Bulletin T.CXLIX de l'Académie serbe des sciences et des arts − 2016
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 41

ABSTRACT DEGENERATE MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS WITH RIEMANN-LIOUVILLE DERIVATIVES

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(Presented at the 8th Meeting, held on November 27, 2015)

A b s t r a c t. *In this paper, we investigate the following abstract multi-term fractional differential equation:*

$$BD_t^{\alpha_n}u(t) + \sum_{j=1}^{n-1} A_j D_t^{\alpha_j}u(t) = AD_t^{\alpha}u(t) + f(t), \quad t \in (0, \tau),$$

where $n \in \mathbb{N} \setminus \{1\}$, A, B *and* $A_1, \ldots, A_{n-1}$ *are closed linear operators on a complex Banach space* E, $0 \leq \alpha_1 < \cdots < \alpha_n$, $0 \leq \alpha < \alpha_n$, $0 < \tau \leq \infty$, $f(t)$ *is an* E*-valued function, and* D_t^{α} *denotes the Riemann-Liouville fractional derivative of order* α (*see* [*E. Bazhlekova, Fractional Evolution Equations in Banach Spaces, Ph.D. Thesis, Eindhoven University of Technology, Eindhoven,* 2001]). *We introduce and further analyze some new types of degenerate* k*-regularized* (C_1, C_2)*-existence and uniqueness* (*propagation*) *families for the previous equation.*

AMS Mathematics Subject Classification (2000): 47D06, 47D60, 47D62, 47D99.

Key Words: Abstract multi-term fractional differential equations, degenerate differential equations, fractional calculus, Riemann-Liouville fractional derivatives.

1. *Introduction and preliminaries*

As already mentioned in [22], the theory of abstract degenerate (multi-term) fractional differential equations is a still-undeveloped subject. Up to now, we have only a few relevant references on abstract degenerate fractional differential equations with Caputo derivatives (cf. [9]–[11] and [15]–[18]). This is, probably, the first paper concerning abstract degenerate (multi-term) fractional differential equations with Riemann-Liouville fractional derivatives. Here we continue our previous research studies [21] (the joint paper with C.-G. Li and M. Li) and [22]; cf. also [4]–[5], [37] and [19]. We study the following abstract multi-term fractional differential equation:

$$BD_t^{\alpha_n}u(t)+\sum_{j=1}^{n-1}A_jD_t^{\alpha_j}u(t)=AD_t^{\alpha}u(t)+f(t),\quad t\in(0,\tau),\tag{1.1}$$

where $n\in\mathbb{N}\setminus\{1\}$, A, B and $A_1,\ldots,A_{n-1}$ are closed linear operators on a complex Banach space E, $0\leq\alpha_1<\cdots<\alpha_n$, $0\leq\alpha<\alpha_n$, $0<\tau\leq\infty$, $f(t)$ is an E-valued function, and D_t^{α} denotes the Riemann-Liouville fractional derivative of order α (see [2]).

As in non-degenerate case, we are burdened with the problem of subjecting initial values to the problem (1.1); we distinguish three important subcases of this problem: (SC1), (SC2) and (SC3). The analysis of subcase (SC3) is very complicated and, because of that, we focus our attention towards the analysis of subcases (SC1) and (SC2). We analyze some new types of degenerate k-regularized (C_1,C_2)-existence and uniqueness (propagation) families for (1.1) in the beginning part of Section 2, while in Subsection 2.1 we analyze some applications of degenerate k-regularized (C_1,C_2)-existence and uniqueness families ([22]) in the study of abstract degenerate multi-term fractional differential equations with Riemann-Liouville derivatives.

For the sake of brevity, we work in the setting of complex Banach spaces. By E and $L(E)$ we denote a non-trivial complex Banach space and the space of all continuous linear mappings from E into E, respectively. The norm of an element $x\in E$ will be denoted by $\|x\|$. By A we denote a closed linear operator with domain and range contained in E; $C\in L(E)$ denotes an injective operator, and the convolution like mapping $*$ is given by $f*g(t):=\int_0^t f(t-s)g(s)\,\mathrm{d}s$. The domain and range of A are denoted by $D(A)$ and $R(A)$, respectively. By $[D(A)]$ and I we denote the Banach space $D(A)$ equipped with the graph norm and the identity operator on E, respectively.

Given $s\in\mathbb{R}$ in advance, set $\lfloor s\rfloor:=\sup\{l\in\mathbb{Z}:s\geq l\}$ and $\lceil s\rceil:=\inf\{l\in\mathbb{Z}:s\leq l\}$. The principal branch is always used to take the powers. Set $\mathbb{N}_l:=\{1,\ldots,l\}$, $\mathbb{N}_l^0:=\{0,1,\ldots,l\}$, $0^\zeta:=0$, $g_\zeta(t):=t^{\zeta-1}/\Gamma(\zeta)$ $(\zeta>0,\,t>0)$ and $g_0(t):=$ the

Dirac δ-distribution. By $\chi_S(\cdot)$ we denote the characteristic function. If $\delta \in (0,\pi]$, then we define $\Sigma_\delta := \{\lambda \in \mathbb{C} : \lambda \neq 0, \ |\arg\lambda| < \delta\}$. Let $0 < \tau \leq \infty$, and let $I = (0,\tau)$. Then the Sobolev space $W^{m,1}(I : E)$ can be introduced in the following way (see e.g. [2, p. 7]):

$$W^{m,1}(I:E) := \Biggl\{ f \ \Bigg| \ \exists \varphi \in L^1(I:E) \ \exists c_k \in \mathbb{C} \ (0 \leq k \leq m-1) \\ f(t) = \sum_{k=0}^{m-1} c_k g_{k+1}(t) + \bigl(g_m * \varphi\bigr)(t) \text{ for a.e. } t \in (0,\tau) \Biggr\}.$$

Then $\varphi(t) = f^{(m)}(t)$ in the distributional sense, and $c_k = f^{(k)}(0)$ $(0 \leq k \leq m-1)$. Let $0 < \tau \leq \infty$ and $k \in L^1_{\mathrm{loc}}([0,\tau))$. Then we say that the function $k(t)$ is a kernel on $[0,\tau)$ iff for each $f \in C([0,\tau))$ the assumption $\int_0^t k(t-s)f(s)\,\mathrm{d}s = 0$, $t \in [0,\tau)$ implies $f(t) = 0$, $t \in [0,\tau)$. If $\tau < \infty$, then the Titchmarsh–Foiaş theorem (see e.g. [13, Theorem 3.4.40]) states that the function $k(t)$ is a kernel on $[0,\tau)$ iff $0 \in \operatorname{supp}(k)$; on the other hand, if $\tau = \infty$ and $k \neq 0$ in $L^1_{\mathrm{loc}}([0,\infty))$, then it is well known that the function $k(t)$ is automatically a kernel on $[0,\infty)$.

