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DISCRIMINANTLY SEPARABLE POLYNOMIALS: AN OVERVIEW

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A b s t r a c t. *The concept of discriminantly separable polynomials has been introduced by the author some years ago. We review the basic notions and several applications to different areas of mathematics and mechanics which arose in the meantime. Some of the results were obtained jointly with Dr. Katarina Kukić, a former author's PhD student.*

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1. *Introduction*

The concept of discriminantly separable polynomials has been introduced by the author some years ago [14]. We review the basic notions and indicate several relationships and applications to different areas of mathematics and mechanics, which arose in the meantime. Some of the results were obtained jointly with Dr. Katarina Kukić, a former author's PhD student.

Although purely algebraic in nature, the concept of discriminantly separable polynomials emerged within author's attempt to develop a novel approach to the classical, celebrated Kowalevski top and a geometrization of the Kowalevski integration procedure from [33]. Thus, one direction of applications goes toward continuous integrable systems and classical mechanics. Geometric applications are related to the fact that the equations of pencils of conics in appropriate coordinates induce discriminantly separable polynomials. Algebraic and algebro-geometric connections lead to so-called Buchstaber-Novikov n-valued groups. Beside continuous integrable systems, discrete integrable systems, namely integrable quad-graphs appear to be closely related to dicriminantly separable polynomials. Moreover, there is a full parallelism between a classification of discriminantly separable polynomials and a well-known ABS classification [2] of quad-graphs. The results presented in this short overview are obtained in [14], [34], [35], [21], [22], [23], [24], [25], [15], [16].

2. *Discriminantly separable polynomials-definition and basic notions*

Before giving a formal definition of the discriminantly separable polynomials, let us recall the equations of a pencil of conics. Denote such an equation as $\mathcal{F}(w, x_1, x_2) = 0$, where w is the pencil parameter; x_1 and x_2 are the Darboux coordinates. The choice of that classical, but mainly forgotten notion of the Darboux coordinates, instead of usual projective coordinates appear to be a subtle point and "an educated guess" which has had important consequences to the development of the theory of discriminantly separable polynomials. These Darboux coordinates [13], (see also [19]), should not be confused with a well-known Darboux coordinates from simplectic geometry, [3]. We recall some of the details: given two conics C_1 and C_2 in a general position by their tangential equations

$$\begin{aligned} &C_1 : a_0w_1^2 + a_2w_2^2 + a_4w_3^2 + 2a_3w_2w_3 + 2a_5w_1w_3 + 2a_1w_1w_2 = 0; \\ &C_2 : w_2^2 - 4w_1w_3 = 0. \end{aligned} \tag{2.1}$$

Then the conics of this general pencil $C(s) := C_1 + sC_2$ have four common tangent lines. Denote the matrix M:

$$M(s, z_1, z_2, z_3) = \begin{bmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & a_0 & a_1 & a_5 - 2s \\ z_2 & a_1 & a_2 + s & a_3 \\ z_3 & a_5 - 2s & a_3 & a_4 \end{bmatrix}. \tag{2.2}$$

The coordinate equations of the conics of the pencil are

$$F(s, z_1, z_2, z_3) := \det M(s, z_1, z_2, z_3) = 0,$$

which determines a quadratic polynomial in the pencil parameter s, namely

$$F := H + Ks + Ls^2,$$

with H, K, and L being quadratic expressions in (z_1, z_2, z_3).

Assume the standard projective coordinates $(z_1 : z_2 : z_3)$ in the plane, and choose, without loss of generality, a rational parametrization of the conic C_2 by $(1, \ell, \ell^2)$. The tangent line to the conic C_2 through a point of the conic with the parameter ℓ_0 is given by the equation

$$t_{C_2}(\ell_0) : z_1\ell_0^2 - 2z_2\ell_0 + z_3 = 0.$$

For a given point P outside the conic in the plane with the coordinates $P = (\hat{z}_1, \hat{z}_2, \hat{z}_3)$, there are two corresponding solutions x_1 and x_2 of the equation quadratic in ℓ

$$\hat{z}_1\ell^2 - 2\hat{z}_2\ell + \hat{z}_3 = 0. \tag{2.3}$$

The two solutions correspond to two tangent lines to the conic C_2 from the point P. We will define the pair (x_1, x_2) as the Darboux coordinates of the point P. One finds immediately the converse formulae $\hat{z}_1 = 1,\ \hat{z}_2 = (x_1 + x_2)/2,\ \hat{z}_3 = x_1x_2$.

