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# LAPLACE TRANSFORM OF FUNCTIONS DEFINED ON A BOUNDED INTERVAL

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*A b s t r a c t. Laplace transform $\dot{\mathcal{L}}$ for functions belonging to $L[0,b],\ 0 < b < \infty$ is defined. This definition is given by using the idea of H. Komatsu* [*J. Fac. Sci. Univ. Tokyo, IA,* **34** (1987), 805–820] *and* [*Structure of solutions of differential equations* (*Katata/Kyoto,* 1995)*, pp.* 227–252*, World Sci. Publishing, River Edge, NJ,* 1996]. *for Laplace hyperfunctions. As an application of $\dot{\mathcal{L}}$ we solve an equation with fractional derivative and an integral equation of the first kind of convolution type.*

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## 1. *Introduction*

Laplace transform of numerical functions has been elaborated as a powerful mathematical theory very useful in practice and many a time applied by engineers. Although it has been belived to have two important shortcomings. First, application of the Laplace transform (In short, LT) not only to functions, but to distributions, ultradistributions, and hyper-functions, calls for some growth conditions of them ([7], [9], [17], [20], [21], [22] and [23]). Secondly, there is no simple characterisation of the functions which are LT of the numerical functions. Hence, we are not always sure whether or not an obtained function $f(s)$ is the LT of a function $g(t)$ of exponential type.

To overcome these difficulties mathematiciens defined LT of functions as classes ([4], [6]) without a rich repercussions, or used algebraic approaches to the Heaviside calculus ([12], [16]).

H. Komatsu [9], [10] overcome successfully all defects of the classical LT. He defined the LT of Laplace hyperfunctions and of hyperfunctions, as well, but in one dimensional case. Since it is a very abstract theory, it cannot be easily accepted by the greater part of people working in applications.

In [20] a definition was developed of LT applicable to locally Bochner integrable Banach space valued functions with arbitrary growth at infinity, based on old ideas (cf. [4], [6]). For $f \subset \mathbb{L}_{\mathrm{loc}}$ this LT coincides with the LT defined by Komatsu.

The aim of the paper [18] was to define and to elaborate the LT of a subset of distributions which contains also the space $\mathbb{L}_{\mathrm{loc}}(\mathbb{R})$, distributions with compact supports and tempered distributions. The methods of Komatsu can not be applied directly because the space of distributions is not a flabby sheaf.

In this paper we define the LT, $\dot{\mathcal{L}}$, of elements of the space $L[a,b]$, $0 < b < \infty$; the function $f \in L[0,b]$ if there exists $\int\limits_0^b f(\tau)\,\mathrm{d}\tau$ in the sense of Lebesgue. This definition is easy accesible to every mathematicien who works with classical LT; because $\dot{\mathcal{L}}$ is defined by the classical Laplace transform $\mathcal{L}$ of a class of functions. In such a way to prove some properties of $\dot{\mathcal{L}}$ we use properties of $\mathcal{L}$.

## 2. *Some spaces we use*

**Vector space $\boldsymbol{L[0,b]}$, $\boldsymbol{0 \le b < \infty}$.** The function $f \in L[0,b]$ if there exists $\int\limits_0^b f(\tau)\,\mathrm{d}\tau$ in the sense of Lebesgue; $\int\limits_0^b f(\tau)\,\mathrm{d}\tau = 0 \Leftrightarrow f(t) = 0$ a.e. on $[0,b]$.

**Vector space of locally integrable functions.** $f \in L_{\mathrm{loc}}[0,\infty)$ if for any $T_1 < T_2 \in [0,\infty)$, $f \in L[T_1,T_2]$. If $f$ and $g$ belong to $L_{\mathrm{loc}}[0,\infty)$, then their convolution, $f*g \in L_{\mathrm{loc}}[0,\infty)$, as well. The convolution is an operation commutative, associative and distributive.

**Vector space $\boldsymbol{L^{\exp}[0,\infty)}$.** The function $f \in L^{\exp}[0,\infty)$ if $f \in L_{\mathrm{loc}}[0,\infty)$ and such that for an $s_0 > 0$ there exists $\int_0^\infty |\mathrm{e}^{-s_0\tau} f(\tau)|\,\mathrm{d}\tau$. With operation addition this is a vector space on $\mathbb{C}$. A function $f \in L^{\exp}[b,\infty)$ if $f \in L^{\exp}[0,\infty)$ and $f(t) = 0,\ t < b,\ b < \infty$. $L^{\exp}[0,\infty)$ is also a vector space.

**Lemma 2.1.** *If $f, g \in L^{\exp}[0,\infty)$, then $f * g \in L^{\exp}[0,\infty)$.*

**Lemma 2.2.** *If* $f \in L^{\exp}[0,\infty)$*, and* $f \notin L^{\exp}[c,\infty)$*, for any* $c > 0$*, but* $g \in L^{\exp}[b,\infty)$*, then* $f * g \in L^{\exp}[b,\infty)$*.*

Proofs of these lemmas can be found in [7], p. 123 and p. 131, respectively.