We employ the following condition:

(P1): $h(t) : [0,\infty) \to E$ is Laplace transformable, i.e., $h \in L^1_{\mathrm{loc}}([0,\infty) : E)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{h}(\lambda) := \mathcal{L}(h)(\lambda) := \lim_{b\to\infty}\int_0^b e^{-\lambda t}h(t)\,\mathrm{d}t := \int_0^\infty e^{-\lambda t}h(t)\,\mathrm{d}t$ exists for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \beta$. Put *abs*(h) :=inf$\{\Re\lambda : \tilde{h}(\lambda)$ exists$\}$, and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

The inclusion $H(\lambda) \in LT_E$ means that there exist a function $h(t) : [0,\infty) \to E$ satisfying (P1) and a number $a >$ *abs*(h) so that $\tilde{h}(\lambda) = H(\lambda)$, $\lambda > a$. Fairly complete information concerning vector-valued Laplace transform can be obtained by consulting the references [1], [34] and [14].

During the past three decades, considerable interest in fractional calculus and fractional differential equations has been stimulated due to their numerous applications in engineering, physics, chemistry, biology and other sciences (for more details about fractional calculus and non-degenerate fractional differential equations, the reader may consult [2], [7], [12]–[14], [26]–[28] and the references cited therein; for the basic source of information on the abstract degenerate differential equations, we refer the reader to [3], [6], [8], [23], [25], [29]–[33] and [35]–[36]).

The most important properties of Riemann-Liouville fractional derivatives are given below. Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined for any function $f \in L^1(I : E)$, by

$$J_t^\alpha f(t) := \bigl(g_\alpha * f\bigr)(t), \quad t > 0.$$

The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined for any function $f \in L^1(I : E)$ satisfying $g_{m-\alpha} * f \in W^{m,1}(I : E)$, by

$$D_t^\alpha f(t) := \frac{\mathrm{d}^m}{\mathrm{d}t^m}\big(g_{m-\alpha} * f\big)(t) = D_t^m J_t^{m-\alpha} f(t), \quad t > 0.$$

Due to [2, Theorem 1.5], the Riemann-Liouville fractional integrals and derivatives satisfy the following equalities:

$$J_t^\alpha J_t^\beta f(t) = J_t^{\alpha+\beta} f(t), \quad D_t^\alpha J_t^\alpha f(t) = f(t),$$

for $f \in L^1(I : E)$ and

$$J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0) g_{\alpha+k+1-m}(t) \tag{1.2}$$

for any $f \in L^1(I : E)$ with $g_{m-\alpha} * f \in W^{m,1}(I : E)$. Denote by $\mathbf{D}_t^\alpha u(t)$ the α-th fractional order Caputo derivative of function $u(t)$ ([2]).

Finally, let us consider the following inhomogeneous equation:

$$Bu(t) + \sum_{j=1}^{n-1} \big(g_{\alpha_n-\alpha_j} * A_j u\big)(t) = f(t) + \big(g_{\alpha_n-\alpha} * Au\big)(t), \quad t \in [0, \tau), \tag{1.3}$$

where $\tau \in (0, \infty]$ and $f \in C([0, \tau) : E)$. Then we say that a function $u \in C([0, \tau) : E)$ is:

(i) a strong solution of (1.3) iff $A_j u \in C([0, \tau) : E)$, $j \in \mathbb{N}_n^0$ and (1.3) holds for every $t \in [0, \tau)$,

(ii) a mild solution of (1.3) iff $(g_{\alpha_n-\alpha_j} * u)(t) \in D(A_j)$, $t \in [0, \tau)$, $j \in \mathbb{N}_n^0$ and

$$Bu(t) + \sum_{j=1}^{n-1} A_j \big(g_{\alpha_n-\alpha_j} * u\big)(t) = f(t) + A\big(g_{\alpha_n-\alpha} * u\big)(t), \quad t \in [0, \tau).$$

2. *Degenerate k-regularized (C_1, C_2)-existence and uniqueness propagation families for* (1.1)

As already mentioned in the abstract, we assume henceforth that E is a complex Banach space, as well as that $n \in \mathbb{N} \setminus \{1\}$, A, B and $A_1, \ldots, A_{n-1}$ are closed linear operators on E, $0 \le \alpha_1 < \cdots < \alpha_n$ and $0 \le \alpha < \alpha_n$. Set $\alpha_0 := \alpha$, $m := \lceil \alpha \rceil$, $A_0 := A$ and $A_n := B$.

Definition 2.1. Suppose $0 < \tau \leq \infty$ and $f \in L^1((0,\tau) : E)$. By a strong solution of (1.1) we mean any function $u \in L^1((0,\tau) : E)$ for which $g_{m_j-\alpha_j} * u \in W^{m_j,1}((0,\tau) : E)$ $(0 \leq j \leq n)$, $A_j D_t^{\alpha_j} u(t) \in L^1((0,\tau) : E)$ $(0 \leq j \leq n)$, and (1.1) holds for a.e. $t \in (0,\tau)$.

Now we would like to introduce the concept mild solution of (1.1), and to endow the equation (1.1) with corresponding inital conditions. In order to do that, let us assume that $u(t)$ is a strong solution of (1.1). Then we can integrate the equation (1.1) α_n-times by using the formula (1.2) and the closedness of the operators A_j for $j \in \mathbb{N}_{n-1}^0$. In such a way, we get

$$\begin{aligned} B\Biggl[u(t) - \sum_{i=0}^{m_n-1} \bigl(g_{m_n-\alpha_n} * u\bigr)^{(i)}(0) g_{\alpha_n+i+1-m_n}(t)\Biggr] \\ + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j \Biggl[u(t) - \sum_{i=0}^{m_j-1} \bigl(g_{m_j-\alpha_j} * u\bigr)^{(i)}(0) g_{\alpha_j+i+1-m_j}(t)\Biggr] \\ = g_{\alpha_n-\alpha} * A \Biggl[u(t) - \sum_{i=0}^{m-1} \bigl(g_{m-\alpha} * u\bigr)^{(i)}(0) g_{\alpha+i+1-m}(t)\Biggr] + \bigl(g_{\alpha_n} * f\bigr)(t), \end{aligned} \tag{2.1}$$

for all $t \in [0,\tau)$. This immediately implies that any strong solution of (1.1) satisfies

$$B\Biggl[u(t) - \sum_{i=0}^{m_n-1} \bigl(g_{m_n-\alpha_n} * u\bigr)^{(i)}(0) g_{\alpha_n+i+1-m_n}(t)\Biggr] \in C([0,\tau) : E), \tag{2.2}$$

whence we may conclude that $u \in C([0,\tau) : E)$ provided that $B^{-1} \in L(E)$.

