal{V}(X)$. Let $\hat{T} = T|_{R^\infty(T)}$ and let $\tilde{T} : X/R^\infty(T) \mapsto X/R^\infty(T)$ be the induced operator by T. Then*

$$r_e(S) < \min(r_-(\hat{T}), r_+(\tilde{T})) \quad \textit{implies} \quad T + S \in \mathcal{V}(X).$$

Corollary 1.9 ([117])**.** *Let $T \in \mathcal{V}(X)$, $S \in B(X)$, $TS = ST$ and S is a Riesz operator (i.e., $r_e(S) = 0$). Then $T + S \in \mathcal{V}(X)$.*

Corollary 1.10 ([117])**.** *Let $T \in B(X)$. Then*

$$\sigma_{gb}(T) = \bigcap \sigma_g(T + S)$$

where the intersection is taken over all Riesz operators in X commuting with T.

Remark 1.2. The reduced minimum modulus of $T \in B(X)$, $\gamma(T)$, plays an important role in the perturbation theory of linear operators. Also the behavior of the sequence $\{\gamma(T^n)^{1/n}\}$, is extremely important.

If $T \in B(X)$ is a semi-Fredholm operator then there is an $\epsilon > 0$ such that both $\dim N(T - \lambda)$ and codim$R(T - \lambda)$ are constant on $0 < |\lambda| < \epsilon$. We can define

$$\delta_+(T) = \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_+(X) \text{ and } \alpha(T - \lambda) = \text{const for } 0 < |\lambda| < \epsilon\},$$

$$\delta_-(T) = \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_-(X) \text{ and } \beta(T - \lambda) = \text{const for } 0 < |\lambda| < \epsilon\}.$$

Let us remark that $r_+(T) \geq \delta_+(T)$ and $r_-(T) \geq \delta_-(T)$.

Recall that an operator $T \in B(X)$ is *bounded below* if and only if $R(T)$ is closed and $N(T) = \{0\}$, i.e., if and only if the minimum modulus of T, $\mu(T) = \inf\{\|Tx\| : x \in X, \|x\| = 1\} > 0$; an operator $T \in B(X)$ is *surjective* if and only if $R(T) = X$, i.e., if and only if $q(T) > 0$.

For $T \in B(X)$ set

$$\mu_r(T) = \sup\{\mu(T + F) : \text{ rank } F < r\},$$

and $g_r(T) = \max\{\mu_r(T), q_r(T)\}$. If T is semi-Fredholm either $\alpha(T+\lambda)$ or $\beta(T+\lambda)$ (the nullity or the defect) is constant $(= n)$ for λ in the semi-Fredholm domain of T except at a discrete set of jump points which may be ordered by their moduli

$$|\lambda_1(T)| \leq |\lambda_2(T)| \leq \cdots < \max\{\delta_+(T), \delta_-(T)\},$$

where each jump appears consecutively according to its multiplicity.

Theorem 1.11 ([167], Theorem 1.1)**.** *Let T be a semi-Fredholm operator. Then for each jumping point $\lambda_r(T)$, $r = 1, 2, \ldots$, we have*

$$|\lambda_r(T)| = \lim_{k \to \infty} g_{kn+r}(T^k)^{1/k},$$

where $g_r(T) = \max\{\mu_r(T), q_r(T)\}$.

1.4. Regular elements in a Calkin algebra

Let A denote a complex Banach algebra with identity 1. An element $a \in A$ is said to be regular provided there is an element $b \in A$ such that $a = aba$. We say that a is decomposably regular provided the b in the preceding equation can be chosen to be an invertible element in A [70]. Let A^{-1} denote the set of all invertible elements in A. Set $\widehat{A} = \{a \in A : a \in aAa\}$ and $A^\bullet = \{a \in A : a^2 = a\}$.

For a subset M of A let δM and $\mathrm{cl}\, M$ denote, respectively, the boundary and the closure of M. We present some results from [149] were we studied the set of regular elements in the Calkin algebra $C(X)$.

Theorem 1.12 ([149])**.** *If X is a Banach space then* $\widehat{B(X)} + K(X) = \pi^{-1}(\widehat{C(X)})$.

Note that the corresponding result for "decomposable regularity" fails: if $T \in B(X)$ is Fredholm with non zero index then

$$\pi(T) \in C(X)^{-1} \subset C(X)^\bullet C(X)^{-1}, \quad \text{but} \quad T \notin B(X)^\bullet B(X)^{-1} + K(X);$$

however

Theorem 1.13 ([149])**.** *If X is a Banach space then*

$$B(X)^\bullet \Phi(X) + K(X) = \pi^{-1}(C(X)^\bullet C(X)^{-1})$$

and

$$\widehat{B(X)} \cap \mathrm{cl}\, \Phi(X) + K(X) = \pi^{-1}(\widehat{C(X)} \cap \mathrm{cl}\, C(X)^{-1}).$$

It follows that

$$\widehat{B(X)} \cap \mathrm{cl}\, \Phi(X) + K(X) = B(X)^\bullet \Phi(X) + K(X).$$

We can be more precise:

Theorem 1.14 ([149])**.** *If X is a Banach space then*

$$\widehat{B(X)} \cap \mathrm{cl}\, \Phi(X) = B(X)^\bullet \Phi(X).$$

Corollary 1.11. *Let X be a Banach space and $A \in \widehat{B(X)}$. Then the following conditions are equivalent:*

$$A \in \delta\Phi(X),$$
$$A = PB, \quad P \in B(X)^\bullet \setminus \Phi(X) \quad \textit{and} \quad B \in \Phi(X),$$
$$A = CQ, \quad Q \in B(X)^\bullet \setminus \Phi(X) \quad \textit{and} \quad C \in \Phi(X),$$

For any Hilbert space X, let $\dim_H X$ denote the Hilbert dimension of X, that is, the cardinality of an orthonormal basis of X. We set $\mathrm{nul}_H(T) = \dim_H N(T)$ and $\mathrm{def}_H(T) = \dim_H R(T)^\perp$ for $T \in B(X)$. If X is a separable Hilbert space, then with connection according to Theorem 1.12 we have

Theorem 1.15 ([149]). *Let X be a separable Hilbert space. Then*

$$\widehat{B(X)} \cap \mathrm{cl}\,\Phi(X)$$
$$= \Phi(X) \cup \{T \in B(X) : \mathrm{nul}_H(T) = \mathrm{def}_H(T) \textit{ and } R(T) \textit{ is closed}\}.$$

2. *Generalized inverses*

If $A \in \Phi(X, Y)$ there exists $B \in \Phi(Y, X)$ such that

$$ABA = A \quad \text{and} \quad BAB = B.$$

Hence,

$$(AB)^2 = AB \quad \text{and} \quad (BA)^2 = BA.$$

The B could be considered as a generalized inverse of A.

2.1. Moore–Penrose inverse

Let $\mathbb{C}^{m\times n}$ be the set of all $m\times n$ complex matrices. Let $A \in \mathbb{C}^{m\times n}$, and consider the Moore-Penrose equations:

$$AGA = A, \quad GAG = G, \quad (AG)^* = AG, \quad \text{and} \quad (GA)^* = GA.$$

Penrose [138, Theorem 1] has proved that the four equations above have a unique solution for any A, which he called the generalized inverse of A and denoted by $G = A^\dagger$. The concept of a generalized inverses of an arbitrary matrix $A \in \mathbb{C}^{m\times n}$ is originally due to Moore [134] 1920 (called by him the general reciprocal). Rado [141] has proved the equivalence of Moore's and Penrose's result, and today this inverse is known as Moore-Penrose inverse, or M–P inverse, for short, of A.

Let us recall that for any $A \in \mathbb{C}^{m\times n}$ we have $(A^\dagger)^\dagger = A$, $(A^*)^\dagger = (A^\dagger)^*$, $(A^*A)^\dagger = A^\dagger(A^\dagger)^*$, but, in general, $A^\dagger A \neq AA^\dagger$.

Let H and K be infinite-dimensional complex Hilbert spaces. It is well known (see eg., [27, 71]) that $A \in B(H, K)$ has closed range if and only if there exists a unique operator $A^\dagger \in B(K, H)$ called the Moore-Penrose inverse (pseudoinverse) of A which satisfies the following properties:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger \quad \text{and} \quad (A^\dagger A)^* = A^\dagger A.$$

Let $\mathcal{A}$ be a complex Banach algebra with identity 1. The element $a \in \mathcal{A}$ is (von Neumann) regular if $a \in a\mathcal{A}a$. The set of all regular elements in $\mathcal{A}$ will be denoted by $\widehat{\mathcal{A}}$. Recall that an element a in $\mathcal{A}$ is *hermitian* if $\|\exp(ita)\| = 1$ for all real t [178]. In connection with the Moore-Penrose generalized inverse, we [145, 148] have studied the set of elements a in $\mathcal{A}$ for which there exists an x in $\mathcal{A}$ satisfying the following conditions:

$$axa = a, \quad xax = x, \quad ax \quad \text{and} \quad xa \quad \text{are hermitian}. \tag{2.1}$$

By [145, Lemma 2.1] there is at most one x such that the equations in (2.1) hold. The unique x is denoted by $a^\dagger$ and called the Moore-Penrose inverse of a. Let $\mathcal{A}^\dagger$ denote the set of all elements in $\mathcal{A}$ which have Moore-Penrose inverses. Clearly $\mathcal{A}^\dagger \subset \widehat{\mathcal{A}}$, and if $\mathcal{A}$ is a C^*-algebra then $\mathcal{A}^\dagger = \widehat{\mathcal{A}}$ [75, Theorem 6]. Given an element $a \in \mathcal{A}$ let L_a denote the left regular representation of a, i.e., $L_a(x) = ax, x \in \mathcal{A}$. If $a \in \mathcal{A}^\dagger$, then it is known that $\|a^\dagger\| = 1/\gamma(L_a)$ (see [145, Theorem 2.3]).

2.2. Continuity of the Moore-Penrose inverse

Contrary to the usual inverse of a square matrix, it is well known that the Moore-Penrose generalized inverse of a matrix is not necessarily a continuous function of the elements of the matrix.

Example 2.1 ([175]). Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For each $\epsilon \neq 0$ we have

$$(A + \epsilon E)^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^{-1} \end{bmatrix}.$$

Hence $A + \epsilon E \to A$, $(\epsilon \to 0)$, but $\lim_{\epsilon \to 0}(A + \epsilon E)^\dagger$ does not exist.

The following theorem gives necessary and sufficient conditions for the continuity of the Moore-Penrose inverse of matrix.

Theorem 2.1 ([175]). *If $A_n \in \mathbb{C}^{m\times n}$, $A \in \mathbb{C}^{m\times n}$, and $A_n \to A$, then*

$$A_n^\dagger \to A^\dagger \quad \Leftrightarrow \quad \exists n_0 : \quad \text{rank}\, A_n = \text{rank}\, A \quad \text{for } n \geq n_0.$$

The continuity of the Moore-Penrose inverse of an operator on Hilbert spaces has been studied by Izumino [87].

Theorem 2.2 ([87], Proposition 2.3)**.** *Let H and K be Hilbert spaces. Let A_n be a sequence in $B(H,K)$, $A \in B(H,K)$, and $A_n \to A$. If $A_n^\dagger$ and $A^\dagger$ exist, then the following conditions are equivalent:*

$$A_n^\dagger \to A^\dagger,$$
$$\sup_n \|A_n^\dagger\| < \infty,$$
$$A_n^\dagger A_n \to A^\dagger A,$$
$$A_n A_n^\dagger \to AA^\dagger.$$

We showed that some of the upper results could be presented in general Banach algebras.