**Vector space $L^{\exp}[0,\infty)/L^{\exp}[b,\infty)$.** In $L^{\exp}[0,\infty)$ we define a two elements relation: $f \sim g = f - g \in L^{\exp}[b,\infty)$, $b > 0$. Since $L^{\exp}[b,\infty)$ is a vector space, a subspace of $L^{\exp}[0,\infty)$, the relation is an equivalence relation in accordance with the vector space $L^{\exp}[0,\infty)$. The equivalence classes are elements of $L_b = L^{\exp}[0,\infty)/L^{\exp}[b,\infty)$. An element $f_b \in L_b$ is defined by $f + L^{\exp}[b,\infty)$, where $f \in L^{\exp}[0,\infty)$. In $L_b$ is defined the addition and product by $r \in \mathbb{R}$.

If $f_b, g_b \in L_b$, and $r \in \mathbb{R}$, then

$$f_b + g_b = f + g + L^{\exp}[b,\infty), \tag{2.1}$$

and

$$rf_b = (rf)_b \,.$$

With this two operation $L_b$ is also a vector space.

## 3. *Properties of the space $L_b$*

**Lemma 3.1.** *Every function* $f \in L[0,b]$ *can be extended to a function* $f \in L^{\exp}[0,b]$*. The space* $L_b$ *is algebraically isomorph to* $L[0,b]$*.*

Proof. One extension of the function $f \in L[0,b]$ is the function $\bar{f}(t) = L^{\exp}[0,\infty)$ such that $\bar{f}(t) = f(t)$, $0 \le t \le b$.

Let $f_1 \in L^{\exp}[0,\infty)$. It defines the class $f_b = f_1 + L^{\exp}[b,\infty) \in L_b$. Then

$$(f_1(t) + L^{\exp}[b,\infty))\big|_{[0,b]} = f(t) \in L[0,b].$$

Here, $\bar{f}\big|_{[0,b]}$ is the restriction of $\bar{f}$ on $[0,b]$.

Conversely, let $f \in L[0,b]$. It can be extended to $\bar{f} \in L^{\exp}[0,\infty)$ which defines the class $f_b = \bar{f} + L^{\exp}[b,\infty)$.

Finally, operation "$\big|_{[0,b]}$" is an isomorphism of $L_b \Leftrightarrow L[0,b]$.

**Convolution in $L_b$.** If $f_b, g_b \in L_b$, $f_b = f + L^{\exp}[0,\infty)$, $g_b = g + L^{\exp}[0,\infty)$, then there exists

$$f_b * g_b = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau, \quad t \ge 0$$

and this convolution belongs to $L^{\exp}[0,\infty)$ (see [7] Theorem 2, p. 123). Consequently,

$$f_b * g_b = \overline{f * g} + L^{\exp}[b,\infty) = \overline{f} * \overline{g} + L^{\exp}[b,\infty).$$

**Lemma 3.2.** *If $f, g \in L[0,b]$ and $\bar{f}, \bar{g}$ be theirs extensions in $L^{\exp}[0,\infty)$, then $f * g \Leftrightarrow \bar{f} * \bar{g} + L^{\exp}[b,\infty) \in L_b$.*

PROOF. By Lemma 2.1 we have

$$f_b * g_b = \overline{f * g} + L^{\exp}[b,\infty) \Leftrightarrow f * g. \qquad \square \tag{3.1}$$

The next two theorems give the extensions of functions having the derivatives.

**Theorem 3.1.** *If the function $f$ has its derivative $f'(t)$ in every point of $(a,b]$ (the derivative in $b$ means the left derivative) and if $f'(t) \in L[0,b]$, then $f'(t)$ can be extended on $[0,\infty)$ so that $\overline{f'}(t) = \overline{f}'(t)$. The function $\bar{f}'$ extends $f'$ on $(0,\infty)$ and $\bar{f}$ extends $f$ on $(0,\infty)$. Consequently $f' \Leftrightarrow \overline{f}' + L^{\exp}[b,\infty)$.*

PROOF. Let $\overline{f}(t)$ and $\overline{f}'(t) \in L^{\exp}[0,\infty)$ extend $f$ and $f'$. If the function $f$ has the left derivative in the point $b$, then it is well-known that there exists

$$\lim_{h\to 0^+} \frac{f(b)-f(b-h)}{h} = f'(b-0^+)$$

and

$$\lim_{t\to b^-} f(t) = f(b-0).$$

First we extended $f$ in such a way that

$$\bar{f}(t) = f(t) \quad (0<t<b) \quad \text{and} \quad \lim_{t\to b^+} \bar{f}(t) = f(b-0).$$