Definition 2.2. Suppose $0 < \tau \leq \infty$ and $f \in L^1((0,\tau) : E)$. By a mild solution of (1.1) we mean any function $u \in L^1((0,\tau) : E)$ for which $g_{m_j-\alpha_j} * u \in W^{m_j,1}((0,\tau) : E)$ $(0 \leq j \leq n)$, $A_j(g_{\alpha_n} * D_t^{\alpha_j} u(\cdot))(\cdot) \in C([0,\tau) : E)$ $(0 \leq j \leq n-1)$, (2.2) holds, and

$$\begin{aligned} B\Biggl[u(t) - \sum_{i=0}^{m_n-1} \bigl(g_{m_n-\alpha_n} * u\bigr)^{(i)}(0) g_{\alpha_n+i+1-m_n}(t)\Biggr] + \sum_{j=1}^{n-1} A_j\Bigl(g_{\alpha_n} * D_t^{\alpha_j} u(\cdot)\Bigr)(t) \\ = A\Bigl(g_{\alpha_n} * D_t^{\alpha} u(\cdot)\Bigr)(t) + \bigl(g_{\alpha_n} * f\bigr)(t), \quad t \in [0,\tau). \end{aligned} \tag{2.3}$$

By the foregoing, any strong solution of (1.1) is also a mild solution of the same problem; the converse statement is not true, in general. Observe that the equation

(2.3) can be written in the following way:

$$B\Bigg[u(t)-\sum_{i=0}^{m_n-1}\big(g_{m_n-\alpha_n}*u\big)^{(i)}(0)g_{\alpha_n+i+1-m_n}(t)\Bigg]$$
$$+\sum_{j=1}^{n-1}A_j\Bigg(g_{\alpha_n-\alpha_j}*\Bigg[u(\cdot)-\sum_{i=0}^{m_j-1}\big(g_{m_j-\alpha_j}*u\big)^{(i)}(0)g_{\alpha_j+i+1-m_j}(\cdot)\Bigg]\Bigg)(t)$$
$$=A\Bigg(g_{\alpha_n-\alpha}*\Bigg[u(\cdot)-\sum_{i=0}^{m-1}\big(g_{m-\alpha}*u\big)^{(i)}(0)g_{\alpha+i+1-m}(\cdot)\Bigg]\Bigg)(t)+\big(g_{\alpha_n}*f\big)(t), \tag{2.4}$$

for all $t \in [0,\tau)$.

Put
$$\mathcal{T}_{(1.1)}:=\begin{cases}1, & \text{if there exists } j\in\mathbb{N}_n^0 \text{ such that } \alpha_j\in\mathbb{N},\\ 0, & \text{otherwise,}\end{cases}$$

and $S := \{j \in \mathbb{N}_n^0 : \alpha_j \in \mathbb{N}\}$. For the sequel, it will be of crucial importance to recognise the following three subcases of (1.1):

(SC1) $\alpha_n > 1$: Then for each number $i \in \mathbb{N}_{m_n-1}$ we define the set $\mathcal{D}_i$ by $\mathcal{D}_i := \{j \in \mathbb{N}_n^0 : m_j - 1 \geq i\}$. Observe that $n \in \mathcal{D}_i$ $(i \in \mathbb{N}_{m_n-1})$ and $\mathcal{D}_{m_n-1} \subseteq \cdots \subseteq \mathcal{D}_1$. Set $S_i := \{m_j - \alpha_j : j \in \mathcal{D}_i\}$ and, after that, $s_i :=$ card(S_i). Then $S_i \subseteq [0,1)$ and S_i can be written in the following way

$$S_i=\big\{a_{i,1},\ldots,a_{i,s_i}\big\},$$

where $0 \leq a_{i,1} < \cdots < a_{i,s_i} \leq 1$ $(i \in \mathbb{N}_{m_n-1})$. Define $\mathcal{D}_i^l := \{j \in \mathcal{D}_i : m_j - \alpha_j = a_{i,l}\}$ $(i \in \mathbb{N}_{m_n-1},\ 1 \leq l \leq s_i)$. Then for each number $i \in \mathbb{N}_{m_n-1}$ we introduce s_i initial values $x_{i,1},\ldots,x_{i,s_i}$ for terms $(g_{m_j-\alpha_j}*u)^{(i)}(0)$, where $j \in \mathcal{D}_i$. In addition, if there exists $j \in \mathbb{N}_n^0$ such that $\alpha_j \in \mathbb{N}$, i.e., if $S \neq \emptyset$, then one has to introduce a new initial value x_0 for term $(g_0 * u)(0) \equiv u(0)$.

(SC2) $\alpha_n = 1$: Then we introduce only one initial value for term $(g_0 * u)(0) \equiv u(0)$.

(SC3) $\alpha_n < 1$: Then we consider the equation (1.1) without initial conditions.

Define
$$\mathcal{B}_{(1)}:=\begin{cases}s_1+\cdots+s_{m_n-1}+\mathcal{T}_{(1.1)}, & \text{if } \alpha_n>1,\\ 1, & \text{if } \alpha_n=1,\\ 0, & \text{if } \alpha_n<1.\end{cases}$$

Then we will have exactly $\mathcal{B}_{(1)}$ initial conditions for (1.1).

The subcase (SC3) is very specific and therefore, not considered henceforth. Consider now, for the sake of illustration and better understanding, the subcase (SC1). Plugging $x_{i,l} = (g_{m_j-\alpha_j} * u)^{(i)}(0)$ in (2.1), where $j \in \mathcal{D}_i^l$, and choosing other initial values to be zeroes, we obtain

$$B\Big[u(t)-\chi_{\mathcal{D}_i^l}(n)g_{\alpha_n+i+1-m_n}(t)x_{i,l}\Big] + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j\Big[u(t) - \chi_{\mathcal{D}_i^l}(j)g_{\alpha_j+i+1-m_j}(t)x_{i,l}\Big] = g_{\alpha_n-\alpha} * A\Big[u(t) - \chi_{\mathcal{D}_i^l}(0)g_{\alpha+i+1-m}(t)x_{i,l}\Big] \quad \text{for all } t \in [0,\tau). \tag{2.5}$$

If $S \neq \emptyset$, then inserting the initial value x_0 for $u(0)$ in (2.1), and choosing $x_{i,l}$ to be zero for $i \in \mathbb{N}_{m_n-1}$ and $1 \leq l \leq s_i$, we obtain similarly that

$$B\Big[u(t)-\chi_S(n)x_0\Big] + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j\Big[u(t) - \chi_S(j)x_0\Big] = g_{\alpha_n-\alpha} * A\Big[u(t) - \chi_S(0)x_0\Big] \quad \text{for all } t \in [0,\tau). \tag{2.6}$$

Suppose now, only for the purpose of further analysis, that $0 < \tau \leq \infty$, $K(t) \neq 0$ in $L^1_{\text{loc}}([0,\tau))$ and $k(t) = \int_0^t K(s)\,\mathrm{d}s$, $t \in [0,\tau)$. Convoluting the above equations with $K(t)$ and using the procedure similar to that already employed for abstract multi-term problems with Caputo fractional derivatives, we come to the following definition.

