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DEGENERATE C-DISTRIBUTION SEMIGROUPS IN LOCALLY CONVEX SPACES

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A b s t r a c t. The main purpose of this paper is to investigate degenerate C-distribution semigroups in the setting of barreled sequentially complete locally convex spaces. In our approach, the infinitesimal generator of a degenerate C-distribution semigroup is a multivalued linear operator and the regularizing operator C is not necessarily injective. We provide a few important theoretical novelties, considering also exponential subclasses of degenerate C-distribution semigroups.

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1. *Introduction and Preliminaries*

In our recent paper [21], we have introduced and systematically analyzed the classes of C-distribution semigroups and C-ultradistribution semigroups in locally

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convex spaces (cf. [4]–[8], [11], [13], [17]–[19], [24]–[26], [30], [35]–[37] and references cited therein). The main aim of this paper is to continue this research by investigating the classes of degenerate C-distribution semigroups in the setting of barreled sequentially complete locally convex spaces (cf. [5], [12], [20], [30] and [36] for further information about well-posedness of abstract degenerate differential equations of first order). As mentioned in the abstract, we consider multivalued linear operators as infinitesimal generators of such semigroups and allow the regularizing operator C to be non-injective (cf. [3], [13], [24], [27] and [30]–[32] for the primary source of information on degenerate distribution semigroups in Banach spaces). In contrast to the analyses carried out in [30, Section 2.2] and [3, Section 3], we do not use any decomposition of the state space E.

The organization of paper can be briefly described as follows. After explaining the basic things about vector-valued generalized function spaces necessary for our further work, in Section 2 we take a preliminary look at multivalued linear operators in locally convex spaces. In Section 3, we repeat some known facts and definitions about fractionally integrated C-semigroups in locally convex spaces and their subgenerators (integral generators). Our main results are contained in Section 4, in which we analyze various themes concerning degenerate C-distribution semigroups in locally convex spaces and further generalize some of our recent results from [21]. The studies of differential and analytical properties of degenerate C-distribution semigroups as well as degenerate q-exponential C-distribution semigroups in locally convex spaces is out of the scope of this paper.

1.1. Notation

Unless specified otherwise, we assume that E is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short. Our standing assumption henceforth will be that the state space E is barreled. By $L(E)$ we denote the space consisting of all continuous linear mappings from E into E. The symbol $\circledast_E$ ($\circledast$, if there is no risk for confusion) denotes the fundamental system of seminorms which defines the topology of E. The Hausdorff locally convex topology on E^*, the dual space of E, defines the system $(|\cdot|_B)_{B\in\mathcal{B}}$ of seminorms on E^*, where $|x^*|_B := \sup_{x\in B}|\langle x^*, x\rangle|$, $x^* \in E^*$, $B \in \mathcal{B}$. The bidual of E is denoted by E^{**}. Recall, the polars of nonempty sets $M \subseteq E$ and $N \subseteq E^*$ are defined as follows $M^\circ := \{y \in E^* : |y(x)| \leq 1 \text{ for all } x \in M\}$ and $N^\circ := \{x \in E : |y(x)| \leq 1 \text{ for all } y \in N\}$.

Now we shall briefly described the main definitions and properties of vector-valued generalized function spaces used henceforth; cf. [2], [4], [11], [14]–[16], [18], [23], [25], [28]–[30], [33]–[34] and references cited therein for more details.

The Schwartz spaces of test functions $\mathcal{D} = C_0^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ and $\mathcal{E} = C^\infty(\mathbb{R})$ carry the usual topologies. If $\emptyset \neq \Omega \subseteq \mathbb{R}$, then the symbol $\mathcal{D}_\Omega$ denotes the subspace of $\mathcal{D}$ consisting of those functions $\varphi \in \mathcal{D}$ for which $\operatorname{supp}(\varphi) \subseteq \Omega$; $\mathcal{D}_0 \equiv \mathcal{D}_{[0,\infty)}$. The spaces $\mathcal{D}'(E) := L(\mathcal{D}, E)$, $\mathcal{E}'(E) := L(\mathcal{E}, E)$ and $\mathcal{S}'(E) := L(\mathcal{S}, E)$ are topologized in the usual way; the symbols $\mathcal{D}'_\Omega(E)$, $\mathcal{E}'_\Omega(E)$ and $\mathcal{S}'_\Omega(E)$ denote the subspaces of $\mathcal{D}'(E)$, $\mathcal{E}'(E)$ and $\mathcal{S}'(E)$, respectively, containing E-valued distributions whose supports are contained in Ω; $\mathcal{D}'_0(E) \equiv \mathcal{D}'_{[0,\infty)}(E)$, $\mathcal{E}'_0(E) \equiv \mathcal{E}'_{[0,\infty)}(E)$, $\mathcal{S}'_0(E) \equiv \mathcal{S}'_{[0,\infty)}(E)$. If $E = \mathbb{C}$, then the above spaces are also denoted by $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{S}'$, $\mathcal{D}'_\Omega$, $\mathcal{E}'_\Omega$, $\mathcal{S}'_\Omega$, $\mathcal{D}'_0$, $\mathcal{E}'_0$ and $\mathcal{S}'_0$. By a regularizing sequence in $\mathcal{D}$ we mean any sequence $(\rho_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_0$ for which there exists a function $\rho \in \mathcal{D}$ satisfying $\int_{-\infty}^{\infty} \rho(t)\,\mathrm{d}t = 1$, $\operatorname{supp}(\rho) \subseteq [0,1]$ and $\rho_n(t) = n\rho(nt)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$. If $\varphi, \psi : \mathbb{R} \to \mathbb{C}$ are locally integrable functions, then we define the convolution products $\varphi * \psi$ and $\varphi *_0 \psi$ by

$$\varphi * \psi(t) := \int_{-\infty}^{\infty} \varphi(t-s)\psi(s)\,\mathrm{d}s \text{ and } \varphi *_0 \psi(t) := \int_0^t \varphi(t-s)\psi(s)\,\mathrm{d}s,\ t \in \mathbb{R}.$$

Notice that $\varphi * \psi = \varphi *_0 \psi$, provided that $\operatorname{supp}(\varphi)$ and $\operatorname{supp}(\psi)$ are subsets of $[0,\infty)$. Given $\varphi \in \mathcal{D}$ and $f \in \mathcal{D}'$, or $\varphi \in \mathcal{E}$ and $f \in \mathcal{E}'$, we define the convolution $f * \varphi$ by $(f * \varphi)(t) := f(\varphi(t - \cdot))$, $t \in \mathbb{R}$. For $f \in \mathcal{D}'$, or for $f \in \mathcal{E}'$, define $\check{f}$ by $\check{f}(\varphi) := f(\varphi(-\cdot))$, $\varphi \in \mathcal{D}$ ($\varphi \in \mathcal{E}$). Generally, the convolution of two distribution f, $g \in \mathcal{D}'$, denoted by $f * g$, is defined by $(f * g)(\varphi) := g(\check{f} * \varphi)$, $\varphi \in \mathcal{D}$. If one of them belongs to $\mathcal{E}'(\mathbb{R})$, then we know that $f * g \in \mathcal{D}'$ and $\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g)$.

Let G be an E-valued distribution, and let $f : \mathbb{R} \to E$ be a locally integrable function. As in the scalar-valued case, we define the E-valued distributions $G^{(n)}$ ($n \in \mathbb{N}$) and hG ($h \in \mathcal{E}$); the regular E-valued distribution $\mathbf{f}$ is defined by $\mathbf{f}(\varphi) := \int_{-\infty}^{\infty} \varphi(t) f(t)\,\mathrm{d}t$ ($\varphi \in \mathcal{D}$). We need the following auxiliary lemma whose proof can be deduced as in the scalar-valued case.

Lemma 1.1. *Suppose that* $0 < \tau \leq \infty$, $n \in \mathbb{N}$. *If* $f : (0,\tau) \to E$ *is a continuous function and*

$$\int_0^\tau \varphi^{(n)}(t) f(t)\,\mathrm{d}t = 0, \quad \varphi \in \mathcal{D}_{(0,\tau)},$$

then there exist elements $x_0, \ldots, x_{n-1}$ *in* E *such that* $f(t) = \sum_{j=0}^{n-1} t^j x_j$, $t \in (0,\tau)$.

Following L. Schwartz [34], it will be said that a distribution $G \in \mathcal{D}'(X)$ is of finite order on the interval $(-\tau,\tau)$ iff there exist an integer $n \in \mathbb{N}_0$ and an X-valued continuous function $f : [-\tau,\tau] \to X$ such that

$$G(\varphi) = (-1)^n \int_{-\tau}^{\tau} \varphi^{(n)}(t) f(t)\,\mathrm{d}t, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)},\ \tau > 0.$$

G is of finite order iff G is of finite order on any finite interval $(-\tau, \tau)$. In the case that X is a quasi-complete (DF)-space, then it is well known that each X-valued distribution is of finite order. The same holds in the case that X is a Banach space.