Theorem 2.3 ([148], Theorem 2.5)**.** *Let $\mathcal{A}$ be a complex Banach algebra, $\{a_n\}$ be a sequence in $\mathcal{A}^\dagger$, and let $a_n \to a \in \mathcal{A}^\dagger$. Then the following conditions are equivalent:*

$$a_n^\dagger \to a^\dagger,$$
$$\sup_n \|a_n^\dagger\| < \infty, \quad \hat{\delta}(N(L_{a_n^\dagger}), N(L_{a^\dagger})) \to 0, \quad \hat{\delta}(R(L_{a_n^\dagger}, R(L_{a^\dagger})) \to 0,$$
$$a_n^\dagger a_n \to a^\dagger a \quad \textit{and} \quad \hat{\delta}(N(L_{a_n^\dagger}), N(L_{a^\dagger})) \to 0,$$
$$a_n a_n^\dagger \to aa^\dagger \quad \textit{and} \quad \hat{\delta}(R(L_{a_n^\dagger}), R(L_{a^\dagger})) \to 0.$$

If $\mathcal{A}$ is a C^*–algebra, then the upper results could be presented in a simpler form.

Theorem 2.4 ([76], Theorem 6 & [153], Theorem 2.2)**.** *Let $\mathcal{A}$ be a C^*–algebra, $\{a_n\}$ be a sequence in $\mathcal{A}^\dagger$, and let $a_n \to a \in \mathcal{A}^\dagger$. Then the following conditions are equivalent:*

$$a_n^\dagger \to a^\dagger,$$
$$\sup_n \|a_n^\dagger\| < \infty,$$
$$a_n^\dagger a_n \to a^\dagger a,$$
$$a_n a_n^\dagger \to aa^\dagger.$$

2.3. Drazin and Koliha Drazin inverses

Let us recall that if S is an algebraic semigroup (or associative ring), then an element $a \in S$ is said to have a Drazin inverse [59] if there exists $x \in S$ such that

$$a^m = a^{m+1}x \qquad \text{for some non-negative integer } m, \tag{2.2}$$
$$x = ax^2, \quad \text{and} \quad ax = xa. \tag{2.3}$$

If a has a Drazin inverse, then the smallest non-negative integer m in (2.2) above is called the *index* $i(a)$ of a. It is well known that there is at most one x such that equations (2.2) and (2.3) hold. The unique x is denoted by a^d and called the Drazin inverse of a. If $i(a) = 1$, then a^d is denoted by $a^\#$ and is called the group inverse of a. Recall that if a has a Drazin inverse, then a^d also has a Drazin inverse, $i(a^d) \leq 1$, $(a^d)^d = a^2a^d$ and $((a^d)^d)^d = a^d$. If S is an associative ring and $a \in S$ has a Drazin inverse then a may always be written as

$$a = c + n,$$

where $c, n \in S$, c has a Drazin inverse, $i(c) \leq 1, cn = nc = 0$, and $n^{i(a)} = 0$. The elements c and n are unique; c is called the ***core*** of a, and n the ***nilpotent*** part of a. Let us mention that in this case

$$c = a^2a^d \quad \text{and} \quad n = a - a^2a^d.$$

We shall refer to $c + n$ as the core nilpotent decomposition of a.

A square matrix always has Drazin inverse. Let X be an infinite-dimensional complex Banach space. Then an operator $T \in B(X)$ has a Drazin inverse T^d if and only if it has finite ascent and descent [99].

The Drazin inverse defined for semigroups is an important theoretical and practical tool in algebra and analysis. When we pass from a semigroup to a ring or a topological algebra A, the Drazin inverse of $a \in A$ in its original form is also restrictive.

We denote by $\mathcal{A}$ a complex Banach algebra with identity e. Following Koliha [100], we say that $a \in \mathcal{A}$ is *generalized Drazin invertible* (*Koliha–Drazin invertible, KD invertible*) if there exists $b \in \mathcal{A}$ such that

$$ab = ba, \quad ab^2 = b, \quad a^2b - a \in \text{qNil}(\mathcal{A}).$$

Such an element b, if it exists, is unique, it is called the generalized Drazin inverse (Koliha–Drazin inverse, KD inverse) of a, and is denoted by a^D, or by a^{KD}. If $a^2b - a$ is in fact nilpotent, then a^D is the conventional Drazin inverse of a. The Koliha–Drazin index $i_k(a)$ of a is equal to $m = i(a)$ if $a^2b - a$ is nilpotent of index m, otherwise $i_k(a) = \infty$.

By $\mathcal{A}^{KD}$ we denote the set of all KD invertible elements of $\mathcal{A}$.

The basic existence results for KD inverse are summarized in the following lemma [100, Theorem 4.2].

Lemma 2.1. *Let $a \in A$. The following conditions on a are equivalent.*

(i) *a is KD invertible.*

(ii) $0 \notin \text{acc}\,\sigma(a)$.

(iii) *There is $p \in \text{Idem}(\mathcal{A})$ commuting with a such that*

$$ap \in \text{qNil}(\mathcal{A}) \quad \textit{and} \quad a + p \in \text{Inv}\,(\mathcal{A}).$$

If (iii) is satisfied, the KD inverse of a is given by

$$a^D = (a+p)^{-1}(e-p).$$

The element p from the preceding lemma is given by

$$p = e - a^D a.$$

It follows that $p = 0$ if $0 \in \rho(a)$; if $0 \in \operatorname{iso}\sigma(a)$, p is the spectral idempotent of a corresponding to 0; we will write $p = a^\pi$.

2.4. Continuity of the Drazin and the KD inverses

It is well known that the Drazin inverse of a matrix is not necessarily a continuous function of the elements of the matrix.

Example 2.2. (a) Let (cf. [25, Example 2])

$$A_n = \begin{bmatrix} 1/n & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$A_n^d = \begin{bmatrix} n & n^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence $A_n \to A$, rank $A_n =$ rank A, and $i\,(A_n) = i\,(A)$, but $A_n^d \not\to A^d$.

(b) In a Banach algebra $\mathcal{A}$ let a be nilpotent of index 3, and therefore Drazin invertible with $a^d = 0$ (cf. [103]). Each $a_n = a + e/n$ is Drazin invertible with $a_n^d = (a+e/n)^{-1} = ne - n^2 a + n^3 a^2$. We have $a_n \to a$, however $a_n^d \not\to a^d$ as the sequence $\|a_n^d\|$ is unbounded. This phenomenon is already well known for matrices.

The following theorem gives necessary and sufficient conditions for the continuity of the Drazin inverse of a matrix.

Theorem 2.5 ([25, 26])**.** *Suppose that $A_n, A \in \mathbb{C}^{m\times m}$, and $A_n \to A$. Let $A = C + N$ and $A_n = C_n + N_n$, $n = 1, 2, \ldots$, be the core nilpotent decompositions of A and A_n, $n = 1, 2, \ldots$, respectively. Then*

$$A_n^d \to A^d \quad \Leftrightarrow \quad \exists n_0 : \quad \operatorname{rank} C_n = \operatorname{rank} C \quad \text{for } n \geq n_0.$$

We study [103] the continuity of the generalized Drazin inverse for elements of Banach algebras and bounded linear operators on Banach spaces. This work extends the results obtained by the second author on the conventional Drazin inverse [156]. The main result on the continuity of the KD inverse in a Banach algebra is expressed in the following theorem.

Theorem 2.6 ([103])**.** *Let a_n and a be KD invertible elements of the Banach algebra $\mathcal{A}$ such that $a_n \to a$, and let p_n and p be the spectral projections of a_n and a corresponding to 0. Then the following conditions are equivalent:*

$a_n^D \to a^D$;

$\sup_n \|a_n^D\| < \infty$;

$\sup_n r(a_n^D) < \infty$;

$\inf_n d(0, \sigma(a_n) \setminus \{0\}) > 0$;

there is $r > 0$ such that $\{\lambda : 0 < |\lambda| < r\} \subset \rho(a) \cap \Big(\bigcap_{n=1}^{\infty} \rho(a_n)\Big)$;

$a_n^D a_n \to a^D a$;

$p_n \to p$.

Now we focus our attention on one aspect of the continuity of the finite index Drazin inverse, namely on a generalization of a theorem due to Campbell and Meyer.

Theorem 2.7 ([103])**.** *Let A_n, $A \in B(X)$ be Drazin invertible operators such that $A_n \to A$, that the indices $i(A_n)$ are bounded, and that the spectral projections P_n, P of A_n, A, corresponding to 0 are of finite rank. Then $A_n^d \to A^d$ if and only if there exists n_0 such that* $\text{rank}P_n = \text{rank}P$ *for all $n \geq n_0$.*

The preceding theorem implies the main result of Campbell and Meyer (Theorem 2.5) on the continuity of the Drazin inverse for matrices.

2.5. Differentiation of the Drazin and the KD inverses

The differentiability of the Drazin inverse for matrices was studied by Campbell [24] and Hartwig and Shoaf [77]. Drazin [60] considered the differentiation of the (finite index) Drazin inverse in associative rings, using a general derivative in the ring. We [111] studied the differentiability of the generalized Drazin inverse for a function $A(t)$ from a real interval into the space $B(X)$ of all bounded linear operators on a Banach space X. As a special case of our results we get a theorem due to Campbell [24].

In this section, J will denote an interval, t_0 an element of J, and $A : J \mapsto \mathcal{B}(X)$ an operator valued function. By $A'(t)$ we denote the derivative of $A(t)$ at t, and by $A^D(t)$ the KD inverse $A(t)^D$. Using the preceding result on the continuity of the Drazin inverse, we proved our main theorem on differentiability.

Theorem 2.8 ([111])**.** *Let the function $A : J \mapsto B(X)$ satisfy the following conditions:*

(i) *$A(t)$ is KD invertible for all $t \in J$,*

(ii) *$A(t)$ is differentiable at t_0.*

Then $A^D(t)$ is differentiable at t_0 if and only if $A^D(t)$ is continuous at t_0. In this case the derivative $(A^D)'(t_0)$ is given by

$$(A^D)'(t_0) = \frac{1}{2\pi i}\int_\Gamma \lambda^{-1} R(\lambda; A(t_0))A'(l_0)R(\lambda; A(t_0))\,\mathrm{d}\lambda,$$

where Γ is a Cauchy cycle relative to $(\mathbb{C}\setminus\{0\}, \sigma(A(t_0))\setminus\{0\}))$, and we obtain the following series expansion for $(A^D)'$:

$$(A^D)' = -A^D A' A^D + \sum_{n=0}^{\infty} PA^n A'(A^D)^{n+2} + \sum_{n=0}^{\infty}(A^D)^{n+2}A'A^n P, \qquad (2.4)$$

where A, A^D, A', P stand for $A(t_0)$, $A^D(t_0)$, $A'(t_0)$, $P(t_0)$, respectively.

Hartwig and Shoaf [77, Eq. (3.10)] used holomorphic calculus to give a formula for the derivative of the Drazin inverse of a complex matrix in terms of the spectral components of $A(t)$.

In the case that the operators $A(t)$ have the conventional Drazin inverse and the indices of $A(t)$ are uniformly bounded, we are able to obtain a stronger result.

Theorem 2.9 ([111])**.** *Let $A(t)$ be an operator valued function defined on an interval J with $A(t)$ Drazin invertible for all $t \in J$ and differentiable at $t_0 \in J$. Suppose that the indices $i(A(t))$ are uniformly bounded and the spectral projections $P(t)$ are of finite rank. Then $A^d(t)$ is differentiable at t_0 if and only if there is $\delta > 0$ such that*

$$\mathrm{rank} P(t) = \mathrm{rank} P(t_0) \quad \textit{whenever} \quad |t - t_0| < \delta.$$

From the preceding theorem we recover the main result of [24] on the differentiability of the matrix Drazin inverse. Recall that $C(t) = A(t)(I - P(t))$ is the so-called core operator of $A(t)$; the rank of $C(t)$ is called the *core rank* of $A(t)$.