Definition 2.3. Let $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $C,\ C_1,\ C_2 \in L(E)$, and let C and C_2 be injective.

(i) (SC1) Suppose that, for every $i \in \mathbb{N}_{m_n-1}$ and $l \in \mathbb{N}_{s_i}$, $(R_{i,l}(t))_{t\in[0,\tau)} \subseteq L(E,[D(B)])$ is strongly continuous, as well as that, for every $t \in [0,\tau)$, $x \in E$, $i \in \mathbb{N}_{m_n-1}$ and $l \in \mathbb{N}_{s_i}$, the following functional equation

$$B\Big[R_{i,l}(t)x - \chi_{\mathcal{D}_i^l}(n)\big(k * g_{\alpha_n+i-m_n}\big)(t)C_1x\Big] + \sum_{j=1}^{n-1} A_j\Big[g_{\alpha_n-\alpha_j} * \Big(R_{i,l}(\cdot)x - \chi_{\mathcal{D}_i^l}(j)\big(k * g_{\alpha_j+i-m_j}\big)(\cdot)C_1x\Big)\Big](t) = A\Big[g_{\alpha_n-\alpha} * \Big(R_{i,l}(\cdot)x - \chi_{\mathcal{D}_i^l}(0)\big(k * g_{\alpha+i-m}\big)(\cdot)C_1x\Big)\Big](t)$$

holds. If $S \neq \emptyset$, then we also introduce a strongly continuous family $(R_{0,1}(t))_{t\in[0,\tau)} \subseteq L(E,[D(B)])$ satisfying that, for every $t \in [0,\tau)$ and $x \in E$,

$$B\Big[R_{0,1}(t)x - \chi_S(n)k(t)C_1x\Big] + \sum_{j=1}^{n-1} A_j\Big[g_{\alpha_n-\alpha_j} * \Big(R_{0,1}(\cdot)x - \chi_S(j)k(\cdot)C_1x\Big)\Big](t) = A\Big[g_{\alpha_n-\alpha} * \Big(R_{0,1}(\cdot)x - \chi_S(0)k(\cdot)C_1x\Big)\Big](t).$$

Then the sequence $((R_{i,l}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ if $S = \emptyset$, resp., $((R_{i,l}(t))_{t\in[0,\tau)},\ (R_{0,1}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ if $S \neq \emptyset$, is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence propagation family for (1.1).
(SC2) A strongly continuous family $(R(t))_{t\in[0,\tau)} \subseteq L(E,[D(B)])$ satisfying that, for every $t \in [0,\tau)$ and $x \in E$,

$$B\Big[R(t)x - k(t)C_1x\Big] + \sum_{j=1}^{n-1} A_j\big(g_{\alpha_n-\alpha_j} * R(\cdot)x\big)(t) = A\big(g_{\alpha_n-\alpha} * R(\cdot)x\big)(t),$$

is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence propagation family for (1.1).

(ii) (SC1) Suppose that, for every $i \in \mathbb{N}_{m_n-1}$ and $l \in \mathbb{N}_{s_i}$, $(W_{i,l}(t))_{t\in[0,\tau)} \subseteq L(E)$ is strongly continuous, as well as that

$$\Big[W_{i,l}(t)Bx - \chi_{\mathcal{D}_i^l}(n)\big(k * g_{\alpha_n+i-m_n}\big)(t)C_2Bx\Big] + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * \Big[W_{i,l}(t)A_jx - \chi_{\mathcal{D}_i^l}(j)\big(k * g_{\alpha_j+i-m_j}\big)(t)C_2A_jx\Big] = g_{\alpha_n-\alpha} * \Big[W_{i,l}(t)Ax - \chi_{\mathcal{D}_i^l}(0)\big(k * g_{\alpha+i-m}\big)(t)C_2Ax\Big],$$

for every $i \in \mathbb{N}_{m_n-1}$, $l \in \mathbb{N}_{s_i}$, $t \in [0,\tau)$ and $x \in \bigcap_{0\leq j\leq n} D(A_j)$. If $S \neq \emptyset$, then we also introduce a strongly continuous family $(W_{0,1}(t))_{t\in[0,\tau)} \subseteq L(E)$ satisfying that, for every $x \in \bigcap_{0\leq j\leq n} D(A_j)$,

$$\Big[W_{0,1}(t)Bx - \chi_S(n)k(t)C_2Bx\Big] + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * \Big[W_{0,1}(t)A_jx - \chi_S(j)k(t)C_2A_jx\Big] = g_{\alpha_n-\alpha} * \Big[W_{0,1}(t)Ax - \chi_S(0)k(t)C_2Ax\Big], \quad t \in [0,\tau).$$

Then the sequence $((W_{i,l}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ if $S=\emptyset$, resp., $((W_{i,l}(t))_{t\in[0,\tau)},\ (W_{0,1}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ if $S\neq\emptyset$, is said to be a (local, if $\tau<\infty$) k-regularized C_2-uniqueness propagation family for (1.1).

(SC2) A strongly continuous family $(W(t))_{t\in[0,\tau)}\subseteq L(E)$ satisfying that, for every $x\in\bigcap_{0\leq j\leq n}D(A_j)$ and $t\in[0,\tau)$,

$$\Big[W(t)Bx-k(t)C_2Bx\Big] + \sum_{j=1}^{n-1}\big(g_{\alpha_n-\alpha_j}*W(\cdot)C_2A_jx\big)(t)=\big(g_{\alpha_n-\alpha}*W(\cdot)C_2Ax\big)(t),$$

is said to be a (local, if $\tau<\infty$) k-regularized C_2-uniqueness propagation family for (1.1).

(iii) Any sequence of strongly continuous families in $L(E)$ that is a k-regularized C-uniqueness propagation family for (1.1) is said to be a k-regularized C-resolvent propagation family for (1.1), in short k-regularized C-propagation family for (1.1), if in addition, any single operator family of such a sequence commutes with the operators A_j ($j\in\mathbb{N}_n^0$) and C, as well as $CA_j\subseteq A_jC$ ($j\in\mathbb{N}_n^0$).