We refer the reader to [21] for some characterizations of vector-valued distributions supported by a point. If the space E satisfies the property that any vector-valued distribution $G \in \mathcal{D}'(E)$ with $\text{supp}(G) \subseteq \{0\}$ can be represented as a finite sum of vector-valued distributions of form $\delta^{(i)} \otimes x_i$, then we say that E is admissible.

2. *Multivalued linear operators*

In this section, we present some definitions and properties of multivalued linear operators that will be necessary for our further work (cf. the monographs [9] by R. Cross and [12] by A. Favini-A. Yagi for more details on the subject). The underlying SCLCSs will be denoted by X and Y; in the third section, we will coming back to our standing notation.

A multivalued map (multimap) $\mathcal{A} : X \to P(Y)$ is said to be a multivalued linear operator (MLO) iff the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a subspace of X;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x + y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

If $X = Y$, then it is also said that $\mathcal{A}$ is an MLO in X. An almost immediate consequence of the definition is that, for every $x,\ y \in D(\mathcal{A})$ and for every $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear manifold in Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. The set $\mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$ and it is denoted by $N(\mathcal{A})$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It is easily seen that $\mathcal{A}^{-1}$ is an MLO in X, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective.

For any mapping $\mathcal{A} : X \to P(Y)$ we define $\check{\mathcal{A}} := \{(x, y) : x \in D(\mathcal{A}),\ y \in \mathcal{A}x\}$. Then $\mathcal{A}$ is an MLO iff $\check{\mathcal{A}}$ is a linear relation in $X \times Y$, i.e., iff $\check{\mathcal{A}}$ is a linear subspace of $X \times Y$. Since no confusion seems likely, we will sometimes identify $\mathcal{A}$ with its graph.

If $\mathcal{A},\ \mathcal{B} : X \to P(Y)$ are two MLOs, then we define its sum $\mathcal{A} + \mathcal{B}$ by $D(\mathcal{A} + \mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A} + \mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A} + \mathcal{B})$. It can be simply checked that $\mathcal{A} + \mathcal{B}$ is likewise an MLO.

Let $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : Y \to P(Z)$ be two MLOs, where Z is an SCLCS. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$

and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. Then $\mathcal{B}\mathcal{A} : X \to P(Z)$ is an MLO and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A} : X \to P(Y)$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$. It is clear that $z\mathcal{A} : X \to P(Y)$ is an MLO and $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, $z,\ \omega \in \mathbb{C}$.

The integer powers of an MLO $\mathcal{A} : X \to P(X)$ is defined recursively as follows: $\mathcal{A}^0 =: I$; if $\mathcal{A}^{n-1}$ is defined, set $D(\mathcal{A}^n) := \{x \in D(\mathcal{A}^{n-1}) : D(\mathcal{A}) \cap \mathcal{A}^{n-1}x \neq \emptyset\}$, and $\mathcal{A}^n x := (\mathcal{A}\mathcal{A}^{n-1})x = \bigcup_{y \in D(\mathcal{A}) \cap \mathcal{A}^{n-1}x} \mathcal{A}y, \quad x \in D(\mathcal{A}^n)$. It is well known that $(\mathcal{A}^n)^{-1} = (\mathcal{A}^{n-1})^{-1}\mathcal{A}^{-1} = (\mathcal{A}^{-1})^n =: \mathcal{A}^{-n}$, $n \in \mathbb{N}$ and $D((\lambda - \mathcal{A})^n) = D(\mathcal{A}^n)$, $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$. Moreover, if $\mathcal{A}$ is single-valued, then the above definitions are consistent with the usual definition of powers of $\mathcal{A}$.

If $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : X \to P(Y)$ are two MLOs, then we write $\mathcal{A} \subseteq \mathcal{B}$ iff $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$. Assume now that a linear single-valued operator $S : D(S) \subseteq X \to Y$ has domain $D(S) = D(\mathcal{A})$ and $S \subseteq \mathcal{A}$, where $\mathcal{A} : X \to P(Y)$ is an MLO. Then S is called a section of $\mathcal{A}$; if this is the case, we have $\mathcal{A}x = Sx + \mathcal{A}0$, $x \in D(\mathcal{A})$ and $R(\mathcal{A}) = R(S) + \mathcal{A}0$.

We say that an MLO operator $\mathcal{A} : X \to P(Y)$ is closed if for any nets (x_τ) in $D(\mathcal{A})$ and (y_τ) in Y such that $y_\tau \in \mathcal{A}x_\tau$ for all $\tau \in I$ we have that $\lim_{\tau\to\infty} x_\tau = x$ and $\lim_{\tau\to\infty} y_\tau = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

If $\mathcal{A} : X \to P(Y)$ is an MLO, then we define the adjoint $\mathcal{A}^* : Y^* \to P(X^*)$ of $\mathcal{A}$ by its graph

$$\mathcal{A}^* := \Big\{ (y^*, x^*) \in Y^* \times X^* : \langle y^*, y \rangle = \langle x^*, x \rangle \text{ for all pairs } (x, y) \in \mathcal{A} \Big\}.$$

It is simply verified that $\mathcal{A}^*$ is a closed MLO, and that $\langle y^*, y \rangle = 0$ whenever $y^* \in D(\mathcal{A}^*)$ and $y \in \mathcal{A}0$.

Concerning the integration of functions with values in SCLCSs, we follow the approach of C. Martinez and M. Sanz [28, pp. 99-102]. Denote by Ω a locally compact and separable metric space and by μ a locally finite Borel measure defined on Ω. Then the following fundamental lemma holds:

Lemma 2.1. *Suppose that $\mathcal{A} : X \to P(Y)$ is a closed* MLO. *Let $f : \Omega \to X$ and $g : \Omega \to Y$ be μ-integrable, and let $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$. Then $\int_\Omega f\,\mathrm{d}\mu \in D(\mathcal{A})$ and $\int_\Omega g\,\mathrm{d}\mu \in \mathcal{A}\int_\Omega f\,\mathrm{d}\mu$.*

In [20], we have recently considered the C-resolvent sets of MLOs in locally convex spaces (where $C \in L(X)$ is injective, $C\mathcal{A} \subseteq \mathcal{A}C$). The C-resolvent set of an MLO $\mathcal{A}$ in X, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which $R(C) \subseteq R(\lambda - \mathcal{A})$ and $(\lambda - \mathcal{A})^{-1}C$ is a single-valued continuous operator on X. The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$). In this paper, we analyze the general situation in which the operator $C \in$

$L(X)$ is not necessarily injective. Then the operator $(\lambda - \mathcal{A})^{-1}C$ is no longer single-valued, which additionally hinders our considerations and work.

3. *Fractionally integrated C-semigroups in locally convex spaces*

In this section, we will collect the most important facts and definitions about (degenerate) fractionally integrated C-semigroups in locally convex spaces. Observe that we do not require the injectiveness of operator $C \in L(E)$. Denote $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $t > 0$.

Definition 3.1 ([22])**.** Let $0 < \alpha < \infty$ and $0 < \tau \leq \infty$. A strongly continuous operator family $(S_\alpha(t))_{t\in[0,\tau)} \subseteq L(E)$ is called a (local, if $\tau < \infty$) α-times integrated C-semigroup iff the following holds:

(i) $S_\alpha(t)C = CS_\alpha(t)$, $t \in [0, \tau)$, and

(ii) For all $x \in E$ and $t,\ s \in [0, \tau)$ with $t + s \in [0, \tau)$, we have

$$S_\alpha(t)S_\alpha(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] g_\alpha(t + s - r)S_\alpha(r)Cx\,\mathrm{d}r.$$

By a C-regularized semigroup (0-times integrated C-regularized semigroup) we mean any strongly continuous operator family $(S_0(t) \equiv S(t))_{t\in[0,\tau)} \subseteq L(E)$ satisfying that $S(t)C = CS(t)$, $t \in [0, \tau)$ and $S(t+s)C = S(t)S(s)$ for all $t,\ s \in [0, \tau)$ with $t + s \in [0, \tau)$. A global C-regularized semigroup $(S(t))_{t\geq 0}$ is said to be entire analytic iff, for every $x \in E$, the mapping $t \mapsto S(t)x$, $t \geq 0$, can be analytically extended to the whole complex plane. We refer the reader to [10] for the most important applications of non-degenerate C-regularized semigroups.