Then,

$$\lim_{t\to b^-} \bar{f}(t) = \lim_{t\to b^+} f(t) = f(b-0).$$

The constructed $\bar{f}$ is a continuous function on $t=b$. Now, we can extend $f'(t)$ on $[0,\infty)$ as

$$\begin{aligned} \overline{f'}(t) &= \overline{f}'(t) = f'(t), \qquad 0<t<b, \\ \lim_{t\to b^+} \overline{f'}(t) &= \lim_{t\to b^-} \overline{f}'(t) = f'(b-0). \end{aligned}$$

In $t=b$ we have

$$\lim_{t\to b^-} \overline{f'}(t) = \lim_{t\to b^+} \overline{f}'(t) = f'(b-0).$$

In such a way we have $\overline{f'}(t) = \overline{f}'(t),\ t \geq 0$,

$$f' \Leftrightarrow \overline{f}'(t) + L^{\exp}[b,\infty) \tag{3.2}$$

and $\lim_{t\to 0^+} f(t) = f(0^+)$ exists (see [7], p. 99).

**Theorem 3.2.** *If the function $f$ has $n$ derivatives: $f^{(1)}(t), \dots, f^{(n)}(t)$, $n \geq 2$, in every point of the interval $(a,b]$ and $f^{(n)}(t) \in L[0,b]$, then also $f, \dots, f^{(n-1)} \in L[0,b]$ and $f^{(n)}(t)$ can be extended on $[0,\infty)$ such that*

$$\overline{f^{(n)}}(t) = \overline{f}^{(n)}(t), \quad t > 0$$

$$f^{(n)}(t) \;\Leftrightarrow\; \overline{f}^{(n)}(t) + L^{\exp}[b,\infty). \tag{3.3}$$

*and* $\lim\limits_{t\to 0^+} f^{(i)}(t)$ *exist for every* $i = 0, 1, \dots, n-1$.

Proof goes just in the same manner as for Theorem 3.1 (see [7], p. 100).

## 4. *Fractional derivatives*

CASE $0 < \alpha < 1$, $\alpha = [\alpha] + \gamma = \gamma \in (0,1)$. By definition, *Riemann-Lionville fractional derivative* ${}_0D_t^\alpha f$, $0 < \alpha < 1$, $f \in L[0,b]$ is

$${}_0D_t^\alpha f = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} * f(\tau)\right)(t). \tag{4.1}$$

**Theorem 4.1.** $1^\circ$ *If $f$ has the derivative $f^{(1)}(t)$ in every point of $(0,b]$ and if $f^{(1)}(t) \in L[0,b]$, then there exists $f(0^+)$ and*

$$\begin{aligned} {}_0D_t^\alpha f \quad &\Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} * \bar{f}(\tau)\right)(t) + L^{\exp}[b,\infty) \\ &\Leftrightarrow \quad \left(\frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} * \bar{f}^{(1)}(\tau)\right)(t) + f(0^+)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + L^{\exp}[b,\infty). \end{aligned} \tag{4.2}$$

(For the proof see Theorem 3.1 and [7], pp. 117–118.)

$2^\circ$ *If the function $\left(\frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} * f(\tau)\right)(t)$ has the derivative in every point $(0,b]$ and this derivative belongs to $L[0,b]$, then*

$$\begin{aligned} {}_0D_t^\alpha f \quad &\Leftrightarrow \quad \overline{\left(\frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} * f(\tau)\right)^{(1)}}(t) + L^{\exp}[b,\infty) \\ &\Leftrightarrow \quad \left(\frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} * \bar{f}(\tau)\right)^{(1)} + L^{\exp}[b,\infty). \end{aligned} \tag{4.3}$$

The proof follows from Theorem 3.1.

CASE $\alpha > 1,\ \alpha = [\alpha] + \gamma,\ \gamma \in (0,1)$.

**Theorem 4.2.** $1^\circ$ *If* $f^{(k)}(t),\ k = 0, 1, \ldots, n,\ n \geq 2$, *exist in every point* $t \in (0, b]$ *and* $f^{(n)} \in L[0, b],\ n = [\alpha] + 1$, *then*

$$
\begin{aligned}
{}_0D_t^\alpha f \ &\Leftrightarrow\ D^{[\alpha]+1}\left(\frac{\tau^{-\gamma}}{\Gamma(1-\gamma)} * \bar{f}(\tau)\right) + L^{\exp}[b, \infty) \\
&\Leftrightarrow\ \left(\frac{\tau^{-\gamma}}{\Gamma(1-\gamma)} * \bar{f}^{([\alpha]+1)}(\tau)\right)(t) + f(0^+)\left(\frac{t^{-\gamma}}{\Gamma(1-\gamma)}\right)^{([\alpha])} \\
&\quad + f^{(1)}(0^+)\left(\frac{t^{-\gamma}}{\Gamma(1-\gamma)}\right)^{([\alpha]-1)} + \cdots \\
&\quad + f^{([\alpha])}(0^+)\frac{t^{-\gamma}}{\Gamma(1-\gamma)} + L^{\exp}[b, \infty).
\end{aligned}
\tag{4.4}
$$

(The proof goes as the proof of Theorem 4.1, part $1^\circ$. To have ${}_0D_t^\alpha f \in L[0, b]$ we must suppose: $f(0) = \cdots = f^{([\alpha]-1)} = 0$.)