If $k(t)=g_{\zeta+1}(t)$, where $\zeta\geq 0$, then a k-regularized C_1-existence propagation family for (1.1) is also said to be ζ-times integrated C_1-existence propagation family for (1.1); 0-times integrated C_1-existence propagation family for (1.1) is simply called C_1-existence propagation family for (1.1); a similar terminological agreement will be used for the classes of C_2-uniqueness propagation families for (1.1) and C-resolvent propagation families for (1.1).

A k-regularized C_1-existence propagation family for (1.1) is said to be locally equicontinuous (exponentially equicontinuous) iff each single operator family of it, considered as an element of the space $L(E,[D(B)])$, is locally equicontinuous (exponentially equicontinuous), while a k-regularized C_1-existence propagation family for (1.1) is said to be an exponentially equicontinuous k-regularized C_1-existence propagation family for (1.1), of angle $\alpha\in(0,\pi/2]$, iff for each single operator family $(R(t))_{t\geq 0}$ of it, the following holds:

(a) For every $x\in E$, the mappings $t\mapsto R(t)x$, $t>0$ and $t\mapsto BR(t)x$, $t>0$ can be analytically extended to the sector Σ_α; since no confusion seems likely, we shall denote these extensions by the same symbols.

(b) For every $x\in E$ and $\beta\in(0,\alpha)$, one has $\lim_{z\to 0,z\in\Sigma_\beta}R(z)x=R(0)x$ and $\lim_{z\to 0,z\in\Sigma_\beta}BR(z)x=BR(0)x$.

(c) For every $\beta \in (0, \alpha)$, there exists $\omega_\beta \geq \max(0, \text{abs}(k))$ ($\omega_\beta = 0$) such that the family $\{e^{-\omega_\beta z}R(z) : z \in \Sigma_\beta\} \subseteq L(E, [D(B)])$ is equicontinuous.

The above terminological agreements and abbreviations can be also understood for the classes of k-regularized C_2-uniqueness propagation families for (1.1) and k-regularized C-propagation families for (1.1).

We define the notion of a mild (strong) solution of problem (2.5), resp. (2.6), in a very natural way:

Definition 2.4. Let $\tau \in (0, \infty]$. By a mild solution of problem (2.5) on $[0, \tau)$ we mean any continuous function $t \mapsto u(t)$, $t \in [0, \tau)$ satisfying that

$$\begin{aligned} B\Big[u(t) - \chi_{\mathcal{D}_i^l}(n)g_{\alpha_n+i+1-m_n}(t)x_{i,l}\Big] \\ + \sum_{j=1}^{n-1} A_j\Big(g_{\alpha_n-\alpha_j} * \Big[u(\cdot) - \chi_{\mathcal{D}_i^l}(j)g_{\alpha_j+i+1-m_j}(\cdot)x_{i,l}\Big]\Big)(t) \\ = A\Big(g_{\alpha_n-\alpha} * \Big[u(\cdot) - \chi_{\mathcal{D}_i^l}(0)g_{\alpha+i+1-m}(\cdot)x_{i,l}\Big]\Big)(t), \quad t \in [0, \tau). \end{aligned}$$

By a strong solution of problem (2.5) on $[0, \tau)$ we mean any continuous function $t \mapsto u(t)$, $t \in [0, \tau)$ satisfying that $t \mapsto A_j[u(t) - \chi_{\mathcal{D}_i^l}(j)g_{\alpha_j+i+1-m_j}(t)x_{i,l}]$, $t \in [0, \tau)$ is continuous for $j \in \mathbb{N}_n^0$, as well as that:

$$\begin{aligned} B\Big[u(t) - \chi_{\mathcal{D}_i^l}(n)g_{\alpha_n+i+1-m_n}(t)x_{i,l}\Big] \\ + \sum_{j=1}^{n-1}\Big(g_{\alpha_n-\alpha_j} * A_j\Big[u(\cdot) - \chi_{\mathcal{D}_i^l}(j)g_{\alpha_j+i+1-m_j}(\cdot)x_{i,l}\Big]\Big)(t) \\ = \Big(g_{\alpha_n-\alpha} * A\Big[u(\cdot) - \chi_{\mathcal{D}_i^l}(0)g_{\alpha+i+1-m}(\cdot)x_{i,l}\Big]\Big)(t), \quad t \in [0, \tau). \end{aligned}$$

The notion of a mild (strong) solution of problem (2.6) on $[0, \tau)$ is defined similarly.

Then the following holds (for the sake of brevity, we shall consider only the subcase (SC1)):

(A) If $S = \emptyset$ and $((R_{i,l}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1, 1\leq l\leq s_i}$ is a C_1-existence propagation family for (1.1), then the function $u_{i,l}(t) := R_{i,l}(t)C_1x$, $t \in [0, \tau)$ is a mild solution of (2.5) with $x_{i,l} = C_1x$ for $1 \leq i \leq m_n - 1, 1 \leq l \leq s_i$. If $S \neq \emptyset$ and $((R_{i,l}(t))_{t\in[0,\tau)},\ (R_{0,1}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1, 1\leq l\leq s_i}$ is a C_1-existence propagation family for (1.1), then the function $u_{0,1}(t) := R_{0,1}(t)C_1x$, $t \in [0, \tau)$ is a mild solution of (2.6) with $x_0 = C_1x$.

(B) If $S = \emptyset$ and $((W_{i,l}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ is a C_2-uniqueness propagation family for (1.1), as well as $A_jW_{i,l}(t)x = W_{i,l}(t)A_jx,\ t \in [0,\tau)$, $x \in \bigcap_{0\leq j\leq n} D(A_j)$ and $C_2A_j \subseteq A_jC_2,\ j \in \mathbb{N}_n^0$, then the function $u_{i,l}(t) := W_{i,l}(t)C_2^{-1}x_{i,l},\ t \in [0,\tau)$ is a strong solution of (2.5) for $1 \leq i \leq m_n-1, 1 \leq l \leq s_i$ and $x_{i,l} \in C_2(\bigcap_{0\leq j\leq n} D(A_j))$.

If $S \neq \emptyset$ and $((W_{i,l}(t))_{t\in[0,\tau)},\ (W_{0,1}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ is a C_2-uniqueness propagation family for (1.1), as well as (in addition to the assumptions employed in the case that $S = \emptyset$) $A_jW_{0,1}(t)x = W_{0,1}(t)A_jx$, $t \in [0,\tau)$, $x \in \bigcap_{0\leq j\leq n} D(A_j)$, then the function $u_0(t) := W_{0,1}(t)C_2^{-1}x_0$, $t \in [0,\tau)$ is a strong solution of (2.5) for $1 \leq i \leq m_n - 1, 1 \leq l \leq s_i$ and $x_0 \in C_2(\bigcap_{0\leq j\leq n} D(A_j))$.