Let $0 < \alpha \leq \infty$. In the case $\tau = \infty$, $(S_\alpha(t))_{t\geq 0}$ is said to be exponentially equicontinuous (equicontinuous) iff there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t}S_\alpha(t) : t \geq 0\}$ is equicontinuous. The integral generator $\hat{\mathcal{A}}$ of $(S_\alpha(t))_{t\in[0,\tau)}$ is defined by its graph

$$\hat{\mathcal{A}} := \left\{(x, y) \in E \times E : S_\alpha(t)x - g_{\alpha+1}(t)Cx = \int_0^t S_\alpha(s)y\,\mathrm{d}s,\ t \in [0, \tau)\right\}.$$

The integral generator $\hat{\mathcal{A}}$ of $(S_\alpha(t))_{t\in[0,\tau)}$ is a closed MLO in E. Furthermore, $\hat{\mathcal{A}} \subseteq C^{-1}\hat{\mathcal{A}}C$ in the MLO sense, with the equality in the case that the operator C is injective.

By a subgenerator of $(S_\alpha(t))_{t\in[0,\tau)}$ we mean any MLO $\mathcal{A}$ in E satisfying the following two conditions:

(A) $S_\alpha(t)x - g_{\alpha+1}(t)Cx = \int_0^t S_\alpha(s)y\,\mathrm{d}s,$ whenever $t \in [0,\tau)$ and $y \in \mathcal{A}x$.

(B) For all $x \in E$ and $t \in [0,\tau)$, we have $\int_0^t S_\alpha(s)x\,\mathrm{d}s \in D(\mathcal{A})$ and $S_\alpha(t)x - g_{\alpha+1}(t)Cx \in \mathcal{A}\int_0^t S_\alpha(s)x\,\mathrm{d}s$.

If $(S_\alpha^1(t))_{t\in[0,\tau)} \subseteq L(E)$, resp. $(S_\alpha^2(t))_{t\in[0,\tau)} \subseteq L(E)$, is strongly continuous and satisfies only (B), resp. (A), then we say that $(S_\alpha^1(t))_{t\in[0,\tau)}$, resp. $(S_\alpha^2(t))_{t\in[0,\tau)}$, is an α-times integrated C-existence family with a subgenerator $\mathcal{A}$, resp., α-times integrated C-uniqueness family with a subgenerator $\mathcal{A}$.

We denote by $\chi(S_\alpha)$ the set consisting of all subgenerators of the α-times integrated C-semigroup $(S_\alpha(t))_{t\in[0,\tau)}$. It is well known that $\chi(S_\alpha)$ can have infinitely many elements; if $\mathcal{A} \in \chi(S_\alpha)$, then $\mathcal{A} \subseteq \hat{\mathcal{A}}$. In general, the set $\chi(S_\alpha)$ can be empty and the integral generator of $(S_\alpha(t))_{t\in[0,\tau)}$ need not be a subgenerator of $(S_\alpha(t))_{t\in[0,\tau)}$ in the case that $\tau < \infty$. In global case, the integral generator $\hat{\mathcal{A}}$ of $(S_\alpha(t))_{t\geq 0}$ is always its subgenerator. If $\mathcal{A}$ is a closed subgenerator of $(S_\alpha(t))_{t\in[0,\tau)}$, defined locally or globally, then we know that $C\mathcal{A} \subseteq \mathcal{A}C$, $\hat{\mathcal{A}} \subseteq C^{-1}\mathcal{A}C$ and that the injectivity of C implies $\hat{\mathcal{A}} = C^{-1}\mathcal{A}C$. Suppose that C is injective and $\mathcal{A}$ is an MLO. Then there exists at most one α-times integrated C-semigroup $(S_\alpha(t))_{t\in[0,\tau)}$ which do have $\mathcal{A}$ as a subgenerator ([22]).

4. *The basic properties of degenerate C-distribution semigroups in locally convex spaces*

Throughout this section, we assume that $C \in L(E)$ is not necessarily injective operator. Since E is barreled, the uniform boundedness principle [29, p. 273] implies that each $\mathcal{G} \in \mathcal{D}'(L(E))$ is boundedly equicontinuous, i.e., that for every $p \in \circledast$ and for every bounded subset B of $\mathcal{D}$, there exist $c > 0$ and $q \in \circledast$ such that $p(\mathcal{G}(\varphi)x) \leq cq(x),\ \varphi \in B,\ x \in E$.

We start this section by introducing the following definition.

Definition 4.1. Let $\mathcal{G} \in \mathcal{D}_0'(L(E))$ satisfy $C\mathcal{G} = \mathcal{G}C$. Then it is said that $\mathcal{G}$ is a pre-(C-DS) iff the following holds:

$$\mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi), \quad \varphi,\ \psi \in \mathcal{D}. \tag{C.S.1}$$

If, additionally,

$$\mathcal{N}(\mathcal{G}) := \bigcap_{\varphi\in\mathcal{D}_0} N(\mathcal{G}(\varphi)) = \{0\}, \tag{C.S.2}$$

then $\mathcal{G}$ is called a C-distribution semigroup, (C-DS) in short. A pre-(C-DS) $\mathcal{G}$ is called dense iff

$$\mathcal{R}(\mathcal{G}) := \bigcup_{\varphi \in \mathcal{D}_0} R(\mathcal{G}(\varphi)) \text{ is dense in } E. \tag{C.S.3}$$

If $C = I$, then we also write pre-(DS),(DS), instead of pre-(C-DS), (C-DS).

Suppose that $\mathcal{G}$ is a pre-(C-DS). Then $\mathcal{G}(\varphi)\mathcal{G}(\psi) = \mathcal{G}(\psi)\mathcal{G}(\varphi)$ for all $\varphi,\ \psi \in \mathcal{D}$, and $\mathcal{N}(\mathcal{G})$ is a closed subspace of E.

The structural characterization of a pre-(C-DS) $\mathcal{G}$ on its kernel space $\mathcal{N}(\mathcal{G})$ is described in the following theorem (cf. [18, Proposition 3.1.1] and the proofs of [24, Lemma 2.2], [18, Proposition 3.5.4]).

Theorem 4.1. *Let $\mathcal{G}$ be a pre-(C-DS), and let the space $L(\mathcal{N}(\mathcal{G}))$ be admissible. Then, with $N = \mathcal{N}(\mathcal{G})$ and G_1 being the restriction of $\mathcal{G}$ to N ($G_1 = \mathcal{G}_{|N}$), we have: There exists an integer $m \in \mathbb{N}$ for which there exist unique operators T_0, $T_1, \ldots, T_m \in L(\mathcal{N}(\mathcal{G}))$ commuting with C so that $G_1 = \sum_{j=0}^{m} \delta^{(j)} \otimes T_j$, $T_iC^i = (-1)^i T_0^{i+1}$, $0 \leq i \leq m-1$ and $T_0T_m = T_0^{m+2} = 0$.*

Let $\mathcal{G} \in \mathcal{D}_0'(L(E))$ and let $T \in \mathcal{E}_0'$ i.e., T is a scalar-valued distribution with compact support contained in $[0,\infty)$. Define

$$G(T) := \Big\{(x,y) \in E \times E : \mathcal{G}(T * \varphi)x = \mathcal{G}(\varphi)y \ \text{ for all } \ \varphi \in \mathcal{D}_0\Big\}.$$

Then it can be easily seen that $G(T)$ is a closed MLO; furthermore, if $\mathcal{G} \in \mathcal{D}_0'(L(E))$ satisfy (C.S.2), then $G(T)$ is a closed linear operator. Assuming that the regularizing operator C is injective, definition of $G(T)$ can be equivalently introduced by replacing the set $\mathcal{D}_0$ with the set $\mathcal{D}_{[0,\epsilon)}$ for any $\epsilon > 0$. In general case, for every $\psi \in \mathcal{D}$, we have $\psi_+ := \psi \mathbf{1}_{[0,\infty)} \in \mathcal{E}_0'$, where $\mathbf{1}_{[0,\infty)}$ stands for the characteristic function of $[0,\infty)$, so that the definition of $G(\psi_+)$ is clear. We define the (infinitesimal) generator of a pre-(C-DS) $\mathcal{G}$ by $\mathcal{A} := G(-\delta')$ (cf. [21] for more details about non-degenerate case, and [3, Definition 3.4] and [13] for some other approaches used in degenerate case). Then $\mathcal{N}(\mathcal{G}) \times \mathcal{N}(\mathcal{G}) \subseteq \mathcal{A}$ and $\mathcal{N}(\mathcal{G}) = \mathcal{A}0$, which simply implies that $\mathcal{A}$ is single-valued iff (C.S.2) holds. If this is the case, then we also have that the operator C must be injective: Suppose that $Cx = 0$ for some $x \in E$. By (C.S.1), we get that $\mathcal{G}(\varphi)\mathcal{G}(\psi)x = 0$, $\varphi,\ \psi \in \mathcal{D}$. In particular, $\mathcal{G}(\psi)x \in \mathcal{N}(\mathcal{G}) = \{0\}$ so that $\mathcal{G}(\psi)x = 0$, $\psi \in \mathcal{D}$. Hence, $x \in \mathcal{N}(\mathcal{G}) = \{0\}$ and therefore $x = 0$.