Corollary 2.1 ([24], Theorem 4)**.** *Let $A(t)$ be a $p \times p$ matrix valued function differentiable at t_0. Then $A^d(t)$ is differentiable at t_0 if and only if the core rank of $A(t)$ is constant in a neighborhood of t_0.*

Let us remark that our approach differs from the one adopted by Campbell in [24], who derived his theorem from the known differentiation result for the Moore–Pen rose inverse and from the relation between the Drazin inverse A^d of a $p \times p$ matrix A and the Moore–Penrose inverse $A^\dagger$ of A:

$$A^d = A^p (A^{2p+1})^\dagger A^p.$$

In the case that the Drazin indices $i(A(t))$ are finite and uniformly bounded, the preceding theorem subsumes the differentiation formula of Campbell [24, Theorem 2] the summation then becomes finite. Let us observe that Campbell's proof is based on the differentiation of the defining equations in the case that A has the Drazin index 1, that is, on the differentiation of the equations

$$AA^dA = A, \quad A^dAA^d = A^d, \quad AA^d = A^dA.$$

Hartwig and Shoaf obtained Campbell's formula from a difference relation [77, (4.16)]. Under the assumption of finite and uniformly bounded indices, formula (2.4) formally agrees with Drazin's result [60, Theorem 2], which is derived for the (finite index) Drazin inverse in associative rings.

We note that if $i(A) \leq 1$, formula (2.4) reduces to

$$(A^d)' = -A^dA'A^d + PAA'(A^d)^2 + (A^d)^2A'AP.$$

For matrices this yields [24, Theorem 1].

Throughout this section let Ω be an open set in the complex plane, and $A(\lambda)$ a continuous function on Ω whose values are bounded linear operators on X.

From Theorem 2.8 we obtain the following main result on holomorphic behavior of $A^D(\lambda)$.

Theorem 2.10 ([112], Theorem 5.1). *Let $A(\lambda)$ be holomorphic at $\lambda_0 \in \Omega$ and KD invertible for all $\lambda \in \Omega_0 \subset \Omega$, where Ω_0 is a neighbourhood of λ_0. Let $C(\lambda)$ be the core part of $A(\lambda)$, $\lambda \in \Omega_0$. Then the following conditions are equivalent:*

(i) *$A^D(\lambda)$ is a holomorphic function at λ_0,*

(ii) *$A^D(\lambda)$ is a continuous function at λ_0,*

(iii) *there exist closed subspaces M_{λ_0} and N_{λ_0} of X and a neighbourhood $U_{\lambda_0} \subset \Omega_0$ of the point λ_0 such that $R(C(\lambda)) \oplus M_{\lambda_0} = X$ and $N(C(\lambda)) \oplus N_{\lambda_0} = X$ for all $\lambda \in U_{\lambda_0}$.*

In the special case when the spectral projections are of finite rank we have the following corollary.

Theorem 2.11 ([112], Theorem 5.2). *Let $A(\lambda)$ be holomorphic and KD invertible for all $\lambda \in \Omega$ with the spectral projections $P(\lambda)$ of finite rank. Then the following conditions are equivalent:*

(i) $A^D(\lambda)$ *is holomorphic,*

(ii) $\dim R(P(\lambda))$ *is constant on each connected component of* Ω.

Let us remark that if $A \in B(X)$ is a finite rank operator, then A has the conventional Drazin inverse. From Theorem 2.11 we deduce the following result.

Theorem 2.12 ([112], Theorem 5.2)**.** *Let* $A(\lambda)$ *be holomorphic in* Ω *and let* $A(\lambda)$ *be a finite rank operator for all* $\lambda \in \Omega$. *Hence* $A(\lambda)$ *is Drazin invertible for all* $\lambda \in \Omega$. *Let* $C(\lambda)$ *be the core part of* $A(\lambda)$, $\lambda \in \Omega_0$. *Then the following conditions are equivalent:*

(i) $A^D(\lambda)$ *is holomorphic,*

(ii) $\dim R(C(\lambda))$ *is constant on each connected component of* Ω.

When the family $A(\lambda)$ is commuting, we have the following specialization.

Theorem 2.13 ([112], Theorem 5.2)**.** *Let* Ω *be open, bounded and connected, and let* $A(\lambda)$ *be holomorphic and Drazin invertible for all* $\lambda \in \Omega$. *Let* $C(\lambda)$ *be the core part of* $A(\lambda)$. *Suppose further that the family* $A(\lambda)$ *is commuting, that is,* $A(\lambda)A(\mu) = A(\mu)A(\lambda)$ *for all* $\lambda, \mu \in \Omega$. *Then the following conditions are equivalent:*

(i) $A^D(\lambda)$ *is holomorphic,*

(ii) $R(C(\lambda))$ *and* $N(C(\lambda))$ *are constant on* Ω.

2.6. Perturbations of the Drazin and KD inverses

Let us point up that the important applications of perturbation of the Drazin inverse are to, e.g. singular perturbations of autonomous linear systems of differential equations and perturbations of continuous semigroups of bounded linear operators.

The perturbation properties of the Drazin inverse for matrices were investigated Yimin Wei and Guorong Wang [176]. In [163] we study the perturbation of the generalized inverse introduced recently by Koliha. We start in the Banach algebra setting, and then move to bounded linear operators. Let $a \in \mathcal{A}$ be KD invertible. Following [176], we say that $b \in \mathcal{A}$ *obeys the condition* $(\mathcal{W})$ *at* a if

$$b - a = aa^D(b - a)aa^D \quad \text{and} \quad \|a^D(b - a)\| < 1.$$

Let us remark that the condition

$$b - a = aa^D(b - a)aa^D$$

is equivalent to the condition

$$b - a = aa^D(b - a) = (b - a)aa^D.$$

The basic auxiliary results are summarized in the following lemma.

Lemma 2.2 ([163]). *Let $a \in \mathcal{A}$ be KD invertible, and let $b \in \mathcal{A}$ obey the condition $(\mathcal{W})$ at a. Then*
(i) $b = a(1 + a^D(b-a))$;
(ii) $b = (1 + (b-a)a^D)a$;
(iii) $1 + a^D(b-a)$ *and* $1 + (b-a)a^D$ *are invertible, and*

$$(1 + a^D(b-a))^{-1}a^D = a^D(1 + (b-a)a^D)^{-1}.$$

Theorem 2.14 ([163]). *Let $a \in \mathcal{A}$ be KD invertible, and let $b \in \mathcal{A}$ obey the condition $(\mathcal{W})$ at a. Then b is KD invertible, $bb^D = aa^D$, $b^D = (1+a^D(b-a))^{-1}a^D = a^D(1 + (b-a)a^D)^{-1}$ and $i_k(a) = i_k(b)$.*

Let us remark that as a direct corollary of Theorem 2.14 we obtain the known result for matrices [176, Theorem 1]. The next corollary is a generalization of [176, Theorem 3.2].

Corollary 2.2 ([163]). *Let $a \in \mathcal{A}$ be KD invertible and let $b \in \mathcal{A}$ obey the condition $(\mathcal{W})$ at a. Then b is KD invertible, and we have*

$$\frac{\|b^D - a^D\|}{\|a^D\|} \le \frac{\|a^D(b-a)\|}{1 - \|a^D(b-a)\|}.$$

Corollary 2.3 ([163]). *Let $a \in \mathcal{A}$ be KD invertible, and let $b \in \mathcal{A}$ obey the condition $(\mathcal{W})$ at a. Then b is KD invertible, and we have*

$$\frac{\|a^D\|}{1 + \|a^D(b-a)\|} \le \|b^D\| \le \frac{\|a^D\|}{1 - \|a^D(b-a)\|}.$$

Corollary 2.4 ([163]). *Let $a \in \mathcal{A}$ be KD invertible, $b \in \mathcal{A}$ obey the condition $(\mathcal{W})$ at a, and $\|a^D(b-a)\| < 1/2$. Then b is KD invertible, and a obeys the condition $(\mathcal{W})$ at b.*

Corollary 2.5 ([163]). *Let $a \in \mathcal{A}$ be KD invertible, let $b \in \mathcal{A}$ obey the condition $(\mathcal{W})$ at a and let $\|a^D\|\|b-a\| < 1$. Then b is KD invertible and we have*

$$\frac{\|b^D - a^D\|}{\|a^D\|} \le \frac{k_D(a)\|b-a\|/\|a\|}{1 - k_D(a)\|(b-a)\|/\|a\|},$$

where $k_D(a) = \|a\|\|a^D\|$ is defined as the condition number with respect to the KD inverse.

We continue with some algebraic properties of KD inverse. Let us remark that it is well known that if $a, b \in \mathcal{A}$ are Drazin (KD) invertible, then ab is not necessary Drazin (KD) invertible, and if ab is Drazin (KD) invertible, then in general $(ab)^D \neq b^D a^D$. In the next propositions we show that if b obeys the condition $(\mathcal{W})$ at a we can be more precise.

Proposition 2.1 ([163]). *Let $a \in \mathcal{A}$ be KD invertible and let $b \in \mathcal{A}$ obey the condition $(\mathcal{W})$ at a. Then:*

ab^D has a group inverse and $(ab^D)^\# = ba^D$;

$b^D a$ has a group inverse and $(b^D a)^\# = a^D b$;

ba^D has a group inverse and $(ba^D)^\# = ab^D$;

$a^D b$ has a group inverse and $(a^D b)^\# = b^D a$.

Proposition 2.2 ([163]). *Let $a \in \mathcal{A}$ be KD invertible and let $b_1, b_2 \in \mathcal{A}$ obey the condition $(\mathcal{W})$ at a. Then $b_1 b_2$ is KD invertible and $(b_1 b_2)^D = b_2^D b_1^D$.*

As a corollary we have the following result.

Corollary 2.6 ([163]). *Let $a \in \mathcal{A}$ be KD invertible and let $b \in \mathcal{A}$ obey the condition $(\mathcal{W})$ at a. Then ab and ba are KD invertible, $(ab)^D = b^D a^D$ and $(ba)^D = a^D b^D$.*

We give some applications of Theorem 2.14, and extend the results obtained by Yimin Wei and Guorong Wang [176]. We shall consider error bounds for the KD inverse in $B(X)$.

Let us consider the equation

$$Ax = b,$$

where A is KD invertible. We study the sensitivity of the solution x to variation in the data b and A, provided that b and x are in $R(A^D)$.