The assertions of [20, Propositions 2.3, 2.5, and 2.6] can be reformulated in our new framework (cf. also [22, Proposition 2.2]). This is also the case with the assertion of [20, Theorem 2.8], as the following theorem shows:

Theorem 2.1. (SC1) *Suppose that* $S = \emptyset$, $C_2 \in L(E)$ *is injective,* $f \in C([0,\tau) : E)$, *and* $((W_{i,l}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ *is a k-regularized* C_2*-uniqueness propagation family for* (1.1). *If a function* $u(t)$ *is a strong solution of problem* (1.3)*, then*

$$\Big[W_{i,l}(\cdot) - \chi_{\mathcal{D}_i^l}(n)\big(k * g_{\alpha_n+i-m_n}\big)(\cdot)C_2\Big] * f$$
$$= \sum_{j=1}^{n-1}\Big[\chi_{\mathcal{D}_i^l}(j)\big(g_{\alpha_n+i-m_j} * kC_2 * A_ju\big)(\cdot)$$
$$- \chi_{\mathcal{D}_i^l}(n)\big(g_{2\alpha_n-\alpha_j+i-m_n} * kC_2 * A_ju\big)(\cdot)\Big]$$
$$- \Big[\chi_{\mathcal{D}_i^l}(0)\big(g_{\alpha_n+i-m} * kC_2 * Au\big)(\cdot)$$
$$- \chi_{\mathcal{D}_i^l}(n)\big(g_{2\alpha_n-\alpha+i-m_n} * kC_2 * Au\big)(\cdot)\Big].$$

If $S \neq \emptyset$ *and* $((W_{i,l}(t))_{t\in[0,\tau)},\ (W_{0,1}(t))_{t\in[0,\tau)})_{1\leq i\leq m_n-1,1\leq l\leq s_i}$ *is a k-regularized* C_2*-uniqueness propagation family for* (1.1)*, then we also have the following equality on* $[0,\tau)$:

$$\Big[W_{0,1}(\cdot)x - \chi_S(n)k(\cdot)C_2\Big] * f$$
$$= \sum_{j=1}^{n-1}\big(\chi_S(j) - \chi_S(n)\big)\big(g_{\alpha_n-\alpha_j} * kC_2 * A_ju\big)(\cdot)$$
$$- \big(\chi_S(0) - \chi_S(n)\big)\big(g_{\alpha_n-\alpha} * kC_2 * Au\big)(\cdot).$$

(SC2) *Suppose that* $C_2 \in L(E)$ *is injective,* $(W(t))_{t\in[0,\tau)}$ *is a* k*-regularized* C_2*-uniqueness propagation family for* (1.1)*, and* $f \in C([0,\tau) : E)$*. If a function* $u(t)$ *is a strong solution of problem* (1.3)*, then the following equality holds on* $[0,\tau)$:

$$\Big[W(\cdot)x - k(\cdot)C_2\Big] * f \\ = \big(g_{\alpha_n-\alpha} * kC_2 * Au\big)(\cdot) - \sum_{j=1}^{n-1}\big(g_{\alpha_n-\alpha_j} * kC_2 * A_j u\big)(\cdot).$$

As explained in a series of our recent papers, the notion of a k-regularized C_1-existence propagation family is probably the best theoretical concept for the investigation of existence of integral solutions of non-degenerate abstract time-fractional equation (1.1) with $A_j \in L(E)$, $1 \leq j \leq n-1$; the Laplace transform cannot be so simply applied in the case that there is an index $j \in \mathbb{N}_{n-1}$ such that $A_j \notin L(E)$. In contrast to the above, it is very simple to reformulate the assertions of [20, Theorem 2.9 (ii)] and [22, Theorem 2.4], concerning the Laplace transform of k-regularized C_2-uniqueness propagation families, to degenerate differential equations with Riemann-Liouville fractional derivatives. Details can be left to the interested reader. The assertions of [20, Theorem 2.10-Theorem 2.12] can be rephrased for abstract degenerate multi-term problems with Riemann-Liouville fractional derivatives, as well. Having these done, it is not difficult to reconsider [20, Example 5.1 (i)] in our new setting. Furthermore, it is not so difficult to construct some examples of local k-regularized I-resolvent propagation families for (1.1); see e.g. [20, Example 5.2] and [22, Example 2.1].

Now we would like to state the following result on the existence of strong solutions of equation (1.1) (cf. [24, Theorem 3.1] and [22, Theorem 3.13] for the corresponding statement in the case of equations with Caputo fractional derivatives).

Theorem 2.2. *Suppose* A, B, $A_1, \ldots, A_{n-1}$ *are closed linear operators on* E, $\omega > 0$, $0 < \tau < \infty$, $C \in L(E)$ *is injective,* $f(t) \equiv 0$, *the operator* P_λ *is injective for* $\lambda > \omega$ *and* $D(P_\lambda^{-1}C) = E$, $\lambda > \omega$.

(SC1) *Suppose* $1 \leq i \leq m_n - 1$, $1 \leq l \leq s_i$, $n \in \mathcal{D}_i^l$, $Cx_{i,l} \in D(P_\lambda^{-1}A_j)$, *provided* $\lambda > \omega$ *and* $j \in \mathbb{N}_n^0 \cap \mathcal{D}_i^l$, *as well as* $\alpha_j - \alpha_n + m_n - 1 - i < 0$, *provided* $j \in \mathbb{N}_{n-1}^0 \setminus \mathcal{D}_i^l$, *and the following holds:*

$$\lambda^{\alpha_n} P_\lambda^{-1}\Bigg[\lambda^{m_n-i-1-\alpha} BCx_{i,l} + \sum_{j=1}^{n-1} A_j\Big(\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1-\alpha}Cx_{i,l}\Big) \\ - A\Big(\chi_{\mathcal{D}_i^l}(0)\lambda^{m-i-1-\alpha}Cx_{i,l}\Big)\Bigg] - \lambda^{m_n-1-i}Cx_{i,l} \in LT_E \qquad (2.7)$$

and

$$A_j\Biggl\{\lambda^{\alpha_j}P_\lambda^{-1}\Biggl[\lambda^{m_n-i-1-\alpha}BCx_{i,l}+\sum_{j=1}^{n-1}A_j\Bigl(\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1-\alpha}Cx_{i,l}\Bigr)$$
$$-A\Bigl(\chi_{\mathcal{D}_i^l}(0)\lambda^{m-i-1-\alpha}Cx_{i,l}\Bigr)\Biggr]-\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1}Cx_{i,l}\Biggr\}\in LT_E. \qquad (2.8)$$