Further on, if $\mathcal{G}$ is a pre-(C-DS), $T \in \mathcal{E}_0'$ and $\varphi \in \mathcal{D}$, then $\mathcal{G}(\varphi)G(T) \subseteq G(T)\mathcal{G}(\varphi)$, $CG(T) \subseteq G(T)C$ and $\mathcal{R}(\mathcal{G}) \subseteq D(G(T))$. If $\mathcal{G}$ is a pre-(C-DS) and $\varphi, \psi \in \mathcal{D}$, then the assumption $\varphi(t) = \psi(t)$, $t \geq 0$, implies $\mathcal{G}(\varphi) = \mathcal{G}(\psi)$. As in the Banach space case, we can prove the following (cf. [18, Proposition 3.1.3, Lemma 3.1.6]): Suppose that $\mathcal{G}$ is a pre-(C-DS). Then $(Cx, \mathcal{G}(\psi)x) \in G(\psi_+)$, $\psi \in \mathcal{D}$, $x \in E$ and $\mathcal{A} \subseteq C^{-1}\mathcal{A}C$, while $C^{-1}\mathcal{A}C = \mathcal{A}$ provided that C is injective. Furthermore, the following holds:

Proposition 4.1. *Let $\mathcal{G}$ be a pre-(C-DS), $S,\ T \in \mathcal{E}_0'$, $\varphi \in \mathcal{D}_0$, $\psi \in \mathcal{D}$ and $x \in E$. Then we have:*

(i) $(\mathcal{G}(\varphi)x,\ \mathcal{G}(\overbrace{T * \cdots * T}^{m} * \varphi)x) \in G(T)^m$, $m \in \mathbb{N}$.

(ii) $G(S)G(T) \subseteq G(S * T)$ *with* $D(G(S)G(T)) = D(G(S * T)) \cap D(G(T))$, *and* $G(S) + G(T) \subseteq G(S + T)$.

(iii) $(\mathcal{G}(\psi)x,\ \mathcal{G}(-\psi')x - \psi(0)Cx) \in G(-\delta')$.

(iv) *If $\mathcal{G}$ is dense, then its generator is densely defined.*

The assertions (ii)–(vi) of [18, Proposition 3.1.2] can be reformulated for pre-(C-DS)'s in locally convex spaces; here it is only worth noting that the reflexivity of state space E implies that the spaces E^* and $E^{**} = E$ are both barreled and sequentially complete:

Proposition 4.2. *Let $\mathcal{G}$ be a pre-(C-DS). Then the following holds:*

(i) $C(\overline{\langle \mathcal{R}(\mathcal{G}) \rangle}) \subseteq \overline{\mathcal{R}(\mathcal{G})}$, *where $\langle \mathcal{R}(\mathcal{G}) \rangle$ denotes the linear span of $\mathcal{R}(\mathcal{G})$.*

(ii) *Assume $\mathcal{G}$ is not dense and $\overline{C\mathcal{R}(\mathcal{G})} = \overline{\mathcal{R}(\mathcal{G})}$. Put $R := \overline{\mathcal{R}(\mathcal{G})}$ and $H := \mathcal{G}_{|R}$. Then H is a dense pre-(C_1-DS) on R with $C_1 = C_{|R}$.*

(iii) *The dual $\mathcal{G}(\cdot)^*$ is a pre-(C^*-DS) on E^* and $\mathcal{N}(\mathcal{G}^*) = \overline{\mathcal{R}(\mathcal{G})}^{\circ}$.*

(iv) *If E is reflexive, then $\mathcal{N}(\mathcal{G}) = \overline{\mathcal{R}(\mathcal{G}^*)}^{\circ}$.*

(v) *The $\mathcal{G}^*$ is a (C^*-DS) in E^* iff $\mathcal{G}$ is a dense pre-(C-DS). If E is reflexive, then $\mathcal{G}^*$ is a dense pre-(C^*-DS) in E^* iff $\mathcal{G}$ is a (C-DS).*

The following proposition has been recently proved in [21] in the case that the operator C is injective (cf. [13, Proposition 2] for a pioneering result in this direction). The argumentation contained in [21] shows that the injectivity of C is superfluous:

Proposition 4.3. *Suppose that $\mathcal{G} \in \mathcal{D}_0'(L(E))$ and $\mathcal{G}(\varphi)C = C\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}$. Then $\mathcal{G}$ is a pre-(C-DS) iff*

$$\mathcal{G}(\varphi')\mathcal{G}(\psi) - \mathcal{G}(\varphi)\mathcal{G}(\psi') = \psi(0)\mathcal{G}(\varphi)C - \varphi(0)\mathcal{G}(\psi)C, \quad \varphi,\ \psi \in \mathcal{D}.$$

In [21], we have proved that every (C-DS) in locally convex space is uniquely determined by its generator. Contrary to the single-valued case, different pre-(C-DS)'s can have the same generator. To see this, we can employ [24, Example 2.3]: Let $C = I$, E is a Banach space and $T \in L(E)$ is nilpotent of order $n \geq 2$. Then the pre-(C-DS)'s $\mathcal{G}_1(\cdot) \equiv \sum_{i=0}^{n-2} \cdot^{(i)}(0)T^{i+1}$ and $\mathcal{G}_2(\cdot) \equiv 0$ have the same generator $\mathcal{A} \equiv E \times E$.

In Theorem 4.2 and Theorem 4.3, we clarify connections between degenerate C-distribution semigroups and degenerate local integrated C-semigroups. For the proof of first theorem, we need some preliminaries from our previous research study of distribution cosine functions (see e.g. [18, Section 3.4]): Let $\eta \in \mathcal{D}_{[-2,-1]}$ be a fixed test function satisfying $\int_{-\infty}^{\infty} \eta(t)\,\mathrm{d}t = 1$. Then, for every fixed $\varphi \in \mathcal{D}$, we define $I(\varphi)$ as follows

$$\mathrm{d}I(\varphi)(x) := \int\limits_{-\infty}^{x} \Bigg[\varphi(t) - \eta(t) \int\limits_{-\infty}^{\infty} \varphi(u)\,\mathrm{d}u\Bigg]\,\mathrm{d}t, \ \ x \in \mathbb{R}.$$

It can be simply verified that, for every $\varphi \in \mathcal{D}$ and $n \in \mathbb{N}$, we have $I(\varphi) \in \mathcal{D}$, $I^n(\varphi^{(n)}) = \varphi$, $\frac{d}{dx}I(\varphi)(x) = \varphi(x) - \eta(x)\int_{-\infty}^{\infty} \varphi(u)\,\mathrm{d}u$, $x \in \mathbb{R}$, as well as that, for every $\varphi \in \mathcal{D}_{[a,b]}$ $(-\infty < a < b < \infty)$, we have:

$$\text{supp}(I(\varphi)) \subseteq [\min(-2, a), \max(-1, b)].$$

This simply implies that, for every $\tau > 2$, $-1 < b < \tau$ and for every $m,\ n \in \mathbb{N}$ with $m \leq n$, we have:

$$I^n\big(\mathcal{D}_{(-\tau,b]}\big) \subseteq \mathcal{D}_{(-\tau,b]} \text{ and } \frac{\mathrm{d}^m}{\mathrm{d}x^m} I^n(\varphi)(x) = I^{m-n}\varphi(x), \quad \varphi \in \mathcal{D},\ x \geq 0, \quad (4.1)$$

where $I^0\varphi := \varphi$, $\varphi \in \mathcal{D}$.

Now we are ready to show the following extension of [24, Proposition 4.3 a)] (E is a Banach space, $C = I$), given here with a different proof.