Theorem 2.15 ([163]). *Let $A \in B(X)$ be KD invertible and let $b, c \in R(A^D)$. If $x, y \in R(A^D)$ satisfy $Ax = b$ and $Ay = c$, then*

$$\frac{\|y - x\|}{\|x\|} \leq k_D(A) \frac{\|c - b\|}{\|b\|}.$$

Theorem 2.16 ([163]). *Let $A \in B(X)$ be KD invertible, let $B \in B(X)$ obey the condition $(\mathcal{W})$ at A, and let $b \in R(A^D)$. If $x, y \in R(A^D)$ satisfy $Ax = b$ and $By = b$, then*

$$\frac{\|y - x\|}{\|x\|} \leq k_D(A) \frac{\|A^D(B - A)\|}{1 - \|A^D(B - A)\|}.$$

Theorem 2.17 ([163]). *Let $A \in B(X)$ be KD invertible, let $B \in B(X)$ obey the condition $(\mathcal{W})$ at A, and let $b, c \in R(A^D)$. If $x, y \in R(A^D)$ satisfy $Ax = b$ and $By = c$, then*

$$\|y - x\| \leq \frac{\|A^D\|}{1 - \|A^D(B - A)\|}(\|A^D(B - A)\|\|b\| + \|c - b\|).$$

Theorem 2.18 ([160]). *Let $a \in \mathcal{A}^{KD}$, and let $b \in \mathcal{A}$. Then the following conditions are equivalent:*

$$b \in \mathcal{A}^{KD} \quad \textit{and} \quad b^\pi = a^\pi.$$

$$a^\pi b = ba^\pi, ba^\pi \in \mathrm{qNil}(\mathcal{A}) \quad \textit{and} \quad b + a^\pi \in \mathrm{Inv}(\mathcal{A}),$$

$$1 + a^D(b-a) \in \mathrm{Inv}(\mathcal{A}), a^\pi b = ba^\pi \quad \textit{and} \quad ba^\pi \in \mathrm{qNil}(\mathcal{A}),$$

$$b \in \mathcal{A}^{KD}, 1 + a^D(b-a) \in \mathrm{Inv}(\mathcal{A}) \quad \textit{and} \quad b^D = (1 + a^D(b-a))^{-1}a^D,$$

$$b^D - a^D = a^D(a-b)b^D.$$

Corollary 2.7 ([160]). *Let $A \in B(X)^{KD}$, and let $B \in B(X)$. Then the following conditions are equivalent:*
(i) $B \in B(X)^{KD}$ *and* $\pi(B^\pi) = \pi(A^\pi)$.
(ii) $\pi(A^\pi B) = \pi(BA^\pi), BA^\pi \in R(X)$ *and* $B + A^\pi \in \Phi(X)$.
(iii) $I + A^D(B-A) \in \Phi(X)$, $\pi(A^\pi B) = \pi(BA^\pi)$ *and* $BA^\pi \in R(X)$.
(iv) $B \in B(X)^{KD}$, $I + A^D(B-A) \in \Phi(X)$ *and*

$$B^D = CA^D + K,$$

where $K \in K(X)$ *and* $C \in B(X)$ *is a Fredholm inverse of* $I + A^D(B-A)$.
(v) $B \in B(X)^{KD}$ *and*

$$B^D - A^D = A^D(A-B)B^D + F,$$

where $F \in K(X)$.

In [29] we studied the Drazin inverse and generalized Drazin inverse of a closed linear operator A and obtain explicit error estimates in terms of the gap between closed operators and in terms of the gap between ranges and nullspaces of operators. The results are used to derive a theorem on the continuity of the Drazin inverse and generalized Drazin inverse for closed operators and to describe the asymptotic behaviour of operator semigroups.

2.7. KD inverse and commuting Riesz perturbations

It is well known in the theory of Fredholm operators that Fredholm, semi-Fredholm, Browder and semi-Browder operators are stable under commuting Riesz perturbations. In [160] we establish similar related results for KD (Koliha–Drazin) invertible operators with finite nullity. Let us point out that whereas Fredholm, semi-Fredholm, Browder and semi-Browder operators have closed range, this need not be the case for KD invertible operators in general.

Theorem 2.19 ([160])**.** *Suppose that $T \in B(X)^{KD}$, $\alpha(T) < \infty$, $S \in R(X)$, and $TS = ST$. Then:*

$$T + S \in B(X)^{KD}.$$

Furthermore, there is a finite rank operator $F \in B(X)$ such that

$$(T + S)^{\pi} = T^{\pi} + F \quad \text{and} \quad T^{\pi}F = FT^{\pi},$$

and

$$(T + S)^{D} = [I + (T + T^{\pi})^{-1}(S + F)]^{-1}T^{D} - (T + T^{\pi} + S + F)^{-1}F.$$

Let us remark that Theorem 2.19 fails to hold if we do not assume that T and S are commuting operators. It can be shown by means of an old and well-known Yood's example [182, p. 599] (see also [160]).

The next example shows that if we assume $\alpha(T) = \infty$ then Theorem 2.19 does not hold in general.

Example 2.3 ([160])**.** Let X be an infinite dimensional Banach space, $T = O \in \mathcal{B}(X)$, and $S \in \mathcal{K}(X)$ be such that $\sigma(S)$ is an infinite subset of $\mathbb{C}$. Now, $\alpha(T) = \dim X = \infty$, $TS = ST$, but $\sigma(T + S) = \sigma(S)$ implies that $T + S$ is not KD invertible.

The next example shows that in general we cannot obtain $\alpha(T + S) < \infty$ in Theorem 2.19.

Example 2.4 ([160])**.** Let $B_W : \ell^2 \mapsto \ell^2$ be the weighted backward shift on ℓ^2 defined by

$$B_W(x_1, x_2, x_3, \dots) = (x_2, 2^{-2}x_3, 3^{-2}x_4, \dots).$$

It is easy to see that B_W is a compact operator, $\alpha(B_W) = 1$ and $\sigma(B_W) = \{0\}$. Thus B_W is KD invertible and $\alpha(B_W) < \infty$, but $\alpha(B_W + (-B_W)) = \alpha(O) = \infty$.

2.8. Idempotents

In the theory of generalized inverses idempotents play a very important role.

Let R and K be subspaces of a Hilbert space H, and let P_R and P_K denote the orthogonal projections of H onto these subspaces. Buckholtz has proved that the operator $P_R - P_K$ is invertible if and only if H is the direct sum of R and K [22, 23]. In this case there exists a linear idempotent M with range R and kernel K, and $\|P_K P_R\| < 1$. In [157] we give a precise value of $\|M\|$, get a sharper estimate of a result of Vidav [179], and prove that $\|P_K P_R\|$ is equal to the gap between R and $K^{\perp}$.

Theorem 2.20 ([157])**.** *Let M be a bounded linear idempotent operator on a Hilbert space H with range R and kernel K. Then*

$$\|M\| = \frac{1}{\sqrt{1 - \|P_K P_R\|^2}}. \tag{2.5}$$

Corollary 2.8. *If P_R and P_K are projections onto the range and the null space of a bounded idempotent operator M then $\|P_R P_K\| = \sqrt{\|M\|^2 - 1}/\|M\|$.*

Following Kato [96, p. 197] for any closed subspaces R, K of H we define the *gap* (or *opening*) *between R and K* by gap $(R, K) = \|P_R - P_K\|$. Now we obtain a relation between the norm of $P_R P_K$ and the gap between R and $K^\perp$.

Corollary 2.9. *If P_R and P_K are projections onto the range and the null space of a bounded idempotent operator M, then $\|P_R P_K\| = \|P_R - P_K^\perp\|$.*

Corollary 2.10. *Let M be a bounded linear idempotent operator on a Hilbert space H. Then $\|M\| = \|I - M\|$.*

Remark 2.1. Theorem 2.20 is not new. Identity 2.5 is due to Ljance [121]; see also [140, 180]. Hence in [157] we offer a new proof of an old theorem. Labrousse [120] was unawere of the work of Ljance [121], and he proved (2.20) with the squared norm of $P_K P_R$ replaced by the square norm of $P_R - P_K^\perp$, not a straightforward exercise. Let us remark that V. Pták [140] mentioned that T. Ando and B. Sz. Nagy called his attention to the fact that this result had been contained in a paper of Ljance [121]. Pták admitted the journal had not been accesible to him, and gave a proof of this result. Finally, 2.10 is due to Del Pasqua [57]; see also [96]. Ljance [121] was unawere of the work of Del Pasqua [57], and he also proved (2.10).

In [109] we study norms of idempotents in C^*-algebras. The results of Ljance, Vidav, Buckholtz and Wimmer on idempotent operators in Hilbert spaces are considered in the setting of C^*-algebras, and simpler new proofs, based on algebraic and spectral–rather than spatial–arguments, are given. We give an application to projections with respect to a-involutions.

We denote by A a C^*-algebra with unit 1 and by A^{-1} the set of all invertible elements in A. For an element $a \in A$ we denote by $\sigma(a)$ the spectrum of a and by $r(a)$ the spectral radius of a.

The term *projection* will be reserved for an element p of a A which is self-adjoint and idempotent, that is, $p^* = p = p^2$.

Let $f \in A$ be an idempotent. Following Koliha [102], we say that $p \in A$ is a *range projection* of f if p is a projection satisfying

$$pf = f \quad \text{and} \quad fp = p. \tag{2.6}$$

If A is a C^*-subalgebra of $B(H)$, the C^*-algebra of all bounded linear operator on a Hilbert space H, then (2.6) holds if and only if p is the (orthogonal) projection onto the range of f. Let us recall Theorem 1.3 of [102] that for every idempotent $f \in A$ there exists a unique range projection of f denoted by $f^\perp$ given explicitly by the Kerzman–Stein formula [97]

$$f^\perp = f(f + f^* - 1)^{-1}. \tag{2.7}$$

If p is a projection, then $p^{\perp} = p$.

Recall that [102, Proposition 1.4]

$$1 - f^{\perp} = (1 - f^{*})^{\perp} \quad \text{and} \quad 1 - (f^{*})^{\perp} = (1 - f)^{\perp}. \tag{2.8}$$

Let $e,\ f \in A$ be idempotents. By $\pi(e, f)$ we denote an idempotent $h \in A$ (if it exists) satisfying the conditions

$$h^{\perp} = e^{\perp}, \quad (1 - h)^{\perp} = f^{\perp}. \tag{2.9}$$

Recall that, for projections $p,\ q \in A$, $\pi(p, q)$ denotes an idempotent $h \in A$ satisfying $p = h^{\perp}$ and $q = (1 - h)^{\perp}$.

Theorem 2.21 ([109]). *Let $h \in A$ be an idempotent. Then*

$$\|h\| = \frac{1}{\sqrt{1 - \|h^{\perp}(1 - h)^{\perp}\|^2}}. \tag{2.10}$$

Corollary 2.11. *Let $h \in A$ be an idempotent. Then*

$$\|h^{\perp}(1 - h)^{\perp}\| = \frac{\sqrt{\|h\|^2 - 1}}{\|h\|}. \tag{2.11}$$

Recall that, for projections $p, q \in A$, $\pi(p, q)$ denotes an idempotent $h \in A$ satisfying $\pi = h^{\perp}$ and $q = (1 - h)^{\perp}$.

Theorem 2.22 ([109]). *Let $p, q \in A$ be nontrivial projections. Then the following conditions are equivalent:*

(i) $A = pA \oplus qA$;
(ii) *The idempotent $\pi(p, q)$ exists;*
(iii) $\|pq\| < 1$ *and* $A = pA + qA$;
(iv) $1 - pq \in A^{-1}$ *and* $A = pA + qA$;
(v) $\|pqp\| < 1$ *and* $A = pA + qA$;
(vi) $1 - pqp \in A^{-1}$ *and* $A = pA + qA$;
(vii) $\|p + q - 1\| < 1$;
(viii) $p - q \in A^{-1}$.

The idempotent $\pi(p, q)$ is given by the formulae

$$p(p, q) = (1 - pqp) - 1(p - pq) = (p - q) - 1(1 - q).$$

Theorem 2.23 ([109]). *Let a be a positive invertible element of A. If $p, q \in A$ are nontrivial idempotents satisfying $ap = p^{*}a$ and $aq = q^{*}a$, then the following conditions are equivalent:*

(i) $A = pA \oplus qA$;

(ii) *There exists an idempotent* $f \in A$ *such that*

$$p = a^{-1/2} f^{\perp} a^{1/2} \text{ and } q = a^{-1/2}(1-f)^{\perp} a^{1/2};$$

(iii) $\|a^{-1/2} pqa^{-1/2}\| < 1$ *and* $A = pA + qA$;
(iv) $1 - pq \in A^{-1}$ *and* $A = pA + qA$;
(v) $\|a^{1/2} pqa^{-1/2}\| < 1$ *and* $A = pA + qA$;
(vi) $1 - pqp \in A^{-1}$ *and* $A = pA + qA$;
(vii) $\|a^{1/2}(p + q - 1)a^{-1/2}\| < 1$;
(viii) $p - q \in A^{-1}$.

The idempotent f *is given by the formula* $f = a^{1/2}(p-q)^{-1}(1-q)a^{-1/2}$.