$2^\circ$ *If the function* $F(t) \equiv \left(\frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} * f(\tau)\right)(t)$ *has* $n$ *derivatives,* $n \geq 2$ *in every point of interval* $(0, b]$ *and* $n$*-th derivative belongs to* $L[0, b]$, *then also* $i$*-th derivative* $F^{(i)},\ i = 0, 1, \ldots, n-1$, *belongs to* $L[0, b]$ *and* $F^{(n)}(t)$ *can be extended in* $[0, \infty)$ *such that* $\overline{F^{(n)}}(t) = \overline{F}^{(n)}(t),\ t > 0$, *and*

$$
{}_0D_t^\alpha f \ \Leftrightarrow\ \overline{\left(\frac{\tau^{-\alpha}}{\Gamma(1-\gamma)} * f(\tau)\right)}^{(n)}(t) + L^{\exp}[b, \infty).
\tag{4.5}
$$

The proof goes as the proof of Theorem 4.1, part $2^\circ$.

**Caputo fractional derivative.** By definition for $\alpha = [\alpha] + \gamma$, and $f \in L[0, b]$, we have for Caputo fractional derivative ${}^cD_{0+}^\alpha f$:

$$
\begin{aligned}
\left({}^cD_{0+}^\alpha f\right)(t) \ &= \ \left(\frac{\tau^{-\gamma}}{(\Gamma(1-\gamma))} * f^{([\alpha]+1)}(\tau)\right)(t) \\
&\Leftrightarrow\ \overline{\left(\frac{\tau^{-\gamma}}{(\Gamma(1-\gamma))} * f^{([\alpha]+1)}(\tau)\right)}(t) + L^{\exp}[b, \infty) \\
&\Leftrightarrow\ \left(\frac{\tau^{-\gamma}}{(\Gamma(1-\gamma))} * \overline{f}^{([\alpha]+1)}(\tau)\right) + L^{\exp}[b, \infty).
\end{aligned}
$$

**Fractional integrals**. We use Rimann-Liouville fraction integral

$$
\left({}_0I_t^\alpha f\right)(t) = \left(\frac{\tau^{\alpha-1}}{\Gamma(\alpha)} * f(\tau)\right) \Leftrightarrow \left(\frac{\tau^{\alpha-1}}{\Gamma(\alpha)} * \bar{f}(\tau)\right)(t) + L^{\exp}[b, \infty).
$$

## 5. *Laplace transform $\dot{\mathcal{L}}$ of elements of $L[0,b]$ given by Laplace transform $\mathcal{L}$ of elements of $L_b$*

If $f \in L[0,b]$, then

$$\dot{\mathcal{L}}\{f\}(s) \Leftrightarrow \mathcal{L}\{\bar{f}\} + \mathcal{L}L^{\exp}[b,\infty),$$

or

$$\dot{\mathcal{L}}\left\{L[0,b]\right\} \Leftrightarrow \mathcal{L}\left\{L^{\exp}[0,\infty)\right\}/\mathcal{L}\{L[b,\infty)\}.$$

The inverse operator of $\dot{\mathcal{L}}$, operator $\dot{\mathcal{L}}^{-1}$, is

$$\dot{\mathcal{L}}^{-1}\left\{\mathcal{L}\left\{\bar{f}\right\} + L^{\exp}[b,\infty)\right\}(t) = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\bar{f}\right\} + \mathcal{L}L^{\exp}[b,\infty)\right\}\Big|_{[0,b]}.$$

Consequently, *every function $f \in L[0,b]$ has its Laplace transform $\dot{\mathcal{L}}\{f\}$.* It is given by the class $\mathcal{L}\{\bar{f}\} + \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}$, where $\bar{f}$ is any extension of $f$ in $L^{\exp}[0,\infty)$.

To find $\dot{\mathcal{L}}\{f\}$ we can use tables of the *classical Laplace transform.* If the function $F(s)$ is the Laplace transform of $f(t) \in L^{\exp}[0,\infty)$, then $f(t)\big|_{[0,b]} \in L[0,b]$ and

$$\dot{\mathcal{L}}\left\{f(t)\big|_{[0,b\}}\right\} \Leftrightarrow F(s) + \mathcal{L}\left\{L^{\exp}\{b,\infty)\right\}.$$

*There exist functions, as it is $F(t) = \exp(t^2),\ t > 0$, which have not the Laplace integrale for any $s$*, but $F(t)\big|_{[0,b]} \in L[0,b]$ has its Laplace transform $\dot{\mathcal{L}}$. Let $\bar{F}(t) = F(t)$, $0 \le t \le 1$, and $\bar{F}(t) = \mathrm{e}$, $t > 1$, then

$$\dot{\mathcal{L}}\left\{F(t)\Big|_{[0,b]}\right\} = \int_0^\infty \mathrm{e}^{-st}\overline{F}(t)\,\mathrm{d}t + \mathcal{L}\left\{L^{\exp}[1,\infty)\right\}.$$