Then there exists a strong solution of (1.1) *on* $(0,\tau)$, *with initial value* $x_{i,l}$ *replaced by* $Cx_{i,l}$ *and other initial values chosen to be zeroes. If* $S\neq\emptyset$ *and the above conditions holds for the initial value* $x_{0,1}=u(0)$ *with set* $\mathcal{D}_i^l$ *replaced by* S $(i=0,\,l=1)$, *then there exists a strong solution of* (1.1) *on* $(0,\tau)$, *with initial value* $x_{0,1}$ *replaced by* $Cx_{0,1}$ *and other initial values chosen to be zeroes.*

(SC2) *Suppose* $Cx_{0,1}\in D(B)$, $\lambda^{1-\alpha}P_\lambda^{-1}BCx_{0,1}-Cx_{0,1}\in LT_E$ *and* $\lambda^{\alpha_j-\alpha}A_jP_\lambda^{-1}BCx_{0,1}\in LT_E$. *Then there exists a strong solution of* (1.1) *on* $(0,\tau)$, *with initial value* $x_{0,1}$ *replaced by* $Cx_{0,1}$ *and other initial values chosen to be zeroes.*

PROOF. We will prove the assertion of theorem only in the case (SC1) with $1\leq i\leq m_n-1$ and $1\leq l\leq s_i$. Let $u_{i,l}\in L^1_{\mathrm{loc}}([0,\infty):E)$ and $F_{i,l,n}\in L^1_{\mathrm{loc}}([0,\infty):E)$ satisfy:

$$\int_0^\infty e^{-\lambda t}u_{i,l}(t)\,dt=P_\lambda^{-1}\Biggl[\lambda^{m_n-i-1-\alpha}BCx_{i,l}+\sum_{j=1}^{n-1}A_j\Bigl(\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1-\alpha}Cx_{i,l}\Bigr)$$
$$-A\Bigl(\chi_{\mathcal{D}_i^l}(0)\lambda^{m-i-1-\alpha}Cx_{i,l}\Bigr)\Biggr]$$

and

$$\int_0^\infty e^{-\lambda t}F_{i,l,n}(t)\,dt=\lambda^{\alpha_n}P_\lambda^{-1}\Biggl[\lambda^{m_n-i-1-\alpha}BCx_{i,l}-A\Bigl(\chi_{\mathcal{D}_i^l}(0)\lambda^{m-i-1-\alpha}Cx_{i,l}\Bigr)$$
$$+\sum_{j=1}^{n-1}A_j\Bigl(\chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1-\alpha}Cx_{i,l}\Bigr)\Biggr]-\lambda^{m_n-1-i}Cx_{i,l},$$

for $\lambda>\omega$ suff. large; cf. (2.7). By performing the Laplace transform, it can be easily checked that:

$$\bigl(g_{m_n}*F_{i,l,n}\bigr)(t)=\bigl(g_{m_n-\alpha_n}*u_{i,l}\bigr)(t)-g_{i+1}(t)Cu_{i,l},\quad t>0.$$

This implies that $D_t^{\alpha_n} u_{i,l}(t)$ is well defined for $t > 0$ (more precisely, on any finite subinterval of $(0, \infty)$) and $F_{i,l,n}(t) = D_t^{\alpha_n} u_{i,l}(t)$, $t > 0$. Keeping in mind that $n \in \mathcal{D}_i^l$ and $\alpha_j - \alpha_n + m_n - 1 - i < 0$ for $j \in \mathbb{N}_{n-1}^0 \setminus \mathcal{D}_i^l$, we can conclude from (2.7) that $\lambda^{\alpha_j} \widetilde{u_{i,l}}(\lambda) - \chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1}Cx_{i,l} \in LT_E$ for all $j \in \mathbb{N}_{n-1}^0$, as well as that $D_t^{\alpha_j} u_{i,l}(t)$ is well defined for $t > 0$, with

$$\int_0^\infty \mathrm{e}^{-\lambda t} D_t^{\alpha_j} u_{i,l}(t)\, \mathrm{d}t = \lambda^{\alpha_j} \widetilde{u_{i,l}}(\lambda) - \chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1}Cx_{i,l} \in LT_E,$$

for all $j \in \mathbb{N}_{n-1}^0$. Using (2.8) and [1, Proposition 1.7.6], it readily follows that $A_j D_t^{\alpha_j} u_{i,l}(t)$ is well defined for $t > 0$, and

$$\int_0^\infty \mathrm{e}^{-\lambda t} A_j D_t^{\alpha_j} u_{i,l}(t)\, \mathrm{d}t = A_j \left[\lambda^{\alpha_j} \widetilde{u_{i,l}}(\lambda) - \chi_{\mathcal{D}_i^l}(j)\lambda^{m_j-i-1}Cx_{i,l} \right] \in LT_E,$$

for all $j \in \mathbb{N}_{n-1}^0$. Finally, a simple calculation yields that:

$$\int_0^\infty \mathrm{e}^{-\lambda t} \Big[BD_t^{\alpha_n} u_{i,l}(t) + A_{n-1} D_t^{\alpha_{n-1}} u_{i,l}(t) + \cdots + A_1 D_t^{\alpha_1} u_{i,l}(t) - AD_t^{\alpha} u_{i,l}(t) \Big]\, dt = 0,$$

which implies by the uniqueness theorem for the Laplace transform that $u_{i,l}(\cdot)$ is a strong solution of the problem (1.1) with initial value $x_{i,l}$ replaced by $Cx_{i,l}$ and other initial values chosen to be zeroes.

Remark 2.1. Consider the subcase (SC1), suppose first that $1 \leq i \leq m_n - 1$, $1 \leq l \leq s_i$, $n \in \mathcal{D}_i^l$, as well as that $\alpha_j - \alpha_n + m_n - 1 - i < 0$, provided $j \in \mathbb{N}_{n-1}^0 \setminus \mathcal{D}_i^l$. Then a straightforward computation shows that the assertion of Theorem 2.2 continues to hold if we replace the term

$$P_\lambda^{-1} \Bigg[\lambda^{m_n-i-1-\alpha} BCx_{i,l} + \sum_{j=1}^{n-1} A_j \Big(\chi_{\mathcal{D}_i^l}(j) \lambda^{m_j-i-1-\alpha} Cx_{i,l} \Big) - A\Big(\chi_{\mathcal{D}_i^l}(0) \lambda^{m-i-1-\alpha} Cx_{i,l} \Big) \Bigg],$$

i.e., the Laplace transform of solution $t \mapsto u_{i,l}(t)$, $t > 0$ with the term