The following theorem is motivated by Wimmer's result [181, Theorem 2.1], proved for finite dimensional Hilbert spaces. Recall that $\pi(u, v) = \pi(u^{\perp}, v^{\perp})$ for idempotents u, $v \in A$.

Theorem 2.24 ([109])**.** *Let* $h \in A$ *be an idempotent and* $f \in A$ *a projection such that*

$$\|h\| \|f - (1-h)^{\perp}\| < 1. \tag{2.12}$$

Then $g := \pi(h, f)$ *exists and*

$$\|g - h\| \leq \frac{\|h\|^2 \|f - (1-h)^{\perp}\|}{1 - \|h\| f - (1-h)^{\perp}\|}. \tag{2.13}$$

From the preceding theorem and its proof we obtain the following result.

Corollary 2.12 ([109])**.** *Let* h, $g \in A$ *be idempotents and* $f \in A$ *be a projection such that* $\|hf\| < 1$ *and* $g = \pi(h, f)$. *Then*

$$\|g - h\| \leq \frac{\|hf\|}{1 - \|hf\|} \|h\|. \tag{2.14}$$

Buckholtz [22, 23] gave necessary and sufficient conditions for the invertibility of the difference of two orthogonal projections in a Hilbert space. We [110] generalize this result by investigating when the difference of such projections is a Fredholm operator, and give an explicit formula for its Fredholm inverse.

Theorem 2.25 ([110])**.** *Let* R *and* K *be closed subspaces of a Hilbert space* H *and let* P *and* Q *be the orthogonal projections with the ranges* R *and* K, *respectively. The following are equivalent:*

(i) $P - Q \in \Phi(H)$;
(ii) $I - PQ \in \Phi(H)$ *and* $I - (I - P)(I - Q) = P + Q - PQ \in \Phi(H)$;
(iii) $R + K$ *is closed in* H *and* $\dim[(R \cap K) \oplus (R^{\perp} \bigcap K^{\perp})] < \infty$;
(iv) $\|P + Q - I\|_e < 1$;
(v) $P + Q \in \Phi(H)$ *and* $I - PQ \in \Phi(H)$.

As a special case of the preceding theorem we now consider the case when the difference $P - Q$ is invertible. This problem is the subject of Buckholtzs papers [22, 23]), and the equivalence of (i), (iii) and (iv) of the following corollary is given in [23, Theorem 1]. In the setting of rings, the equivalence of (i), (ii), (iii) and (v) was proved in [108].

Corollary 2.13 ([110])**.** *Let R and K be closed subspaces of a Hilbert space H and let P and Q be the orthogonal projections with the ranges R and K, respectively. The following are equivalent:*
(i) $P - Q$ *is invertible;*
(ii) $I - PQ$ *and* $I - (I - P)(I - Q) = P + Q - PQ$ *are invertible;*
(iii) $H = R \oplus K$*;*
(iv) $\|P + Q - I\| < 1$*;*
(v) $P + Q$ *and* $I - PQ$ *are invertible.*

We recall that $G \in B(H)$ is a projection if and only if H is the topological direct sum $H = R \oplus N$, where $R(G) = R$ and $N(G) = N$. We call G the projection onto R along N, and write $G = P_{R,N}$.

Theorem 2.26 ([110])**.** *Let $P, Q \in B(H)$ be orthogonal projections with the ranges R and K, respectively, and let $R(P - Q)$ be closed. Then*

$$(P - Q)^\dagger = P_{M,N} - P_{L,S},$$

where

$$M = R \cap (R^\perp + K^\perp), \qquad N = K \oplus (R^\perp \cap K^\perp),$$
$$L = R^\perp \cap (R + K), \qquad S = K^\perp \oplus (R \cap K).$$

Theorem 2.27 ([110])**.** *Let $P, Q \in B(H)$ be orthogonal projections with the ranges R and K, respectively, and let $P - Q \in \Phi(H)$. Then*

$$(P - Q)^\Phi = U + U^* - I,$$

where $U = ((I-Q)P)^\dagger$ is the projection onto $R \cap (R^\perp + K^\perp)$ along $K \oplus (R^\perp \cap K^\perp)$.

From the following corollary we recover the result of Buckholtz (cf. [22, 23]).

Corollary 2.14 ([110])**.** *Let $P, Q \in B(H)$ be orthogonal projections with the ranges R and K, respectively. Then $P - Q$ is invertible if and only if $H = R \oplus K$, in which case*

$$(P - Q)^{-1} = U + U^* - I,$$

where $U = P_{R,K}$ and $U^* = P_{K^\perp, R^\perp}$.

We proved [114] a stability theorem for the nullity of a linear combination $c_1P_1 + c_2P_2$ of two idempotent operators P_1, P_2 on a Banach space provided c_1, c_2 and $c_1 + c_2$ are nonzero. We then show that for $c_1P_1 + c_2P_2$ the property of being upper semi-Fredholm, lower semi-Fredholm and Fredholm, respectively, is independent of the choice of c_1, c_2, and that the nullity, defect and index of $c_1P_1 + c_2P_2$ are stable. For convenience, we define a subset Γ of $\mathbb{C}^2$ by

$$\Gamma = \{(c_1, c_2) \in \mathbb{C} : c_1 \neq 0, c_2 \neq 0, c_1 + c_2 \neq 0\}.$$

Theorem 2.28 ([114]). *Let $P_1, P_2 \in B(X)$ be idempotents. Then:*

(1) *If $c_1P_1 + c_2P_2$ is upper semi-Fredholm for some $(c_1, c_2) \in \Gamma$, then it is upper semi-Fredholm for all $(c_1, c_2) \in \Gamma$, and $\alpha(c_1P_1 + c_2P_2)$ is constant on Γ.*

(2) *If $c_1P_1 + c_2P_2$ is lower semi-Fredholm for some $(c_1, c_2) \in \Gamma$, then it is lower semi-Fredholm for all $(c_1, c_2) \in \Gamma$, and $\beta(c_1P_1 + c_2P_2)$ is constant on Γ.*

(3) *If $c_1P_1 + c_2P_2$ is Fredholm for some $(c_1, c_2) \in \Gamma$, then it is Fredholm for all $(c_1, c_2) \in \Gamma$, and $\alpha(c_1P_1 + c_2P_2), \beta(c_1P_1 + c_2P_2)$ and $i(c_1P_1 + c_2P_2)$ are constant on Γ.*

Corollary 2.15 ([114]). *Let P_1 and P_2 be two idempotents in $B(X)$. Then the invertibility of $c_1P_1 + c_2P_2$ is independent of the choice of $(c_1, c_2) \in \Gamma$.*

Corollary 2.16 ([114]). *Let p_1, p_2 be two idempotents in a Banach algebra A. Then the invertibility of $c_1p_1 + c_2p_2$ is independent of the choice of $(c_1, c_2) \in \Gamma$.*

3. *Measures of noncompactness*

The first measure of noncompactness, the function α, was defined and studied by Kuratowski [124] in 1930. It is surprising that later in 1955 Darbo [146] was the first who continued to use the function α. Darbo proved that if T is a continuous self-mapping of a nonempty, bounded, closed and convex subset C of a Banach space X such that

$$\alpha(T(Q)) \leq k\alpha(Q) \quad \text{for all } Q \subset C, \tag{3.1}$$

($k \in (0, 1)$ is a constant) then T has at least one fixed point in the set C. Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem and it includes the existence part of Banach's fixed point theorem.

Let (X, d) be a metric space and Q a bounded subset of X. Then the *Kuratowski measure of noncompactness* (*the set- measure of noncompactness, α–measure*) of Q, denoted by $\alpha(Q)$, is the infimum of the set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters $< \epsilon$, that is,

$$\alpha(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} S_i,\ S_i \subset X,\ \text{diam}(S_i) < \epsilon\,(i = 1, \ldots, n;\ n \in \mathbb{N})\right\}. \tag{3.2}$$

The function α is called the Kuratowski measure of noncompactness. Clearly

$$\alpha(Q) \leq \text{diam}(Q) \quad \text{for each bounded subset } Q \text{ of } X.$$

Usually it is complicated to find the exact value of $\alpha(Q)$. Another measure of noncompactness, which is more applicable in many cases, was introduced and studied by Goldenstein, Gohberg and Markus (the ball or Hausdorff measure of noncompactness) [67] in 1957 (later studied by Goldenstein and Markus [68] in 1965). It is given in the next definition.

Let (X, d) be a metric space and Q a bounded subset of X. Then the Hausdorff measure of noncompactness (the ball measure of noncompactness, χ–measure) of the set Q, denoted by $\chi(Q)$ is defined to be the infimum of the set of all reals $\epsilon > 0$ such that Q can be covered by a finite number of balls of radii $< \epsilon$, that is,

$$\chi(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in X,\ r_i < \epsilon\,(i = 1, \ldots, n)\, n \in \mathbb{N}\right\}. \tag{3.3}$$

The function χ is called Hausdorff measure of noncompactness.

We present some our results (with coauthors) on matrix transformations on certain sequence spaces using the Hausdorff measure of noncompactness.

Let μ_1 and μ_2 be measures of noncompactness on the Banach spaces X and Y, respectively. An operator $A : X \to Y$ is said to be (μ_1, μ_2)*–bounded* if

$$A(Q) \in \mathcal{M}_Y \quad \text{for each} \quad Q \in \mathcal{M}_X \tag{3.4}$$

and there exists a real k with $0 \leq k < \infty$ such that

$$\mu_2(AQ) \leq k\mu_1(Q) \quad \text{for each } Q \in \mathcal{M}_X. \tag{3.5}$$

If an operator A is (μ_1, μ_2)-*bounded* then the number $\|A\|_{\mu_1,\mu_2}$ is defined by

$$\|A\|_{\mu_1,\mu_2} = \inf\{k \geq 0 : \mu_2(AQ) \leq k\mu_1(Q) \quad \text{for each } Q \in \mathcal{M}_X\} \tag{3.6}$$

and called (μ_1, μ_2)*–operator norm of* A, or (μ_1, μ_2)*–measure of noncompactness of* A, or simply *measure of noncompactness of* A.

If $\mu_1 = \mu_2 = \mu$ then we write $\|A\|_\mu$ instead of $\|A\|_{\mu,\mu}$.

Let us mention that if $A \in B(X, Y)$, then

$$\|A\|_\chi = \chi(AS_X) = \chi(AB_X). \tag{3.7}$$

Let $\mathcal{A}$ be a unital C^*– algebra and Inv $(\mathcal{A})$ the set of invertible elements of $\mathcal{A}$. If $\mathcal{I}$ is a closed two-sided ideal in $\mathcal{A}$, let $x + \mathcal{I}$ denote the coset in the quotient algebra $\mathcal{A}/\mathcal{I}$ containing x. For $x \in \mathcal{A}$ denote by $r(x)$ $(r(x+\mathcal{I}))$ the spectral radius of the element x $(x + \mathcal{I})$.

We proved the next result.