*As regards convolution* defined in Lemma 3.2 we have

$$\dot{\mathcal{L}}\left(f * g\right) \Leftrightarrow \mathcal{L}\{\bar{f}\}\cdot\mathcal{L}\{\bar{g}\} + \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}.$$

**Laplace transform of fractional derivatives.** We analyzed the fractional derivative ${}_0D_t^\alpha f$, in previous pages, dividing $\alpha$ in two cases: $0 < \alpha < 1$ and $1 \le [\alpha]$. To find the Laplace transform of ${}_0D_t^\alpha f$ we do the same.

CASE $0 < \alpha < 1$. Starting from (4.2) (Theorem 4.1 gives the conditions that (4.2) is valid),

$$\begin{aligned}
\dot{\mathcal{L}}\left\{{}_0D_t^\alpha f\right\} \quad &\Leftrightarrow \quad \mathcal{L}\left\{\frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}\right\}(s)\mathcal{L}\{\bar{f}^{(1)}\}(s) + f(0^+)s^{\alpha-1} + \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}(s) \\
&\Leftrightarrow \quad s^{\alpha-1}\left(s\mathcal{L}\{\bar{f}\}(s) - f(0^+)\right) + f(0^+)s^{\alpha-1} + \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}(s) \\
&\Leftrightarrow \quad s^\alpha\mathcal{L}\{\bar{f}\}(s) + \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}(s). \qquad (5.1)
\end{aligned}$$

We have only to prove that $s^\alpha \bar{f}(t) \in L^{\exp}[0,\infty)$. Since $\bar{f} \in L^{\exp}[0,b]$, then the integral $\int_0^\infty |\mathrm{e}^{-s_0 t} f(t)|\,\mathrm{d}t$ converges for an $s_0 > 0$.

Let $0 < \varepsilon < s_0$ for $s_0 > 0$. Then for $s_0 - \varepsilon > 0$ we have

$$\begin{aligned}\lim_{\substack{\omega_1\to\infty\\ \omega_2>\omega_1}} \int_{\omega_1}^{\omega_2} s^\alpha \mathrm{e}^{-s_0 t} f(t)\,\mathrm{d}t &\leq \lim_{\substack{\omega_1\to\infty\\ \omega_2>\omega_1}} \int_{\omega_1}^{\omega_2} |s^\alpha \mathrm{e}^{-\varepsilon t}||\mathrm{e}^{-(s_0-\varepsilon)t} f(t)|\,\mathrm{d}t \\ &\leq \lim_{\substack{\omega_1\to\infty\\ \omega_2>\omega_1}} \int_{\omega_1}^{\omega_2} \mathrm{e}^{-(s_0-\varepsilon)t}|f(t)|\,\mathrm{d}t.\end{aligned}$$

Let us now start with (4.3), then we have:

$$\begin{aligned}\dot{\mathcal{L}}\left\{{}_0D_t^\alpha f\right\}(s) &= s\mathcal{L}\left\{\frac{\tau^{-\gamma}}{\Gamma(1-\gamma)} * \bar{f}(\tau)\right\}(s) \\ &\quad -\left(\frac{\tau^{-\gamma}}{\Gamma(1-\gamma)} * f(\tau)\right)(0^+) + \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}(s). \qquad (5.2)\end{aligned}$$

If $[\alpha] \geq 1,\ \alpha > 1$, then we start with (4.4) (Theorem 3.1 gives the conditions that (4.4) is invalid)

$$\begin{aligned}{}_0D_t^\alpha f \quad &\Leftrightarrow \quad \left(\frac{\tau^{-\gamma}}{\Gamma(1-\gamma)} * \bar{f}^{([\alpha]+1)}(\tau)\right)(t) + f(0)\left(\frac{\tau^{-\gamma}}{\Gamma(1-\gamma)}\right)^{([\alpha])} \\ &\quad + f^{(1)}(0)\left(\frac{\tau^{-\gamma}}{\Gamma(1-\gamma)}\right)^{([\alpha]-1)} + \cdots + f^{([\alpha])}(0)\frac{t-\gamma}{\Gamma(1-\gamma)} \\ &\quad + \left\{L^{\exp}[b,\infty)\right\}. \qquad (5.3)\end{aligned}$$