Theorem 4.2. *Let $\mathcal{G}$ be a pre-(C-DS) generated by $\mathcal{A}$, and let $\mathcal{G}$ be of finite order. Then, for every $\tau > 0$, there exist a number $n_\tau \in \mathbb{N}$ and a local n_τ-times integrated C-semigroup $(S_{n_\tau}(t))_{t\in[0,\tau)}$ such that*

$$\mathcal{G}(\varphi)x = (-1)^{n_\tau} \int_0^\infty \varphi^{(n_\tau)}(s) S_{n_\tau}(s)x\,\mathrm{d}t, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)},\ x \in E. \qquad (4.2)$$

Furthermore, $(S_{n_\tau}(t))_{t\in[0,\tau)}$ is an n_τ-times integrated C-existence family with a subgenerator $\mathcal{A}$, and the admissibility of space $L(\mathcal{N}(\mathcal{G}))$ implies that $S_{n_\tau}(t)x = 0$, $t \in [0, \tau)$ for some $x \in \mathcal{N}(\mathcal{G})$ iff $T_i x = 0$ for $0 \leq i \leq n_\tau - 1$; see Theorem 4.1(i) *with $m \geq n_\tau - 1$.*

PROOF. Let $\tau > 2$ and $\rho \in \mathcal{D}_{[0,1]}$ with $\int \rho\,\mathrm{d}m = 1$ be fixed. Set $\rho_n(\cdot) := n\rho(n\cdot)$, $n \in \mathbb{N}$. Then, for every $t \in [0, \tau)$, the sequence $\rho_n^t(\cdot) := \rho_n(\cdot - t)$ converges to δ_t as $n \to +\infty$ (in the space of scalar-valued distributions). Since $\mathcal{G} \in \mathcal{D}_0'(L(E))$ and $\mathcal{G}$ is of finite order, we know that there exist a number $n_\tau \in \mathbb{N}$ and a strongly continuous operator family $(S_{n_\tau}(t))_{t\in[0,\tau)} \subseteq L(E)$ such that (4.2) holds good. We will first prove that $(S_{n_\tau}(t))_{t\in[0,\tau)}$ is a local n_τ-times integrated C-existence family commuting with C and having $\mathcal{A}$ as a subgenerator. In order to do that, observe that the commutation of $\mathcal{G}(\cdot)$ and C yields

$$\mathrm{d}\int_0^\infty \varphi^{(n_\tau)}(s)CS_{n_\tau}(s)x\,\mathrm{d}t = \int_0^\infty \varphi^{(n_\tau)}(s)S_{n_\tau}(s)Cx\,\mathrm{d}t, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)},\ x \in E.$$

Plugging $\varphi = I^{n_\tau}(\rho_n^t)$ in this expression (cf. also (4.1)), we get that

$$\int_0^\infty \rho_n^t(s)CS_{n_\tau}(s)x\,\mathrm{d}t = \int_0^\infty \rho_n^t(s)S_{n_\tau}(s)Cx\,\mathrm{d}t, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)},\ x \in E,\ t \in [0,\tau).$$

Letting $n \to +\infty$ we obtain $CS_{n_\tau}(t)x = S_{n_\tau}(t)Cx$, $x \in E$, $t \in [0,\tau)$. Now we will prove that the condition (B) hold with the number α replaced with the number n_τ therein. By Proposition 4.1(iii), we have $(\mathcal{G}(\varphi)x, \mathcal{G}(-\varphi')x - \varphi(0)Cx) \in \mathcal{A}$, $\varphi \in \mathcal{D}$, $x \in E$. Applying integration by parts and multiplying with $(-1)^{n_\tau+1}$ after that, the above implies

$$\left(\int_0^\infty \varphi^{(n_\tau+1)}(s)\int_0^s S_{n_\tau}(r)x\,\mathrm{d}r\,\mathrm{d}s, \int_0^\infty \varphi^{(n_\tau+1)}(s)S_{n_\tau}(s)x\,\mathrm{d}s + (-1)^{n_\tau}\varphi(0)Cx\right) \in \mathcal{A},$$

for any $\varphi \in \mathcal{D}_{(-\tau,\tau)}$ and $x \in E$. Plugging $\varphi = I^{n_\tau+1}(\rho_n^t)$ in this expression, we get that

$$\left(\int_0^\infty \rho_n^t(s)\int_0^s S_{n_\tau}(r)x\,\mathrm{d}r\,\mathrm{d}s, \int_0^\infty \rho_n^t(s)S_{n_\tau}(s)x\,\mathrm{d}s + (-1)^{n_\tau}I^{n_\tau+1}(\rho_n^t)(0)Cx\right) \in \mathcal{A}, \tag{4.3}$$

for any $t \in [0,\tau)$ and $x \in E$. Let us prove that

$$\lim_{n\to+\infty} I^{n_\tau+1}(\rho_n^t)(x) = (-1)^{n_\tau+1}g_{n_\tau+1}(t-x), \quad t \in [0,\tau),\ 0 \le x \le t. \tag{4.4}$$

Let $t \in [0,\tau)$ and $x \in [0,t]$ be fixed. Then a straightforward integral computation shows that

$$I^{n_\tau+1}(\varphi)(x) = (-1)^{n_\tau+1}\int_x^\infty \int_{x_{n_\tau}}^\infty \int_{x_{n_\tau-1}}^\infty \ldots \int_{x_2}^\infty \varphi(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2\ldots\mathrm{d}x_{n_\tau+1}$$

for any $\varphi \in \mathcal{D}$. For $\varphi = I^{n_\tau+1}(\rho_n^t)$, we have

$$\begin{aligned} I^{n_\tau+1}\big(\rho_n^t\big)(0) &= (-1)^{n_\tau+1} \int_x^{t+(1/n)} \int_{x_{n_\tau}}^{t+(1/n)} \int_{x_{n_\tau-1}}^{t+(1/n)} \cdots \int_{x_2}^{t+(1/n)} \\ &\quad \times \rho_n^t(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_{n_\tau+1} \\ &= (-1)^{n_\tau+1} \int_x^{t+(1/n)} \int_{x_{n_\tau}}^{t+(1/n)} \int_{x_{n_\tau-1}}^{t+(1/n)} \cdots \int_{x_3}^{t+(1/n)} \\ &\quad \times \left[1 - \int_0^{nx_2-nt} \rho(x_1)\,\mathrm{d}x_1\right] \mathrm{d}x_2 \ldots \mathrm{d}x_{n_\tau+1} \\ &= (-1)^{n_\tau+1} \int_x^{t+(1/n)} \int_{x_{n_\tau}}^{t+(1/n)} \int_{x_{n_\tau-1}}^{t+(1/n)} \cdots \int_{x_3}^{t+(1/n)} \\ &\quad \times \mathrm{d}x_2 \cdots \mathrm{d}x_{n_\tau+1} \\ &\quad - (-1)^{n_\tau+1} \int_x^{t+(1/n)} \int_{x_{n_\tau}}^{t+(1/n)} \int_{x_{n_\tau-1}}^{t+(1/n)} \cdots \int_t^{t+(1/n)} \\ &\quad \times \int_0^{nx_2-nt} \rho(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_{n_\tau+1} \\ &:= (-1)^{n_\tau+1}\big[I_1(t,x) - I_2(t,x)\big], \quad t \in [0,\tau). \end{aligned}$$

Since

$$\int_t^{t+(1/n)} \int_0^{nx_2-nt} \rho(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \leq 1/n, \quad t \in [0,\tau),\ n \in \mathbb{N},$$

we have that $\lim_{n\to+\infty} I_2(t,x) = 0$, $t \in [0,\tau)$. Clearly,

$$\lim_{n\to+\infty} I_1(t,x) = \int_x^t \int_{x_{n_\tau}}^t \int_{x_{n_\tau-1}}^t \cdots \int_{x_3}^t \mathrm{d}x_2 \cdots \mathrm{d}x_{n_\tau+1} = g_{n_\tau+1}(t-x).$$

This gives (4.4). Keeping in mind this equality and letting $n \to +\infty$ in (4.3), we obtain (B). It remains to be proved the semigroup property of $(S_{n_\tau}(t))_{t\in[0,\tau)}$. Toward this end, let us recall that

$$\big(\varphi *_0 \psi\big)^{(n_\tau)}(u) = \big(\varphi^{(n_\tau)} *_0 \psi\big)(u) + \sum_{j=0}^{n_\tau-1} \varphi^{(j)}(0)\psi^{(n_\tau-1-j)}(u), \quad \varphi,\ \psi \in \mathcal{D},\ u \in \mathbb{R}. \tag{4.5}$$

Fix $x \in E$ and $t,\ s \in [0,\tau)$ with $t+s \in [0,\tau)$. Using (4.5), (C.S.1) and the foregoing arguments, we get that, for every $m,\ n \in \mathbb{N}$ sufficiently large:

$$\int_0^t \int_0^s \rho_n^t(u)\rho_m^s(v)S_{n_\tau}(u)S_{n_\tau}(v)x\,\mathrm{d}u\,\mathrm{d}v$$
$$= (-1)^{n_\tau} \int_0^{t+s} \Bigg[\Big(\rho_n^t *_0 I^{n_\tau}(\rho_m^s) \Big)(u)$$
$$+ \sum_{j=0}^{n_\tau - 1} I^{n_\tau - j}(\rho_n^t)(0) I^{j+1}(\rho_m^s)(u) \Bigg] S_{n_\tau}(u)Cx\,\mathrm{d}u.$$