Theorem 3.1 ([147]). *Let $\mathcal{A}$ be a unital C^*– algebra and $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$. Then*

$$r(x+\mathcal{I}) = \inf_{s \in Inv(\mathcal{A})} \|s^{-1}xs + \mathcal{I}y\|. \tag{3.8}$$

Now as a corollary we get the main result of Mau–Hsiang Shin [173]

Corollary 3.1 ([173]). *Let X be a Hilbert space. Then*

$$r_e(T) = \inf_{S \in Inv(B(X))} \|S^{-1}TS\|_\alpha. \tag{3.9}$$

3.1. Operators on the spaces $s_\alpha, s_\alpha^\circ, s_\alpha^{(c)}, \ell_\alpha^p$

A Banach space E of complex sequences $x = (x_n)_{n\geq 1}$ with the norm $\|\cdot\|_E$ is a *BK space* if each projection $x \mapsto x_n$ is continuous. A BK space E is said to have AK, if for every $b = (b_n)_{n\geq 1} \in E$, $b = \sum_{m=1}^\infty b_m e_m$ (with $e_n = (0, \ldots, 1, \ldots)$ and 1 being in the $n-$th position). We will write s for the set of all complex sequences, ℓ_∞, c, c_0 for the sets of bounded, convergent and null sequences, respectively. Recall that ℓ^p, for $1 \leq p < \infty$ is the set of sequences $x = (x_n)_{n\geq 1}$ such that $\sum_{n=1}^\infty |x_n|^p < \infty$. Put now $U = \{x = (x_n)_{n\geq 1} \in s \ : \ x_n \neq 0 \text{ for all } n\}$ and $U^+ = \{x = (x_n)_{n\geq 1} \in s \ : \ x_n > 0 \text{ for all } n\}$. For any given $\alpha = (\alpha_n)_{n\geq 1} \in U^+$ and $p \geq 1$ real we write $\ell_\alpha^p = (1/\alpha)^{-1} * \ell^p = \{x \in s \ : \ \sum_{n=1}^\infty (|x_n|/\alpha_n)^p < \infty\}$. Define the diagonal matrix $D_\xi = (\xi_n \delta_{nm})_{n,m\geq 1}$, (where $\delta_{nm} = 0$ for all $n \neq m$ and $\delta_{nm} = 1$ otherwise), we then have $D_\alpha \ell^p = \ell_\alpha^p$. In the same way we will define the sets $s_\alpha = (1/\alpha)^{-1} * l_\infty = \{x \in s \ : \ x_n/\alpha_n = O(1) \ (n \to \infty)\}, \overset{\circ}{s_\alpha} = \{x \in s \ : \ x_n/\alpha_n = o(1) \ \ (n \to \infty)\}$ and $s_\alpha^{(c)} = \{x \in s \ : \ x_n/\alpha_n \to x_{0\alpha} \quad (n \to \infty) \text{ for some } x_{0\alpha}\}$. If $\alpha = (r^n)_{n\geq 1}$, s_α, s_α^0, $s_\alpha^{(c)}$ and ℓ_α^p are defined by s_r, $\overset{0}{s_r}$, $s_r^{(c)}$ and ℓ_r^p, respectively. When $r = 1$, we obtain $s_1 = \ell_\infty$, $\overset{0}{s_1} = c_0$, $s_1^{(c)} = c$ and $\ell_1^p = \ell^p$. Let us remark that the spaces $s_\alpha, s_\alpha^\circ, s_\alpha^{(c)}$ and ℓ_α^p have been studied by de Malafosse [45, 46, 47] and by de Malafosse and Malkowsky [48], and they find their applications in a number of areas.

In [49], we characterize some operators and matrix transformations in the sequence spaces $s_\alpha, s_\alpha^\circ, s_\alpha^{(c)}, \ell_\alpha^p$. Moreover, using the Hausdorff measure of noncompactness necessary and sufficient conditions are formulated for a linear operator between the mentioned spaces to be compact. Among other things, some results of Cohen and Dunford [30] are recovered. We now present some results from [49].

Theorem 3.2 ([49]). *Each bounded linear operator A on $s_\alpha^{(c)}$ into s_β, $s_\beta^{(c)}$ or s_β^0 determines and is determined by a matrix of scalars a_{nm}, $n = 1, 2, \ldots$, $m = 0, 1, 2, \ldots$, $y = Ax$, is defined by the equations*

$$y_n = a_{n0} x_{0\alpha} + \sum_{m=1}^\infty a_{nm} x_m, \quad n = 1, 2, \ldots \ ,$$

where $x = (x_n)$ *in* $s_\alpha^{(c)}$, *and* $\lim_n x_n/\alpha_n = x_{0\alpha}$.

The norm of A *is defined by*

$$\|A\| = \sup_{n\geq 1} \frac{1}{\beta_n}\left(|a_{n0}| + \sum_{m=1}^{\infty} |a_{nm}|\alpha_m\right). \tag{3.10}$$

(i) *For* $A \in B(s_\alpha^{(c)}, s_\beta)$ *the only condition on the matrix* (a_{nm}) *is that the expression in* (3.10) *is finite.*

(ii) *For* $A \in B(s_\alpha^{(c)}, s_\beta^{(c)})$ *the additional condition is*

$$\lim_{n\to\infty} \frac{1}{\beta_n}\left(a_{n0} + \sum_{m=1}^{\infty} a_{nm}\alpha_m\right) = \omega \tag{3.11}$$

and

$$\lim_{n\to\infty} \frac{a_{nm}}{\beta_n}\alpha_m = \omega_m \tag{3.12}$$

if $m = 1, 2, \ldots$.

(iii) *Finally,* $A \in B(s_\alpha^{(c)}, s_\beta^0)$ *if and only if the expression in* (3.10) *is finite,*

$$\lim_{n\to\infty} \frac{1}{\beta_n}\left(a_{n0} + \sum_{m=1}^{\infty} a_{nm}\alpha_m\right) = 0$$

and

$$\lim_{n\to\infty} \frac{a_{nm}}{\beta_n}\alpha_m = 0 \tag{3.13}$$

exists if $m = 1, 2, \ldots$.

Theorem 3.3 ([49]). *Let* $\alpha = (\alpha_n)_{n\geq 1}$, $\beta = (\beta_n)_{n\geq 1} \in U^+$ *and let* $A = (a_{nm})_{n,m\geq 1}$ *be an infinite matrix. Then*
(i) $A \in (s_\alpha, s_\beta)$ *if and only if*

$$\|A\| = \sup_{n\geq 1}\left(\frac{1}{\beta_n}\sum_{m=1}^{\infty} |a_{nm}|\alpha_m\right) < \infty. \tag{3.14}$$

Furthermore, $(s_\alpha, s_\beta) = (s_\alpha^{(c)}, s_\beta) = (s_\alpha^0, s_\beta)$.
(ii) $A \in (s_\alpha^0, s_\beta^0)$ *if and only if* (3.13) *and* (3.14) *hold.*
(iii) $A \in (s_\alpha^0, s_\beta^{(c)})$ *if and only if (3.12) and (3.14) hold.*
(iv) $A \in (s_\alpha^{(c)}, s_\beta^{(c)})$ *if and only if* (3.12), (3.14) *and* (3.15) *hold*

$$\lim_{n\to\infty} \frac{1}{\beta_n}\sum_{m=1}^{\infty} a_{nm}\alpha_m = \psi. \tag{3.15}$$

(v) $A \in (s_\alpha^{(c)}, s_\beta^{0)})$ *if and only if* (3.13), (3.14) *and* $\psi = 0$ *in* (3.15) *hold.*

Theorem 3.4 ([49])**.** *Let* $\alpha = (\alpha_n)_{n\geq 1}$, $\beta = (\beta_n)_{n\geq 1} \in U^+$ *and let* $A = (a_{nm})_{n,m\geq 1}$ *be an infinite matrix.*
(i) *If* $A \in (\ell^1_\alpha, \ell^p_\beta)$, $1 \leq p < \infty$, *then*

$$\|A\|_\chi = \lim_{n\to\infty} \sup_{m\geq 1} \alpha_m \left(\sum_{k=n+1}^{\infty} \left(\frac{|a_{km}|}{\beta_k} \right)^p \right)^{1/p}.$$

(ii) *If* $A \in (\ell^1_\alpha, s^0_\beta)$, *then*

$$\|A\|_\chi = \limsup_{n\to\infty} \left(\sup_{m\geq 1} |a_{nm}| \frac{\alpha_m}{\beta_n} \right).$$

(iii) *If* $A \in (\ell^1_\alpha, s^{(c)}_\beta)$, *then*

$$\frac{1}{2} \limsup_{n\to\infty} \left(\sup_{m\geq 1} \left| \frac{a_{nm}\alpha_m}{\beta_n} - \omega_m \right| \right) \leq \|A\|_\chi \leq \limsup_{n\to\infty} \left(\sup_{m\geq 1} \left| \frac{a_{nm}\alpha_m}{\beta_n} - \omega_m \right| \right).$$

(iv) *If* $A \in (\ell^1_\alpha, s_\beta)$, *then*

$$0 \leq \|A\|_\chi \leq \limsup_{n\to\infty} \left(\sup_{m\geq 1} |a_{nm}| \frac{\alpha_m}{\beta_n} \right).$$

Theorem 3.5 ([49])**.** *Let* $\alpha = (\alpha_n)_{n\geq 1}$, $\beta = (\beta_n)_{n\geq 1} \in U^+$, $1 < p < \infty$, $1/p + 1/q = 1$ *and let* $A = (a_{nm})_{n,m\geq 1}$ *be an infinite matrix. Then*
(i) $A \in (\ell^p_\alpha, s_\beta)$, $1 < p < \infty$, *if and only if*

$$\|A\| = \sup_{n\geq 1} \frac{1}{\beta_n} \left(\sum_{m=1}^{\infty} |a_{nm}\alpha_m|^q \right)^{1/q} < \infty.$$

(ii) *If* $A \in (\ell^p_\alpha, s_\beta)$, $1 < p < \infty$, *then*

$$\frac{1}{2} \lim_{m\to\infty} \sup_{n\geq 1} \frac{1}{\beta_n} \left(\sum_{k=m}^{\infty} |a_{nk}\alpha_k|^q \right)^{1/q} \leq \|A\|_\chi \leq 2 \lim_{m\to\infty} \sup_{n\geq 1} \frac{1}{\beta_n} \left(\sum_{k=m}^{\infty} |a_{nk}\alpha_k|^q \right)^{1/q}.$$

(iii) *If* $A \in (\ell^p_\alpha, s_\beta)$, $1 < p < \infty$, *then* A *is compact if and only if*

$$\lim_{m\to\infty} \sup_{n\geq 1} \frac{1}{\beta_n} \left(\sum_{k=m}^{\infty} |a_{nk}\alpha_k|^q \right)^{1/q} = 0.$$

(iv) *If* $A \in (s_\alpha, s_\beta)$, *then* A *is compact if and only if*

$$\lim_{k\to\infty} \sup_{n\geq 1} \frac{1}{\beta_n} \sum_{m=k}^{\infty} |a_{nm}| \alpha_m = 0. \tag{3.16}$$

Theorem 3.6 ([49]). *Let* $A \in B(s_\alpha^{(c)}, s_\beta^{(c)})$, ω *be as in* (3.11) *and* ω_n, $n = 1, 2, \dots$, *be as in* (3.12). *Then*

$$\frac{1}{2} \limsup_{n\to\infty} \left(\left| \frac{a_{n0}}{\beta_n} - \omega + \sum_{m=1}^{\infty} \omega_m \right| + \sum_{m=1}^{\infty} \left| \frac{a_{nm}}{\beta_n} \alpha_m - \omega_m \right| \right) \le \|A\|_\chi$$
$$\le \limsup_{n\to\infty} \left(\left| \frac{a_{n0}}{\beta_n} - \omega + \sum_{m=1}^{\infty} \omega_m \right| + \sum_{m=1}^{\infty} \left| \frac{a_{nm}}{\beta_n} \alpha_m - \omega_m \right| \right).$$

Corollary 3.2 ([49]). *Let* $A \in B(s_\alpha^{(c)}, s_\beta^{(c)})$, $y = Ax$, *and* $y_{0\beta} = x_{0\alpha}$ *for every choice of* x. *Then* A *is compact if and only if* $\omega = 1$ *and* $\omega_1 = \omega_2 = \cdots = 0$ *and*

$$\lim_{n\to\infty} \left(\left| \frac{a_{n0}}{\beta_n} - 1 \right| + \frac{1}{\beta_n} \sum_{m=1}^{\infty} |a_{nm}| \, \alpha_m \right) = 0.$$

In the special case, when $\alpha_n = \beta_n = 1$, $n = 1, 2, \dots$, Corollary 3.2 implies the next well known result of Cohen and Dunford (Corollary 3 of [30]).