To have the existence of $\dot{\mathcal{L}}$ of ${}_0D_t^\alpha f$ we must have $f(0) = \cdots = f^{([\alpha]-1)}(0) = 0$ (and $f^{([\alpha])}(0)$ can be $\neq 0$), then

$$\begin{aligned}\dot{\mathcal{L}}\left\{{}_0D_t^\alpha f\right\}(s) &\Leftrightarrow \mathcal{L}\left\{\bar{f}^{([\alpha]+1)}(\tau) * \frac{\tau^{-\gamma}}{\Gamma(1-\gamma)}\right\}(s) + f^{([\alpha])}(0)s^{\gamma-1} \qquad (5.4) \\ &\quad + \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}(s) \\ &\Leftrightarrow s^{\gamma-1}s^{[\alpha]+1}\mathcal{L}\{\bar{f}\} - f^{([\alpha])}s^{\gamma-1} + f^{([\alpha])}(0)s^{\gamma-1} + \mathcal{L}\left\{L^{\exp}[b,\infty)\right\} \\ &\Leftrightarrow s^\alpha\mathcal{L}\{\bar{f}\} + \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}.\end{aligned}$$

Let us start with (4.5), then (Theorem 4.1 gives the conditions that (4.5) is valid)

$$\begin{aligned}\dot{\mathcal{L}}\left\{{}_0D_t^{\alpha} f\right\}(s) \quad &\Leftrightarrow \quad \mathcal{L}\left\{\left(\overline{\frac{\tau^{-\gamma}}{\Gamma(1-\alpha)} * f(\tau)}\right)^{(n)}(t)\right\}(s) + \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}(s) \\ &\Leftrightarrow \quad s^n\left[\mathcal{L}\left\{\left(\frac{\tau^{-\gamma}}{\Gamma(1-\gamma)} * \overline{f}(\tau)\right)\right\}(s)\right. \\ &- \quad \left.\sum_{k=0}^{n-1}\left(\frac{\tau^{-\gamma}}{\Gamma(1-\gamma)} * f(\tau)\right)^{(k)}(0^+)s^{-k-1}\right] + \mathcal{L}L^{\exp}[b,\infty)(s).\end{aligned} \tag{5.5}$$

In this way we have $\dot{\mathcal{L}}\left\{{}_0D_t^{\alpha} f\right\}$ in both cases, $0<\alpha<1$ and $\alpha>1$.

**Laplace transform of Caputo fractional derivative.** Here

$$\begin{aligned}\left(\dot{\mathcal{L}}({}^cD_{0+}^{\alpha} f)\right)(s) \;\Leftrightarrow\; & s^{\alpha}\left(\mathcal{L}\overline{f}\right)(s) - f(0)s^{\alpha-1} - f^{(1)}(0)s^{\alpha-2} - \cdots \\ & - f^{([\alpha])}(0)s^{\gamma-1} + \mathcal{L}L^{\exp}[b,\infty)(s).\end{aligned}$$

## 6. *Applications of the Laplace transform of functions defined on the bounded interval*

Laplace transform is often used by processes observed in time, but in this case with limitation which bring Laplace transform. The equation

$$y'(t) + y(t) = \exp(t^2), \quad t>0,$$

can not be treated by the Laplace transform because the function $\exp(t^2)$, but it can be treated by the Laplace transform defined on bounded interval $[0,b]$, for any $b>0$.

We show two different cases of application of Laplace transform defined on an bounded interval.

### *6.1. Laplace transform and equation with fractional derivatives*

The procedure is the following. If we have an equation on a bounded interval $[0,b]$, we construct the corresponding equation in $L_b = L^{\exp}[0,\infty)/L^{\exp}[b,\infty)$, taking care that

$$L^{\exp}[b,\infty) + L^{\exp}[b,\infty) = L^{\exp}[b,\infty),$$

which means: *If* $f,g \in L^{\exp}[b,\infty)$, *then* $f+g \in L^{\exp}[b,\infty)$. The solutions of the constructed equation have to be of the form $y_b = \tilde{y} + L^{\exp}[b,\infty)$, where $(\bar{y})$ is the extension of $y \in L[0,b]$ in $L^{\exp}[0,\infty)$. Then the solution $y$ of the equation on $[0,b]$ is

$$y(t) = \bar{y}(t)|_{[0,b]} + L^{\exp}[b,\infty)|_{[0,b]} = \bar{y}(t)|_{[0,b]}.$$

As an example we consider Bagley-Torvik equation ([2], [3], [13], [14], [15], [19], [24]):

$$\big(D^2 + AD_{0^+}^{3/2} + B\big)y(t) = f(t), \quad 0 \le t \le b, \quad A, B > 0, \tag{6.1}$$

with initial conditions

$$y(0^+) = y_0, \quad y'(0^+) = y_0'. \tag{6.2}$$

The fractional derivative $D_{0^+}^{3/2}$ is used to describe the damping force studing the forced motion of the rigid plate immersed in the Newtonian fluid.