Letting $n \to +\infty$, we obtain with the help of (4.4) that

$$\int_0^s \rho_m^s(v)S_{n_\tau}(t)S_{n_\tau}(v)x\,\mathrm{d}v = (-1)^{n_\tau} \lim_{n\to+\infty} \int_0^{t+s} \Bigg[\Big(\rho_n^t *_0 I^{n_\tau}(\rho_m^s) \Big)(u)$$
$$+ \sum_{j=0}^{n_\tau - 1} I^{n_\tau - j}(\rho_n^t)(0) I^{j+1}(\rho_m^s)(u) \Bigg] S_{n_\tau}(u)Cx\,\mathrm{d}u$$
$$= (-1)^{n_\tau} \int_0^t \Bigg[\sum_{j=0}^{n_\tau - 1} (-1)^{n_\tau - j} g_{n_\tau - j}(t) I^{j+1}(\rho_m^s)(u) \Bigg] S_{n_\tau}(u)Cx\,\mathrm{d}u$$
$$+ (-1)^{n_\tau} \int_t^{t+s} \Bigg[I^{n_\tau}(\rho_m^s)(u-t) + \sum_{j=0}^{n_\tau - 1} (-1)^{n_\tau - j} g_{n_\tau - j}(t) I^{j+1}(\rho_m^s)(u) \Bigg]$$
$$\times S_{n_\tau}(u)Cx\,\mathrm{d}u$$
$$= \sum_{j=0}^{n_\tau - 1} (-1)^j g_{n_\tau - j}(t) \int_0^s I^{j+1}(\rho_m^s)(u)S_{n_\tau}(u)Cx\,\mathrm{d}u$$
$$+ (-1)^{n_\tau} \int_t^{t+s} I^{n_\tau}(\rho_m^s)(u-t)S_{n_\tau}(u)Cx\,\mathrm{d}u.$$

The semigroup property now easily follows by letting $m \to +\infty$ in the above expresion, with the help of (4.4) and the identity

$$\sum_{j=0}^{n_\tau - 1} g_{n_\tau - j}(t) g_{j+1}(s-u) = g_{n_\tau}(t+s-u), \quad u > 0.$$

Let $x \in \mathcal{N}(\mathcal{G})$. Then there are $x_0, x_1, \ldots, x_{n_\tau - 1} \in E$, such that

$$S_{n_\tau}(t)x = \sum_{i=0}^{n_\tau - 1} \frac{t^i}{i!} x_i,$$

for $t \in [0, \tau)$ and $x \in E$. For $\varphi \in \mathcal{D}$, such that $\varphi = 1$ on a neighborhood of zero and integrating by parts n_τ-times we have

$$T_i x = \mathcal{G}(\varphi)x = (-1)^{n_\tau} \int_0^\infty \varphi^{(n_\tau)}(t) S_{n_\tau}(t)x \, \mathrm{d}t = \varphi(0)\Big(S_{n_\tau}(t)\Big)x_{(n_\tau - 1)} \,|_{t=0} = x_{n_\tau - 1}.$$

Now, for x is not an element in $N(T_i)$, $i = 0, 1, \ldots, n_\tau - 1$, $m \geq n_\tau - 1$, we have that x is not an element in $N(S_{n_\tau}(t))$. But for $x \in N(T_i)$, $i = 0, 1, 2, \ldots, n_\tau - 1$, we have that $\mathcal{G}(\varphi)x = 0$ holds for all $\varphi \in \mathcal{D}_{(-\infty,\tau]}$ and this implies that $S_{n_\tau}(t)x = 0$, $t \in [0, \tau)$.

Remark 4.1. (i) We have already seen that $\mathcal{G}(\cdot) \equiv 0$ is a degenerate pre-distribution semigroup with the generator $\mathcal{A} \equiv E \times E$. Then, for every $\tau > 0$ and for every number $n_\tau \in \mathbb{N}$, there exists only one local n_τ-times integrated semigroup $(S_{n_\tau}(t) \equiv 0)_{t \in [0,\tau)}$ so that (4.2) holds. It is clear that the condition (B) holds and that condition (A) does not hold here. Denote by $\mathcal{A}_\tau$ the integral generator of $(S_{n_\tau}(t) \equiv 0)_{t \in [0,\tau)}$. Then $\mathcal{A}_\tau = \{0\} \times E$ is strictly contained in the integral generator $\mathcal{A}$ of $\mathcal{G}$. Furthermore, if $C \neq 0$, then there do not exist $\tau > 0$ and $n_\tau \in \mathbb{N}$ such that $\mathcal{A}$ is the integral generator (subgenerator) of a local n_τ-times integrated C-semigroup.

(ii) A similar line of reasoning as in the final part of the proof of [18, Theorem 3.1.9] shows that for each $(x, y) \in \mathcal{A}$ there exist elements $x_0, x_1, \ldots, x_{n_\tau}$ in E such that

$$S_{n_\tau}(t)x - g_{n_\tau+1}(t)Cx - \int_0^t S_{n_\tau}(s)y \, \mathrm{d}s = \sum_{j=0}^{n_\tau} g_{j+1}(t)x_j, \quad t \in [0, \tau)$$

and $x_j \in \mathcal{A}x_{j-1}$ for $1 \leq j \leq n_\tau$. In purely multivalued case, it is not clear how we can prove that $x_j = 0$ for $0 \leq j \leq n_\tau$ without imposing some additional unpleasant conditions.

(iii) Using dualization, we can simply reformulate the second equality appearing on the second line after the equation [24, (11)] in our context.

The proof of subsequent theorem can be deduced by using the argumentation contained in the proof of [18, Theorem 3.1.8].

Theorem 4.3. *Suppose that there exists a sequence $((p_k, \tau_k))_{k \in \mathbb{N}_0}$ in $\mathbb{N}_0 \times (0, \infty)$ such that $\lim_{k \to \infty} \tau_k = \infty$, $(p_k)_{k \in \mathbb{N}_0}$ and $(\tau_k)_{k \in \mathbb{N}_0}$ are strictly increasing, as well as that* *for each $k \in \mathbb{N}_0$ there exists a local p_k-times integrated C-semigroup $(S_{p_k}(t))_{t\in[0,\tau_k)}$ on E so that*

$$S_{p_m}(t)x = \big(g_{p_m-p_k} *_0 S_{p_k}(\cdot)x\big)(t), \quad x \in E,\ t \in [0, \tau_k), \tag{4.6}$$

provided $k < m$. Define

$$\mathcal{G}(\varphi)x := (-1)^{p_k} \int_0^\infty \varphi^{(p_k)}(t) S_{p_k}(t)x \,\mathrm{d}t, \quad \varphi \in \mathcal{D}_{(-\infty,\tau_k)},\ x \in E,\ k \in \mathbb{N}_0.$$

Then $\mathcal{G}$ is well-defined and $\mathcal{G}$ is a pre-(C-DS).

Remark 4.2. (i) Denote by $\mathcal{A}_k$ the integral generator of $(S_{p_k}(t))_{t\in[0,\tau_k)}$ ($k \in \mathbb{N}_0$). Then $\mathcal{A}_k \subseteq \mathcal{A}_m$ for $k > m$ and $\bigcap_{k\in\mathbb{N}_0} \mathcal{A}_k \subseteq \mathcal{A}$, where $\mathcal{A}$ is the integral generator of $\mathcal{G}$. Even in the case that $C = I$, $\bigcup_{k\in\mathbb{N}_0} \mathcal{A}_k$ can be a proper subset of $\mathcal{A}$.

(ii) Suppose that $\mathcal{A}$ is a subgenerator of $(S_{p_k}(t))_{t\in[0,\tau_k)}$ for all $k \in \mathbb{N}_0$. Then (4.6) automatically holds.

(iii) In the case that $C = I$, then it suffices to suppose that there exists an MLO $\mathcal{A}$ such that $\mathcal{A}$ is a subgenerator of a local p-times integrated semigroup $(S_p(t))_{t\in[0,\tau)}$ for some $p \in \mathbb{N}$ and $\tau > 0$ ([22]).

Let $\alpha \in (0,\infty) \setminus \mathbb{N}$, $f \in \mathcal{S}$ and $n = \lceil \alpha \rceil$. Let us recall that the Weyl fractional derivative W_+^α of order α is defined by

$$W_+^\alpha f(t) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_t^\infty (s-t)^{n-\alpha-1} f(s)\,\mathrm{d}s,\ t \in \mathbb{R}.$$

If $\alpha = n \in \mathbb{N}_0$, then we set $W_+^n := (-1)^n \frac{d^n}{dt^n}$. It is well known that the following equality holds: $W_+^{\alpha+\beta} f = W_+^\alpha W_+^\beta f$, α, $\beta > 0$, $f \in \mathcal{S}$.