Corollary 3.3. *Let* $A \in B(c, c)$ *be regular transformation. Then* A *is compact if and only if*

$$\lim_{n\to\infty} \left(|a_{n0} - 1| + \sum_{m=1}^{\infty} |a_{nm}| \right) = 0.$$

Theorem 3.7 ([49]). *Let* $A \in B(s_\alpha^{(c)}, s_\beta^{0})$. *Then*

$$\|A\|_\chi = \limsup_{n\to\infty} \frac{1}{\beta_n} \left(|a_{n0}| + \sum_{m=1}^{\infty} |a_{nm}| \, \alpha_m \right)$$

and A *is compact if and only if*

$$\lim_{n\to\infty} \frac{1}{\beta_n} \left(|a_{n0}| + \sum_{m=1}^{\infty} |a_{nm}| \, \alpha_m \right) = 0.$$

Theorem 3.8 ([49]). *Let* $\alpha = (\alpha_n)_{n\ge1}$, $\beta = (\beta_n)_{n\ge1} \in U^+$ *and let* $A = (a_{nm})_{n,m\ge1}$ *be an infinite matrix.*

(i) *If* $A \in (s_\alpha, s_\beta) = (s_\alpha^{(c)}, s_\beta) = (s_\alpha^0, s_\beta)$, *then* A *is compact if and only if* (3.16) *is satisfied.*

(ii) *Let* $A \in (s_\alpha^{(c)}, s_\beta^{(c)})$, ψ *be as in* (3.15) *and* $\omega_n, n = 1, 2, \dots$, *be as in* (3.12). *Then*

$$\frac{1}{2} \limsup_{n\to\infty} \left(\left| \psi - \sum_{m=1}^{\infty} \omega_m \right| + \sum_{m=1}^{\infty} \left| \frac{a_{nm}}{\beta_n} \alpha_m - \omega_m \right| \right) \le \|A\|_\chi$$
$$\le \limsup_{n\to\infty} \left(\left| \psi - \sum_{m=1}^{\infty} \omega_m \right| + \sum_{m=1}^{\infty} \left| \frac{a_{nm}}{\beta_n} \alpha_m - \omega_m \right| \right),$$

and A is compact if and only if

$$\lim_{n\to\infty}\left(\left|\psi-\sum_{m=1}^{\infty}\omega_m\right|+\sum_{m=1}^{\infty}\left|\frac{a_{nm}}{\beta_n}\alpha_m-\omega_m\right|\right)=0.$$

(iii) *If $A\in(s_\alpha^{(c)},s_\beta^0)$ or $A\in(s_\alpha^0,s_\beta^0)$, then*

$$\|A\|_\chi=\limsup_{n\to\infty}\frac{1}{\beta_n}\sum_{m=1}^{\infty}|a_{nm}|\,\alpha_m,$$

and A is compact if and only if

$$\lim_{n\to\infty}\frac{1}{\beta_n}\sum_{m=1}^{\infty}|a_{nm}|\,\alpha_m=0.$$

(iv) *If $A\in(s_\alpha^0,s_\beta^{(c)})$, then*

$$\frac{1}{2}\limsup_{n\to\infty}\sum_{m=1}^{\infty}\left|\frac{a_{nm}}{\beta_n}\alpha_m-\omega_m\right|\le\|A\|_\chi\le\limsup_{n\to\infty}\sum_{m=1}^{\infty}\left|\frac{a_{nm}}{\beta_n}\alpha_m-\omega_m\right|,$$

and A is compact if and only if

$$\lim_{n\to\infty}\sum_{m=1}^{\infty}\left|\frac{a_{nm}}{\beta_n}\alpha_m-\omega_m\right|=0.$$

We give some examples to illustrate some of the applications of the results given in the preceding sections.

Let $\rho=(\rho_n)_{n\ge1},\xi=(\xi_n)_{n\ge1}\in U$ and consider the infinite matrix

$$M(\rho,\xi)=\begin{bmatrix} 1 & -\xi_1 & 0 & & & & \dots & 0\\ -\rho_1 & 1 & -\xi_2 & 0 & & & \dots & 0\\ 0 & -\rho_2 & 1 & -\xi_3 & 0 & & \dots & 0\\ \vdots & & & \ddots & & \ddots & & \\ 0 & & \dots & -\rho_n & 1 & -\xi_{n-1} & \dots & 0\\ \vdots & & & & \ddots & & \ddots & \vdots \end{bmatrix}$$

We will consider $M(\rho,\xi)$ as operator from s_τ^0 into itself. Then we have the following result.

Theorem 3.9 ([49]). *Let $\tau \in U^{+}$. If*

$$\sup_{n}\left(|\rho_{n-1}|\,\frac{\tau_{n-1}}{\tau_{n}}+|\xi_{n}|\,\frac{\tau_{n+1}}{\tau_{n}}\right)<\infty,$$

then $M(\rho,\xi)\in\left(s_{\tau}^{0},s_{\tau}^{0}\right)$,

$$\|M(\rho,\xi)\|=\sup_{n}\left(|\rho_{n-1}|\,\frac{\tau_{n-1}}{\tau_{n}}+1+|\xi_{n}|\,\frac{\tau_{n+1}}{\tau_{n}}\right),$$

$$\|M(\rho,\xi)\|_{\chi}=\limsup_{n\to\infty}\left(|\rho_{n-1}|\,\frac{\tau_{n-1}}{\tau_{n}}+1+|\xi_{n}|\,\frac{\tau_{n+1}}{\tau_{n}}\right),$$

$$\|I-M(\rho,\xi)\|=\sup_{n}\left(|\rho_{n-1}|\,\frac{\tau_{n-1}}{\tau_{n}}+|\xi_{n}|\,\frac{\tau_{n+1}}{\tau_{n}}\right),$$

$$\|I-M(\rho,\xi)\|_{\chi}=\limsup_{n\to\infty}\left(|\rho_{n-1}|\,\frac{\tau_{n-1}}{\tau_{n}}+|\xi_{n}|\,\frac{\tau_{n+1}}{\tau_{n}}\right).$$

Corollary 3.4. *Let $\tau \in U^{+}$. If*

$$\limsup_{n\to\infty}\left(|\rho_{n-1}|\,\frac{\tau_{n-1}}{\tau_{n}}+|\xi_{n}|\,\frac{\tau_{n+1}}{\tau_{n}}\right)<1,$$

then $M(\rho,\xi)\in\left(s_{\tau}^{0},s_{\tau}^{0}\right)$, $\|I-M(\rho,\xi)\|_{\chi}<1$, $M(\rho,\xi)$ is a Fredholm operator and $i(M(\rho,\xi))=0$.

Theorem 3.10 ([49]). *Let $\tau \in U^{+}$. If $\lim_{n\to\infty}\rho_{n-1}(\tau_{n-1}/\tau_{n})$ exists, then $\Delta_{\rho}=M(\rho,0)\in\left(s_{\tau}^{(c)},s_{\tau}^{(c)}\right)$,*

$$\|\Delta_{\rho}\|=\sup_{n}\left(|\rho_{n-1}|\,\frac{\tau_{n-1}}{\tau_{n}}+1\right),$$

$$\frac{1}{2}\cdot\left(\left|\lim_{n\to\infty}\rho_{n-1}\frac{\tau_{n-1}}{\tau_{n}}+1\right|+\lim_{n\to\infty}|\rho_{n-1}|\frac{\tau_{n-1}}{\tau_{n}}+1\right)$$
$$\leq\|\Delta_{\rho}\|_{\chi}\leq\left|\lim_{n\to\infty}\rho_{n-1}\frac{\tau_{n-1}}{\tau_{n}}+1\right|+\lim_{n\to\infty}|\rho_{n-1}|\frac{\tau_{n-1}}{\tau_{n}}+1,$$

$$\|I-\Delta_{\rho}\|=\sup_{n}\left(|\rho_{n-1}|\,\frac{\tau_{n-1}}{\tau_{n}}\right),$$

$$\lim_{n\to\infty}|\rho_{n-1}|\,\frac{\tau_{n-1}}{\tau_{n}}\leq\|I-\Delta_{\rho}\|_{\chi}\leq 2\cdot\lim_{n\to\infty}|\rho_{n-1}|\,\frac{\tau_{n-1}}{\tau_{n}}.$$

Now as a corollary we have the next result.

Corollary 3.5. *Let $\tau \in U^{+}$. If $\lim_{n}\rho_{n-1}(\tau_{n-1}/\tau_{n})$ exists, then $\Delta_{\rho}\in\left(s_{\tau}^{(c)},s_{\tau}^{(c)}\right)$; if in addition $\lim_{n}|\rho_{n-1}|(\tau_{n-1}/\tau_{n})<1/2$, then $\|I-\Delta_{\rho}\|_{\chi}<1$, Δ_{ρ} is a Fredholm operator and $i(\Delta_{\rho})=0$.*

Now we will consider the operator $\Sigma = (\sigma_{nm})_{n,m\geq 1}$ defined by $\sigma_{nm} = 1$ for $m \leq n$ and $\sigma_{nm} = 0$ otherwise, and put $C(\tau) = D_{1/\tau}\Sigma$. Let us remark that the operator $C(\tau)$ is a generalization of the Cesàro operator.

Theorem 3.11 ([49]). *Let $\tau \in U^+$. If*

$$\sup_{n\geq 1} \frac{1}{\tau_n} \sum_{k=1}^{n} \tau_k < \infty,$$

then $\Sigma \in (s_\tau^0, s_\tau^0)$,

$$\|\Sigma\| = \sup_{n\geq 1} \frac{1}{\tau_n} \sum_{k=1}^{n} \tau_k, \qquad \|\Sigma\|_\chi = \limsup_{n\to\infty} \frac{1}{\tau_n} \sum_{k=1}^{n} \tau_k,$$

$$\|I - \Sigma\| = \sup_{n\geq 1} \frac{1}{\tau_n} \sum_{k=1}^{n-1} \tau_k, \quad \|I - \Sigma\|_\chi = \limsup_{n\to\infty} \frac{1}{\tau_n} \sum_{k=1}^{n-1} \tau_k.$$

Corollary 3.6. *Let $\tau \in U^+$. If*

$$\limsup_{n\to\infty} \frac{1}{\tau_n} \sum_{k=1}^{n-1} \tau_k < 1,$$

then $\Sigma \in \left(s_\tau^0, s_\tau^0\right)$, $\|I - \Sigma\|_\chi < 1$, Σ is a Fredholm operator and $i(\Sigma) = 0$.

Theorem 3.12 ([49]). *Let $\tau \in U^+$. If*

$$\left(\frac{1}{\tau_n} \sum_{k=1}^{n} \tau_k\right)_n \in c,$$

then $\Sigma \in (s_\tau^{(c)}, s_\tau^{(c)})$,

$$\|\Sigma\| = \sup_{n\geq 1} \frac{1}{\tau_n} \sum_{k=1}^{n} \tau_k,$$

$$\lim_{n\to\infty} \frac{1}{\tau_n} \sum_{k=1}^{n} \tau_k \leq \|\Sigma\|_\chi \leq 2 \lim_{n\to\infty} \frac{1}{\tau_n} \sum_{k=1}^{n} \tau_k,$$

$$\|I - \Sigma\| = \sup_{n\geq 1} \frac{1}{\tau_n} \sum_{k=1}^{n-1} \tau_k,$$

$$\lim_{n\to\infty} \frac{1}{\tau_n} \sum_{k=1}^{n-1} \tau_k \leq \|I - \Sigma\|_\chi \leq 2 \lim_{n\to\infty} \frac{1}{\tau_n} \sum_{k=1}^{n-1} \tau_k.$$

Corollary 3.7. *Let* $\tau \in U^+$. *If*

$$\lim_{n\to\infty} \frac{1}{\tau_n} \sum_{k=1}^{n-1} \tau_k < \frac{1}{2},$$

then $\Sigma \in (s_\tau^{(c)}, s_\tau^{(c)})$, $\|I - \Sigma\|_\chi < 1$, Σ *is a Fredholm operator and* $i(\Sigma) = 0$.