First of all we have to construct the corresponding equation in $L_b$ (see Section 2 of this text). In Theorem 4.2 we have two possibilities, denoted by (a) and (b). By equation (6.1) we have first to suppose that $y^{(2)}(t) \in L[0,b]$, then

$$\begin{aligned}
y(t) \quad &\Leftrightarrow \quad \bar{y}(t) + L^{\exp}[b,\infty);\\
D_{0^+}^{3/2} y(t) \quad &\Leftrightarrow \quad D^{1+1}\left(\frac{\tau^{-1/2}}{\Gamma(1/2)} * y(\tau)\right)(t) + L^{\exp}[b,\infty)\\
&\Leftrightarrow \quad \left(\frac{\tau^{-1/2}}{\Gamma(1/2)} * y(\tau)\right)(t) + y(0^+)\left(\frac{\tau^{-1/2}}{\Gamma(1/2)}\right)^{(1)}\\
&\qquad\qquad + y^{(1)}\frac{t^{-1/2}}{\Gamma(1/2)} + L^{\exp}[b,\infty); \qquad (6.3)\\
D^2 y(t) \quad &\Leftrightarrow \quad D^2\bar{y}(t) + L^{\exp}[b,\infty).
\end{aligned}$$

Second, we have to add supposition that relation (6.3) are valid and that they are in $L_b$ That means that $y^{(2)}$ exists for $t > 0$ and $y(0^+) = 0$. The corresponding equation in $L_b$ is

$$D^2\bar{y}(t) + A\left(\frac{\tau^{-1/2}}{\Gamma(1/2)} * \bar{y}(\tau)\right)(t) + B\bar{y}(t) = \bar{g}(t) + y^{(1)}(0^+)\frac{t^{-1/2}}{\Gamma(1/2)} + L^{\exp}[b,\infty). \tag{6.4}$$

Now, we can apply the classical Laplace transform to (6.3) (see (5.4)),

$$(s^2 + As^{3/2} + B)(\mathcal{L}\bar{y})(s) = (\mathcal{L}\bar{q})(s) + y(0)s + y^{(1)}(0) + L^{\exp}[b,\infty)(s), \tag{6.5}$$

where $y(0) = 0$.

In (b) the supposition we have to take may be lees as in (a), but they are also less visible.

The function $\left(\frac{\tau^{-1/2}}{\Gamma(1/2)} * y(\tau)\right)^{(2)}(t)$ has to belong to $L[0,b]$ and instead of initial condition (6.2) we will have

$$\left(\frac{\tau^{-1/2}}{\Gamma(1/2)} * y(\tau)\right)(0) = \xi_0, \quad \left(\frac{\tau^{-1/2}}{\Gamma(1/2)} * y(\tau)\right)^{(1)}(0) = \xi_1.$$

We return to equation (6.5)

$$(\mathcal{L}y)(s) = \frac{(\mathcal{L}\bar{g})(s) + y_0'}{s^2 + As^{3/2} + B} + \frac{(\mathcal{L}L^{\exp}[b,\infty))(s) + y_0'}{s^2 + As^{3/2} + B}. \tag{6.6}$$

Function

$$\frac{1}{s^2 + As^{3/2} + B} = (\mathcal{L}(\tau + (\tau * \Phi(\tau))))(s), \tag{6.7}$$

where

$$\Phi(t) = \sum_{r=1}^{\infty} (-1)^r \Phi_r(t), \quad \Phi_r(t) = \left(A\frac{\tau^{-1/2}}{\Gamma(1/2)} + B(t)\right)^{*r},$$

and $(\cdot)^{*r}$ denotes $r-$ time convolution $(\cdot) * (\cdot) \cdots (\cdot)$ (see [1]).

The solution $y_b = (\bar{y}) + L^{\exp}[b,\infty)$ of equation (6.5) is

$$\begin{aligned} y_b(t) &= \mathcal{L}^{-1}\mathcal{L}[(\tau + (\tau * \Phi(\tau))) * (\bar{g}(\tau) + y^1(0^+))](t) \\ &= \mathcal{L}^{-1}\mathcal{L}[(\tau + (\tau * \Phi(\tau)) * L^{\exp}[b,\infty))](t) \end{aligned}$$

Since

$$[\tau + (\tau * \Phi(\tau)) * L^{\exp}[b,\infty)]\,(t) \in L^{\exp}[b,\infty),$$

(see Lemma 2.2) we have

$$y_b(t) = \left[\tau + (\tau * \Phi(\tau))) * (\bar{g}(\tau) + y_0^1))\right](t) + L^{\exp}[b,\infty))](t).$$

The solution of equation (6.1) with initial conditions $y(0^+) = 0$ and $y^{(1)}(0^+) = y_0^1$ is

$$y_b(t)|_{[a,b]} = \left[\tau + (\tau * \Phi(\tau))) * (\bar{g}(\tau) + y_0^1))\right](t).$$

*6.2. Equation of the convolution type as an other examples*

Solve an integral equation of the first kind of convolution type

$$\int_0^t K(t-\tau)X(\tau)\,\mathrm{d}\tau = G(t), \quad 0 \le t \le b < \infty. \tag{6.8}$$

This equation (6.8) can be solved only for some special cases. So, for example, if $K(t)$ and $G(t)$ have derivatives and $K(0) \neq 0$; also if $K(t) = t^{-\alpha}$, $0 < \alpha < 1$, it is Abel singular equation. If the interval in (6.8) is the half axis, $0 \le t < \infty$, one can use the classical $\mathcal{L}$-transform to give a solution to equation (6.8).