Suppose now that $\alpha \in (0,\infty) \setminus \mathbb{N}$ and $\mathcal{A}$ is the integral generator of a global α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$ on E. Then $\mathcal{A}$ is the integral generator of a global n-times integrated C-semigroup $(S_n(t))_{t\geq 0}$ on E, where $n = \lceil \alpha \rceil$ and $S_n(t)x := (g_{n-\alpha} * S_\alpha(\cdot)x)(t)$, $x \in E$, $t \geq 0$ ([22]). Arguing as in [21], we have that:

$$\int_0^\infty W_+^\alpha \varphi(t) S_\alpha(t)x \,\mathrm{d}t = (-1)^n \int_0^\infty \varphi^{(n)}(t) S_n(t)x \,\mathrm{d}t, \quad x \in E,\ \varphi \in \mathcal{D}.$$

Keeping in mind the proof of [18, Theorem 3.1.8], we obtain the following:

Theorem 4.4. *Assume that $\alpha \geq 0$ and $\mathcal{A}$ is the integral generator of a global α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$ on E. Set*

$$\mathcal{G}_\alpha(\varphi)x := \int_0^\infty W_+^\alpha \varphi(t) S_\alpha(t)x \, \mathrm{d}t, \quad x \in E, \ \varphi \in \mathcal{D}.$$

Then $\mathcal{G}$ is a pre-(C-DS) whose integral generator contains $\mathcal{A}$.

We will accept the following definition of an exponential pre-(C-DS).

Definition 4.2. Let $\mathcal{G}$ be a pre-(C-DS). Then $\mathcal{G}$ is said to be an exponential pre-(C-DS) iff there exists $\omega \in \mathbb{R}$ such that $e^{-\omega t}\mathcal{G} \in \mathcal{S}'(L(E))$. We use the shorthand pre-(C-EDS) to denote an exponential pre-(C-DS).

We have the following fundamental result:

Theorem 4.5. *Assume that $\alpha \geq 0$ and $\mathcal{A}$ generates an exponentially equicontinuous α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$. Define $\mathcal{G}$ through $\mathcal{G}_\alpha(\varphi)x := \int_0^\infty W_+^\alpha \varphi(t) S_\alpha(t)x \, \mathrm{d}t$, $x \in E$, $\varphi \in \mathcal{D}$. Then $\mathcal{G}$ is a pre-(C-EDS) whose integral generator contains $\mathcal{A}$.*

Remark 4.3. (i) Suppose that $\mathcal{G}$ is a pre-(C-EDS) generated by $\mathcal{A}$, $\omega \in \mathbb{R}$ and $e^{-\omega t}\mathcal{G} \in \mathcal{S}'(L(E))$. Suppose, further, that there exists a non-negative integer n and a continuous function $V : \mathbb{R} \to L(E)$ satisfying that

$$\left\langle e^{-\omega t}\mathcal{G}, \varphi \right\rangle = (-1)^n \int_{-\infty}^\infty \varphi^{(n)}(t) V(t) \, \mathrm{d}t, \quad \varphi \in \mathcal{D},$$

and that there exists a number $r \geq 0$ such that the operator family $\{(1 + t^r)^{-1} V(t) : t \geq 0\} \subseteq L(E)$ is equicontinuous. Since $e^{-\omega \cdot}\mathcal{G}$ is a pre-(C-EDS) generated by $\mathcal{A} - \omega$, the proof of Theorem 4.2 shows that $(V(t))_{t\geq 0}$ is an exponentially equicontinuous n-times integrated C-semigroup; by Theorem 4.5, the integral generator $\hat{\mathcal{A}}^\omega$ of $(V(t))_{t\geq 0}$ is contained in $\mathcal{A} - \omega$. Define

$$S_n(t)x := e^{\omega t} V(t)x + \int_0^t \sum_{k=1}^\infty \binom{n}{k} \frac{(-1)^k \omega^k (t-s)^{k-1}}{(k-1)!} e^{\omega s} V(s)x \, \mathrm{d}s.$$

Arguing as in the proof of [18, Theorem 2.5.1, Theorem 2.5.3], we can prove that the MLO $\hat{\mathcal{A}}^\omega + \omega$ $(\subseteq \mathcal{A})$ is the integral generator of an exponentially equicontinuous n-times integrated C-semigroup $(S_n(t))_{t\geq 0}$.

(ii) The conclusions from Theorem 4.5 and the first part of this remark can be re-word for the classes of q-exponentially equicontinuous integrated C-semigroups and q-exponentially equicontinuous pre-(C-DS)'s; cf. [21] for the notion.

Remark 4.4. Suppose that $\mathcal{G} \in \mathcal{D}_0'(L(E))$, $\mathcal{G}(\varphi)C = C\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}$ and $\mathcal{A}$ is a closed MLO on E satisfying that $\mathcal{G}(\varphi)\mathcal{A} \subseteq \mathcal{A}\mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}$ and

$$\mathcal{G}\big(-\varphi'\big)x - \varphi(0)Cx \in \mathcal{A}\mathcal{G}(\varphi)x, \quad x \in E,\ \varphi \in \mathcal{D}. \tag{4.7}$$

In [21], we have proved the following:

(i) If $\mathcal{A} = A$ is single-valued, then $\mathcal{G}$ satisfies (C.S.1).

(ii) If $\mathcal{G}$ satisfies (C.S.2) holds, C is injective and $\mathcal{A} = A$ is single-valued, then $\mathcal{G}$ is a (C-DS) generated by $C^{-1}AC$.

(iii) If E is admissible and $\mathcal{A} = A$ is single-valued, then the condition (C.S.2) automatically holds for $\mathcal{G}$.

As we have already seen, the conclusion from (ii) immediately implies that $\mathcal{A} = A$ must be single-valued and that the operator C must be injective.

Concerning the assertion (i), its validity is not true in multivalued case: Let $C = I$, let $\mathcal{A} \equiv E \times E$, and let $\mathcal{G} \in \mathcal{D}_0'(L(E))$ be arbitrarily chosen. Then $\mathcal{G}$ commutes with $\mathcal{A}$ and (4.7) holds but $\mathcal{G}$ need not satisfy (C.S.1).

Concerning the assertion (iii) in multivalued case, we can prove that the admissibility of state space E implies that for each $x \in \mathcal{N}(\mathcal{G})$ there exist an integer $k \in \mathbb{N}$ and a finite sequence $(y_i)_{0\leq i\leq k-1}$ in $D(\mathcal{A})$ such that $y_i \in \mathcal{A}y_{i+1}$ $(0 \leq i \leq k-1)$ and $Cx \in \mathcal{A}y_0 \subseteq \mathcal{A}^{k+2}0$.

Now we will reconsider some conditions introduced by J. L. Lions [26] in our new framework. Suppose that $\mathcal{G} \in \mathcal{D}_0'(L(E))$ and $\mathcal{G}$ commutes with C. We analyze the following conditions for $\mathcal{G}$:

(d_1) $\mathcal{G}(\varphi * \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi)$, $\varphi,\ \psi \in \mathcal{D}_0$,

(d_3) $\mathcal{R}(\mathcal{G})$ is dense in E,

(d_4) for every $x \in \mathcal{R}(\mathcal{G})$, there exists a function $u_x \in C([0,\infty) : E)$ so that $u_x(0) = Cx$ and $\mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)u_x(t)\,\mathrm{d}t$, $\varphi \in \mathcal{D}$,

(d_5) $(Cx, \mathcal{G}(\psi)x) \in G(\psi_+)$, $\psi \in \mathcal{D}$, $x \in E$.

Suppose that $\mathcal{G} \in \mathcal{D}_0'(L(E))$ is a pre-(C-DS). Then it is clear that $\mathcal{G}$ satisfies (d_1), our previous considerations shows that $\mathcal{G}$ satisfies (d_5); by the proof of [18, Proposition 3.1.24], we have that $\mathcal{G}$ also satisfies (d_4). On the other hand, it is well known that (d_1), (d_4) and (C.S.2) taken together do not imply (C.S.1), even in the case that $C = I$; see e.g. [18, Remark 3.1.20]. Furthermore, let (d_1), (d_3) and (d_4) hold.