4. *Fixed point theory*

Fixed point theory is a major branch of nonlinear functional analysis because of its wide applicability. Numerous questions in physics, chemistry, biology, and economics lead to various nonlinear differential and integral equations.

The classical Banach contraction principle [11] is one of the most useful results in metric fixed point theory. Due to its applications in mathematics and other related disciplines, this principle has been generalized in many directions. Extensions of the Banach contraction principle have been obtained either by generalizing the distance properties of underlying domain or by modifying the contractive condition on the mappings.

4.1. Extensions of Banach's theorem to partial metric space

Matthews [132] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, showing that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification.

A nonnegative mapping $p : X \times X \to \mathbb{R}$, where X is a nonempty set, is said to be a *partial metric on* X if for any $x, y, z \in X$ the following four conditions hold true:

(P1) $x = y$ if and only if $p(x,x) = p(y,y) = p(x,y)$;

(P2) $p(x,x) \leq p(x,y)$;

(P3) $p(x,y) = p(y,x)$;

(P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is then called a partial metric space. A sequence $\{x_m\}_{m=0}^\infty$ of elements of X is called *p-Cauchy* if the limit $\lim_{m,n} p(x_n, x_m)$ exists and is finite. The partial metric space (X,p) is called complete if for each p-Cauchy sequence $\{x_m\}_{m=0}^\infty$ there is some $z \in X$ such that

$$p(z,z) = \lim_n p(z,x_n) = \lim_{n,m} p(x_n,x_m). \tag{4.1}$$

It can be shown that if (X,p) is a partial metric space then by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, for $x, y \in X$, a metric p^s is defined on the set X such that $\{x_n\}_{n\geq 1}$ converges to $z \in X$ with respect to p^s if and only if (4.1) holds.

Set $r_p := \inf\{p(x,y) : x, y \in X\} = \inf\{p(x,x) : x \in X\}$ and $R_p := \{x \in X : p(x,x) = r_p\}$. Notice that R_p may be empty. Ĩf p is a metric then $r_p = 0$ and $R_p = X$.

In [82] we studied fixed point results of new extensions of Banach's contraction principle on partial metric space, and we give some generalized versions of the fixed point theorem of Matthews. The theory was illustrated by some examples.

Theorem 4.1 ([82]). *Let (X,p) be a complete partial metric space, $\alpha \in [0,1)$ and $T : X \to X$ be a given mapping. Suppose the following condition holds for each $x, y \in X$*

$$p(Tx,Ty) \leq \max\{\alpha p(x,y), p(x,x), p(y,y)\}. \tag{4.2}$$

Then the set R_p is nonempty. There is a unique $u \in R_p$ such that $Tu = u$. For each $x \in R_p$ the sequence $\{T^n x\}_{n\geq 1}$ converges with respect to the metric p^s to u.

If the condition (4.2) is replaced by the somewhat stronger condition bellow then the uniqueness of the fixed point is guaranteed.

Theorem 4.2 ([82]). *Let (X,p) be a complete partial metric space, $\alpha \in [0,1)$ and $T : X \to X$ be a given mapping. Suppose the following condition holds for each $x, y \in X$*

$$p(Tx,Ty) \leq \max\left\{\alpha p(x,y), \frac{p(x,x)+p(y,y)}{2}\right\}. \tag{4.3}$$

Then there is a unique $z \in X$ such that $Tz = z$. Furthermore $z \in R_p$ and for each $x \in R_p$ the sequence $\{T^n x\}_{n\geq 1}$ converges with respect to the metric p^s to z.

As a corollary we obtain the already mentioned result of Matthews.

Corollary 4.1 ([132]). *Let (X,p) be a complete partial metric space, $\alpha \in [0,1)$ and $T : X \to X$ be a given mapping. Suppose the following condition holds for each $x, y \in X$*

$$p(Tx,Ty) \leq \alpha p(x,y). \tag{4.4}$$

Then there is a unique $z \in X$ such that $Tz = z$. Furthermore $z \in R_p$ and for each $x \in R_p$ the sequence $\{T^n x\}_{n\geq 1}$ converges with respect to the metric p^s to z.

Example 4.1 ([82]). Let $X := [0,1] \cup [2,3]$ and define $p : X^2 \to \mathbb{R}$ by $p(x,y) = \max\{x,y\}$ if $\{x,y\} \cap [2,3] \neq \emptyset$ and $p(x,y) = |x-y|$ if $\{x,y\} \subseteq [0,1]$. Then (X,p) is a complete partial metric space.

Define $T : X \to X$ by $Tx = \frac{x+1}{2}$ if $x \in [0,1]$, $Tx = \frac{2+x}{2}$ if $x \in (2,3]$ and $T2 = 1$.

Then $p(Tx,Ty) \leq \frac{1}{2}p(x,y)$ holds whenever $\{x,y\} \subseteq [0,1]$ and $p(Tx,Ty) \leq \frac{p(x,x)+p(y,y)}{2}$ holds whenever $\{x,y\} \cap [2,3] \neq \emptyset$.

Given any $\gamma \in [0,1)$ observe that if $\gamma \leq \frac{1}{2}$ then (4.4) fails for any $x, y \in (2,3]$ and if $\gamma \in (\frac{1}{2}, 1)$ then (4.4) fails for any x, y such that $2 < y \leq x < \frac{2}{2\gamma - 1}$.

By Theorem 4.2 there is a unique fixed point $z = 1$ and we have $p(1,1) = 0 = r_p$.

Here $R_p = [2,3]$ and whether Picard sequences $\{T^n x\}_{n \geq 0}$ of points $x \in X \setminus R_p$ converge to the fixed point or not depends on the particular point x chosen: if $x \in (2,3]$ the answer is negative and if $x = 2$ the answer is positive.

4.2. Perov contractive condition in fixed point theory

Kurepa [119] in 1934, initiated the idea of more general concept of metric space that was later introduced by Zabreiko [183] as K-metric space and by Huang and Zhang [78] as cone metric space. There were published many results concerning fixed point theorems on both normal and non-normal cone metric spaces in the sense of Huang and Zhang.

In 1964, Perov [139] studied the Banach contraction principle on a generalized metric space. He replaced the contractive constant with a matrix with nonnegative entries and spectral radius less than 1, and obtained some fixed point theorems with various applications in coincidence problems, coupled fixed point problems and systems of semilinear differential inclusions. Let us remark that his generalized metric space is a special case of a normal cone metric space. We study fixed point results for the new extensions of Banach's contraction principle to cone metric spaces, and give some generalized versions of the fixed point theorem of Perov. As corollaries we generalized some results of Zima [184] and Borkowski, Bugajewski and Zima [21] for a Banach space with a non normal cone. The theory is illustrated with some examples.

Let E be a real Banach space. A subset P of E is called a cone if:

(i) P is closed, nonempty and $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define the partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$ (interior of P).

There exist two kinds of cones: normal and non-normal ones.

The cone P in a real Banach space E is called normal if

$$\inf\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0 \tag{4.5}$$

or, equivalently, if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\,\|y\|\,. \tag{4.6}$$

The least positive number satisfying (4.6) is called the normal constant of P. It is clear that $K \geq 1$.

Let X be a nonempty set, and let P be a cone on a real ordered Banach space E. Suppose that the mapping $d : X \times X \mapsto E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space [78].

Let $\{x_n\}$ be a sequence in X, and $x \in X$. If for every c in E with $0 \ll c$, there is n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$, then $\{x_n\}$ is said to converge to x, denoted by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$, $n \to \infty$. If for every c in E with $0 \ll c$, there is n_0 such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$, then $\{x_n\}$ is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space [78].

Theorem 4.3 ([35])**.** *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, $f : X \mapsto X$, $A \in B(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, such that*

$$d(f(x), f(y)) \leq Ad(x, y), \quad x, y \in X. \tag{4.7}$$

Then:

(i) *f has a unique fixed point $z \in X$;*

(ii) *For any $x_0 \in X$ the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$ converges to z and*

$$d(x_n, z) \leq A^n(I - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N};$$

(iii) *Suppose that $g : X \mapsto X$ satisfies the condition $d(f(x), g(x)) \leq c$ for all $x \in X$ and some $c \in P$. Then if $y_n = g^n(x_0), n \in \mathbb{N}$, we have*

$$d(y_n, z) \leq (I - A)^{-1}(c) + A^n(I - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

Theorem 4.4 ([35])**.** *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, and let T be a set-valued d–Perov contractive mapping (i.e. there exists $A \in B(E)$, such that $r(A) < 1$, $A(P) \subseteq P$ and for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$ there is $y_2 \in Tx_2$ with $d(y_1, y_2) \leq A(d(x_1, x_2))$) from X into itself such that for any $x \in X$, Tx is a nonempty closed subset of X. Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$, i.e., x_0 is a fixed point of T.*

Remark 4.1. Let us remark that the initial assumption $A \in \mathcal{M}_{n,n}(\mathbb{R}_+)$, in Perov theorem, is unnecessary. The last remark will be illustrated by the following example.

Example 4.2 ([35])**.** Let

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$X = \left\{ x = \begin{bmatrix} x_1 \\ 1 \\ x_3 \end{bmatrix} : x_i \in \mathbb{R}, i = 1,3 \right\} \text{ and } f : X \mapsto X,\, f\left(\begin{bmatrix} x_1 \\ 1 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} \frac{x_1+1}{2} \\ 1 \\ \frac{x_3+2}{3} \end{bmatrix}.$$

Let us define $d\ : X \times X \mapsto \mathbb{R}^3$ by

$$d(x,y) = \max\{|x_1 - y_1|, 0, |x_3 - y_3|\}, \quad x, y \in X,$$

and set $\|x\| = \max\{|x_1|, |x_2|, |x_3|\}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$.

For arbitrary $x \in \mathbb{R}^3$, we have

$$\begin{aligned} \|Ax\| &= \max\left\{ |\frac{1}{2}x_1 - \frac{1}{4}x_2|, |\frac{1}{4}x_1 - \frac{1}{2}x_2|, |\frac{1}{2}x_3| \right\} \\ &\leq \max\left\{ \frac{1}{2}\|x\| + \frac{1}{4}\|x\|, \frac{1}{4}\|x\| + \frac{1}{2}\|x\|, \frac{1}{2}\|x\| \right\} \\ &= \frac{3}{4}\|x\|. \end{aligned}$$

Thus, $\|A\| \leq \frac{3}{4}$. If $x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\|x\| = 1$, then $\|Ax\| = \frac{3}{4}$. Hence, $\|A\| = \frac{3}{4}$.

Now $r(A) \leq \|A\| = 3/4$ and

$$d(f(x), f(y)) \leq A(d(x,y)), \quad x, y \in X.$$

Clearly, $A(P) \nsubseteq P$, and $(1,1,1)$ is a unique fixed point of f in X.

Based on the previous comments, we obtain the next result, where we do not suppose that $A(P) \subset P$.

Theorem 4.5 ([35]). *Let (X,d) be a complete cone metric space, $d : X \times X \mapsto E$, P a normal cone with normal constant K, $A \in B(E)$ and $K \cdot \|A\| < 1$. If the condition (4.7) holds for a mapping $f : X \mapsto X$, then f has a unique fixed point $z \in X$ and the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$ converges to z for any $x_0 \in X$.*

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University of Niš
Faculty of Science and Mathematics
Višegradska 33
18000 Niš, Serbia
e-mail: vrakoc@sbb.rs