$$P_\lambda^{-1}\Biggl[\lambda^{m_n-i-1-\alpha_n}P_\lambda C x_{i,l} - \sum_{j\in\mathbb{N}_{n-1}\setminus\mathcal{D}_i^l} A_j\Bigl(\lambda^{m_n-i-1-\alpha_n+\alpha_j-\alpha}Cx_{i,l}\Bigr) + A\Bigl(\chi_{\mathbb{N}_{n-1}\setminus\mathcal{D}_i^l}(0)\lambda^{m_n-i-1-\alpha_n}Cx_{i,l}\Bigr)\Biggr],$$

and suppose that $Cx_{i,l} \in D(P_\lambda^{-1}A_j)$ for $\lambda > \omega$ and $j \in \mathbb{N}_{n-1}^0 \setminus \mathcal{D}_i^l$, instead of $Cx_{i,l} \in D(P_\lambda^{-1}A_j)$ for $\lambda > \omega$ and $j \in \mathbb{N}_{n-1}^0 \cap \mathcal{D}_i^l$. If $S \neq \emptyset$, $i = 0$ and $l = 1$, then one has to replace the set $\mathcal{D}_i^l$ with S. The corresponding analysis of the subcase (SC2) is left to the interested reader.

2.1. k-regularized (C_1, C_2)-existence and uniqueness families for (1.1)

In this subsection, we assume that X (the state space) and Y are complex Banach spaces; the norm of an element $y \in Y$ will be denoted by $\|y\|_Y$, while the abbreviation $L(Y, X)$ stands for the space which consists of all bounded linear operators from Y into X. Now the closed linear operators A, B, $A_1, \ldots, A_{n-1}$ are acting on X.

We need the following definition from [22].

Definition 2.5. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $C_1 \in L(Y,X)$, and $C_2 \in L(X)$ is injective.

(i) A strongly continuous operator family $(E(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence family iff, for every $y \in Y$, the following holds: $E(\cdot)y \in C^{m_n-1}([0,\tau) : [D(B)])$, $E^{(i)}(0)y = 0$ for every $i \in \mathbb{N}_0$ with $i < m_n - 1$, $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(\cdot)y \in C([0,\tau) : X)$ for $0 \leq j \leq n$, and

$$BE^{(m_n-1)}(t)y + \sum_{j=1}^{n-1} A_j\bigl(g_{\alpha_n-\alpha_j} * E^{(m_n-1)}\bigr)(t)y - A\bigl(g_{\alpha_n-\alpha} * E^{(m_n-1)}\bigr)(t)y = k(t)C_1y,$$

for any $t \in [0,\tau)$.

(ii) A strongly continuous operator family $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_2-uniqueness family iff, for every $\tau \in [0,\tau)$

and $x \in \bigcap_{0\leq j\leq n} D(A_j)$, the following holds:

$$U(t)Bx+\sum_{j=1}^{n-1}\bigl(g_{\alpha_n-\alpha_j}*U(\cdot)A_jx\bigr)(t) \\ -\bigl(g_{\alpha_n-\alpha}*U(\cdot)Ax\bigr)(t)y=\bigl(k*g_{m_n-1}\bigr)(t)C_2x.$$

If $k(t)=g_{\zeta+1}(t)$, where $\zeta\geq 0$, then it is also said that $(E(t))_{t\in[0,\tau)}$ is a ζ-times integrated C_1-existence family; 0-times integrated C_1-existence family is also said to be a C_1-existence family. A similar notion can be introduced for all other classes of uniqueness and resolvent families introduced in Definition 2.5.

The first part of subsequent theorem can be proved following the analysis carried out in [21, Section 3]; the second part of this theorem can be proved by using [22, Theorem 3.4 (ii)] and the fact that $D_t^{\zeta}u(t)=\mathbf{D}_t^{\zeta}u(t)$, $t>0$, provided that $\zeta>0$, $\mathbf{D}_t^{\zeta}u(t)$ is defined and $u^{(i)}(0)=0$ for all $i\in\mathbb{N}^0_{\lceil\zeta\rceil-1}$. Observe also that we can reformulate the final conclusions from [16, Theorem 2.2] for degenerate multi-term problems with Riemann-Liouville fractional derivatives and that it is very difficult to state, in contrast to the equations with Caputo fractional derivatives, some satisfactory results on the existence of strong solutions of (1.1), provided that there exists a (local) C_1-existence family.

Theorem 2.3. *Let $0<\tau\leq\infty$, let $C_1\in L(Y,X)$, and let $C_2\in L(X)$ be injective.*

(i) *Suppose that $(E(t))_{t\in[0,\tau)}$ is a (local) C_1-existence family, $1\leq i\leq m_n-1$ and $1\leq l\leq s_i$. Then there exists a mild solution of the problem (2.5) provided that $Bx_{i,l}\in R(C_1)$, if $n\in\mathcal{D}_i^l$, and that, for every $j\in\mathbb{N}^0_{n-1}\cap\mathcal{D}_i^l$, there exists an element $y_{i,l,j}\in Y$ such that $A_jx_{i,l}=C_1y_{i,l,j}$. If $S\neq\emptyset$, $i=0$ and $l=1$, then the above holds with the set $\mathcal{D}_i^l$ replaced by S.*

(ii) *Suppose that $(U(t))_{t\in[0,\tau)}$ is a k-regularized C_2-uniqueness family and $k(t)$ is a kernel on $[0,\tau)$. Then every two strong (mild) solutions of the equation (1.1) possessing the same initial conditions (cf. (2.1) and (2.4)) are identically equal on $[0,\tau)$.*

In [22, Theorem 3.6], we have considered inhomogeneous degenerate multi-term problems with Caputo fractional derivatives. Similarly we can consider the inhomogeneous degenerate multi-term problems with Riemann-Liouville fractional derivatives.

As indicated in [22], our results can be applied in the analysis of some atypical multi-term differential equations in L^p-type spaces. For example, making use of Theorem 2.9 and the analysis contained in [22, Example 3.2], we can prove certain results about the existence and uniqueness of mild solutions of the integral equations associated with fractional analogons of the following equation:

$$\begin{cases} (\Delta - I)u_{tt}(t,x) + F_1 u_t(t,x) = \Big(\sum_{|\eta|\leq Q} a_\eta D^\eta + F_0\Big)u(t,x), \\ u(0,x) = u_0(x) = \phi(x),\ u_t(0,x) = u_1(x) = \psi(x), \end{cases}$$

involving Riemann-Liouville fractional derivatives ($1 < p < \infty$, $X = L^p(\mathbb{R}^n)$, $F_0 \in L(X)$ and $F_1 \in L(X)$ satisfy certain properties). Details can be left to the interested reader.

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