In this paper we apply $\dot{\mathcal{L}}$-transform for function belonging to $L[0,b]$, $b < \infty$ to solve equation (6.8), as an application.

**Proposition 6.1.** *If the equation* (6.8) *has a solution belonging to* $L[0,b]$*,* $b < \infty$*, and if there is no* $\delta > 0$ *such that* $K(t) = 0$*,* $0 \le t \le \delta$*, then this solution is unique in* $L[0,b]$.

PROOF. Let us suppose that there exist two solutions $X_1$ and $X_2$ of (6.8). Then

$$[K * (X_1 - X_2)](t) = 0, \quad 0 \le t \le b. \tag{6.9}$$

To equation (6.9) in $L_b$ corresponds

$$\left[\bar{K} * \left(\bar{X}_1 - \bar{X}_2\right)\right](t) = L^{\exp}[b,\infty), \quad 0 \le t \le b \tag{6.10}$$

From (6.10) it follows that

$$\left[\overline{K} * \left(\overline{X_1 - X_2}\right)\right] = 0, \quad 0 \le t \le b. \tag{6.11}$$

By [7], p. 131, if $\left[\bar{K} * \left(\overline{X_1 - X_2}\right)\right](t) = 0$, $0 \le t \le b$, then $K_1(t) = 0$, $0 \le t \le a$, and $(X_1 - X_2)(t) = 0$, $0 \le t \le c$, where $a + c = b$. By our supposition that there is no $\delta > 0$ such that $K(t) = 0$, $0 \le t \le \delta$, it follows that $X_1(t) = X_2(t)$, $0 \le t \le b$.

**Proposition 6.2.** *If* $K(t) = 0$*,* $0 \le t \le \delta$ *for no* $\delta > 0$ *and if*

$$\mathcal{L}\left\{\bar{G}\right\}(s) = \frac{g(s)}{s^{1+\varepsilon}}\mathcal{L}\left\{\bar{K}\right\}(s),$$

*where* $g(s)$ *is analytical in* $\operatorname{Re} s > X_1 \ge 0$ *and bounded for* $\operatorname{Re} s \ge X_1 + \delta > X_1$*, then equation* (6.8) *has a solution*

$$X(t) = \mathcal{L}^{-1}\left[\mathcal{L}\{\bar{G}\}/\mathcal{L}\{\bar{K}\}\right](t) = \mathcal{L}^{-1}\frac{g(s)}{s^{1+\varepsilon}}, \quad 0 \le t \le b.$$

PROOF. Let us suppose that

$$\mathcal{L}\left\{\bar{G}\right\}(s) = \frac{g(s)}{s}\mathcal{L}\{\bar{K}\}(s). \tag{6.12}$$

To the equation (6.8) corresponds in $L_b$ the equation

$$\left[\bar{K}(\tau) * \bar{X}(\tau)\right](t) - \bar{G}(t) = L^{\exp}[b,\infty), \quad 0 < t. \tag{6.13}$$

The $\mathcal{L}$-transformation of equation (6.13) is

$$\mathcal{L}\left\{\bar{K}(\tau) * \bar{X}(\tau)\right\}(s) - \mathcal{L}\left\{\bar{G}\right\}(s) = \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}(s), \quad \operatorname{Re} s > X_1,$$

or

$$\mathcal{L}\{\bar{K}\}(s) \cdot \mathcal{L}\{\bar{X}\}(s) - \mathcal{L}\{\bar{K}\}(s)\frac{g(s)}{s^{1+\varepsilon}} = \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}(s), \quad \operatorname{Re} s > X_1,$$

i.e.,

$$\mathcal{L}\left\{\bar{K}\right\}(s)\left[\mathcal{L}\{\bar{X}\}(s) - \frac{g(s)}{s^{1+\varepsilon}}\right] = \mathcal{L}\left\{L^{\exp}[b,\infty)\right\}(s),\ Re\ s > X_1. \tag{6.14}$$

By Theorem 4, p. 263, from [4], $\mathcal{L}^{-1}\frac{g(s)}{s^{1+\varepsilon}} = G_1(t)$, $t \geq 0$, and $G_1(t) \in L^{\exp}[0,\infty)$. From (6.14) we have

$$\left\{\bar{K}(\tau) * \left[\bar{X}(\tau) - G_1(\tau)\right]\right\}(t)\Big|_{[0,b]} = 0, \quad 0 \leq t \leq b.$$

By the cited theorem from [4] (p. 131) and supposition that $K(t) = 0$, $0 \leq t \leq \delta$, for no one $\delta > 0$ it follows that $X(t) = G_1(t)$, $0 \leq t \leq b$.

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