Then (d_5) holds, as well. In order to see this, fix $x \in \mathcal{R}(\mathcal{G})$ and $\varphi \in \mathcal{D}$; then it suffices to show that $(Cx, \mathcal{G}(\varphi)x) \in G(\varphi_+)$. Suppose that (ρ_n) is a regularizing sequence and $u_x(t)$ is a function appearing in the formulation of the property (d_4). The arguments contained in the proof of [18, Proposition 3.1.19] shows that, for every $\eta \in \mathcal{D}_0$, one has

$$\begin{aligned}\mathcal{G}(\rho_n)\mathcal{G}(\varphi_+ * \eta)x &= \mathcal{G}((\varphi_+ * \rho_n) * \eta)Cx = \mathcal{G}(\eta)\mathcal{G}(\varphi_+ * \rho_n)x \\ &= \mathcal{G}(\eta)\int_0^\infty (\varphi_+ * \rho_n)(t)u_x(t)\,\mathrm{d}t \\ &\to \mathcal{G}(\eta)\int_0^\infty \varphi(t)u_x(t)\,\mathrm{d}t = \mathcal{G}(\eta)\mathcal{G}(\varphi)x,\ n \to \infty; \\ \mathcal{G}(\rho_n)\mathcal{G}(\varphi_+ * \eta)x &= \mathcal{G}(\varphi_+ * \eta * \rho_n)Cx \to \mathcal{G}(\varphi_+ * \eta)Cx,\ n \to \infty.\end{aligned}$$

Hence, $\mathcal{G}(\varphi_+ * \eta)Cx = \mathcal{G}(\eta)\mathcal{G}(\varphi)x$ and (d_5) holds, as claimed. On the other hand, (d_1) is a very simple consequence of (d_5); to verify this, observe that for each $\varphi \in \mathcal{D}_0$ and $\psi \in \mathcal{D}$ we have $\psi_+ * \varphi = \psi *_0 \varphi = \varphi *_0 \psi$, so that (d_5) is equivalent to say that $\mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi)$ ($\varphi \in \mathcal{D}_0$, $\psi \in \mathcal{D}$). In particular,

$$\mathcal{G}(\varphi)\mathcal{G}(\psi) = \mathcal{G}(\psi)\mathcal{G}(\varphi), \quad \varphi \in \mathcal{D}_0,\ \psi \in \mathcal{D}. \tag{4.8}$$

Suppose now that (d_5) holds. Let $\varphi \in \mathcal{D}_0$ and $\psi,\ \eta \in \mathcal{D}$. Observing that $\psi_+ * \eta_+ * \varphi = (\psi *_0 \eta)_+ * \varphi$, we have (cf. also [24, Remark 3.13]):

$$\begin{aligned}\mathcal{G}(\varphi)\mathcal{G}(\eta)\mathcal{G}(\psi) &= C\mathcal{G}(\eta_+ * \varphi)\mathcal{G}(\psi) \\ &= C\mathcal{G}(\psi_+ * \eta_+ * \varphi) = C\mathcal{G}\big((\psi *_0 \eta)_+ * \varphi\big)C \\ &= C\mathcal{G}(\varphi)\mathcal{G}(\psi *_0 \eta) = \mathcal{G}(\varphi)\mathcal{G}(\psi *_0 \eta)C.\end{aligned} \tag{4.9}$$

By (4.8)-(4.9), we get

$$\mathcal{G}(\eta)\mathcal{G}(\psi)\mathcal{G}(\varphi) = \mathcal{G}(\psi *_0 \eta)C\mathcal{G}(\varphi). \tag{4.10}$$

Due to (4.8)-(4.10), we have the following:

(i) (d_5) and (d_3) together imply (C.S.1); in particular, (d_1), (d_3) and (d_4) together imply (C.S.1). This is an extension of [18, Proposition 3.1.19].

(ii) (d_5) and (d_2) together imply that $\mathcal{G}$ is a (C-DS); in particular, $\mathcal{A} = A$ must be single-valued and C must be injective.

On the other hand, (d_5) does not imply (C.S.1) even in the case that $C = I$. A simple counterexample is $\mathcal{G} \in \mathcal{D}_0'(L(E))$ given by $\mathcal{G}(\varphi)x := \varphi(0)x$, $x \in E$, $\varphi \in \mathcal{D}$.

The exponential region $E(a, b)$ has been defined for the first time by W. Arendt, O. El–Mennaoui and V. Keyantuo in [1]:

$$E(a,b) := \Big\{\lambda \in \mathbb{C} : \Re\lambda \geq b,\ |\Im\lambda| \leq e^{a\Re\lambda}\Big\}\ (a,\ b > 0).$$

Now we are able to state the following theorem:

Theorem 4.6. *Let $a > 0$, $b > 0$ and $\alpha > 0$. Suppose that $\mathcal{A}$ is a closed* MLO *and, for every λ which belongs to the set $E(a, b)$, there exists an operator $F(\lambda) \in L(E)$ so that $F(\lambda)\mathcal{A} \subseteq \mathcal{A}F(\lambda)$, $\lambda \in E(a, b)$, $F(\lambda)x \in (\lambda-\mathcal{A})^{-1}Cx$, $\lambda \in E(a, b)$, $x \in E$, $F(\lambda)C = CF(\lambda)$, $\lambda \in E(a, b)$, $F(\lambda)x - Cx = F(\lambda)y$, whenever $\lambda \in E(a, b)$ and $(x, y) \in \mathcal{A}$, and that the mapping $\lambda \mapsto F(\lambda)x$ is analytic on $\Omega_{a,b}$ and continuous on $\Gamma_{a,b}$, where $\Gamma_{a,b}$ denotes the upwards oriented boundary of $E(a, b)$ and $\Omega_{a,b}$ the open region which lies to the right of $\Gamma_{a,b}$. Let the operator family $\{(1+|\lambda|)^{-\alpha}F(\lambda) : \lambda \in E(a, b)\} \subseteq L(E)$ be equicontinuous. Set*

$$\mathcal{G}(\varphi)x := (-i)\int_{\Gamma_{a,b}} \hat{\varphi}(\lambda)F(\lambda)x\,\mathrm{d}\lambda,\ \ x \in E,\ \varphi \in \mathcal{D}.$$

Then $\mathcal{G}$ is a pre-(C-DS) generated by an extension of $\mathcal{A}$.

PROOF. Arguing as in non-degenerate case [21], we can prove with the help of Lemma 2.1 that $\mathcal{G} \in \mathcal{D}_0'(L(E))$ as well as that $\mathcal{G}$ commutes with C and $\mathcal{A}$. The prescribed assumptions imply by [22, Theorem 3.23] (cf. also [18, Theorem 2.7.2(iv)]) that for each $n \in \mathbb{N}$ with $n > \alpha + 1$ the MLO $\mathcal{A}$ subgenerates a local n-times integrated C-semigroup $(S_n(t))_{t\in[0,a(n-\alpha-1))}$. It is straightforward to prove [21] that

$$\mathcal{G}(\varphi)x = (-1)^n \int_{-\infty}^{\tau} \varphi^{(n)}(t)S_n(t)x\,\mathrm{d}t, \quad x \in E,\ \varphi \in \mathcal{D}_{(-\infty,a(n-\alpha-1))}.$$

Now the conclusion directly follows from Theorem 4.3 and Remark 4.2(i)-(ii).

Remark 4.5. (i) If C is injective, $\mathcal{A} = A$ is single-valued, $\rho_C(A) \subseteq E(a, b)$ and $F(\lambda) = (\lambda - \mathcal{A})^{-1}C$, $\lambda \in E(a, b)$, then $\mathcal{G}$ is a (C-DS) generated by $C^{-1}AC$ ([21]). Even in the case that $C = I$, the integral generator $\mathcal{A}$ of $\mathcal{G}$, in multivalued case, can strictly contain $C^{-1}\mathcal{A}C$; see Remark 4.1(i).

(ii) Let $\mathcal{A}$ be a closed MLO, let C be injective and commute with $\mathcal{A}$, and let $\rho_C(\mathcal{A}) \subseteq E(a, b)$. Then the choice $F(\lambda) = (\lambda - \mathcal{A})^{-1}C$, $\lambda \in E(a, b)$ is always possible; in this case, we have $\mathcal{A}0 \subseteq N(\mathcal{G}(\varphi))$, $\varphi \in \mathcal{D}$ ([20]).

Local integrated semigroups generated by multivalued linear operators (see e.g. [20, Example 3.2.11(i)]) can be used for construction of pre-(DS)'s. In [20, Theorem 3.2.21] and [20, Example 3.2.23], we have investigated the entire solutions of backward heat Poisson equation, showing the existence of an entire C-regularized semigroup ($C \in L(L^p(\Omega))$ non-injective) generated by the multivalued linear operator $\Delta \cdot m(x)^{-1}$ in $L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$. This example can serve us to construct an important example of a pre-(C-DS); cf. also [22, Example 3.24]. Examples of exponentially bounded integrated semigroups generated by multivalued linear operators can be found in [12, Section 5.8] and these examples can be used for construction of exponential pre-(DS)'s. Also by Proposition 4.2(iii) the duals of non-dense pre-(C-DS)'s are pre-(C^*-DS)'s on E^*, so this is another way of constructing of degenerate C-distribution semigroups.

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