Changing the variables in the polynomial F from the projective coordinates $(z_1 : z_2 : z_3)$ to the Darboux coordinates, we rewrite its expression in the form

$$\mathcal{F}(s, x_1, x_2) = L(x_1, x_2)s^2 + K(x_1, x_2)s + H(x_1, x_2).$$

The key algebraic property of the pencil polynomial written in this form, as a quadratic polynomial in each of the three variables s, x_1, x_2 is: *all three of its discriminants are expressed as products of two polynomials in one variable each*:

$$\mathcal{D}_w(\mathcal{F})(x_1, x_2) = P(x_1)P(x_2),\ \mathcal{D}_{x_i}(\mathcal{F})(w, x_j) = J(w)P(x_j), \quad i, j = 1, 2,$$

where J and P are polynomials of degree 3 and 4 respectively, and the elliptic curves

$$\Gamma_1 : y^2 = P(x), \quad \Gamma_2 : y^2 = J(s)$$

appear to be isomorphic (see Proposition 1 of [14]). Here, and below, we denote by $\mathcal{D}_{x_i}\mathcal{F}(x_j, x_k)$, the discriminant of $\mathcal{F}$ considered as a quadratic polynomial in x_i.

As a geometric interpretation of $F(s, x_1, x_2) = 0$ we may say that the point P in the plane, with the Darboux coordinates with respect to C_2 equal to (x_1, x_2) belongs to two conics of the pencil, with the pencil parameters equal to s_1 and s_2, such that

$$\mathcal{F}(s_i, x_1, x_2) = 0, \quad i = 1, 2.$$

Now we provide a general definition of the discriminantly separable polynomials. With $\mathcal{P}_m^n$ denote the polynomials of m variables of the degree n in each variable.

Definition 2.1 ([14])**.** A polynomial $F(x_1,\dots,x_n)$ is *discriminantly separable* if there exist polynomials $f_i(x_i)$ such that for every $i=1,\dots,n$

$$\mathcal{D}_{x_i}F(x_1,\dots,\hat{x}_i,\dots,x_n)=\prod_{j\neq i}f_j(x_j).$$

It is *symmetrically discriminantly separable* if $f_2=f_3=\cdots=f_n$, while it is *strongly discriminantly separable* if $f_1=f_2=f_3=\cdots=f_n$. It is *weakly discriminantly separable* if there exist polynomials $f_i^j(x_i)$ such that for every $i=1,\dots,n$,

$$\mathcal{D}_{x_i}F(x_1,\dots,\hat{x}_i,\dots,x_n)=\prod_{j\neq i}f_j^i(x_j).$$

2.1. Two-valued groups

The idea of n-valued groups, on a local level, goes back to Buchstaber and Novikov (see [9]), to their 1971 study of characteristic classes of vector bundles. That concept was significantly developed further by Buchstaber and his collaborators ([11] and references therein). An n-valued group on X can be defined as a map:

$$m:\ X\times X\to(X)^n,$$
$$m(x,y)=x*y=[z_1,\dots,z_n],$$

where $(X)^n$ denotes the symmetric n-th power of X and z_i coordinates therein. Such a map should satisfy the following axioms. *Associativity:* the condition of equality of two n^2-sets

$$[x*(y*z)_1,\dots,x*(y*z)_n],$$
$$[(x*y)_1*z,\dots,(x*y)_n*z]$$

for all triplets $(x,y,z)\in X^3$. Similarly, an element $e\in X$ is *a unit* if

$$e*x=x*e=[x,\dots,x],$$

for all $x\in X$. A map $\mathrm{inv}:X\to X$ is *an inverse* if it satisfies

$$e\in\mathrm{inv}(x)*x,\quad e\in x*\mathrm{inv}(x),$$

for all $x\in X$. Buchstaber says that m defines *an n-valued group structure* (X,m,e,inv) if it is associative, with a unit and an inverse.

An n-valued group X acts on a set Y if there is a mapping

$$\phi:\ X\times Y\to(Y)^n,$$
$$\phi(x,y)=x\circ y,$$

such that the two n^2-multisubsets $x_1\circ(x_2\circ y),\ (x_1*x_2)\circ y$ of Y are equal for all $x_1,x_2\in X, y\in Y$. It is also assumed $e\circ y=[y,\dots,y]$ for all $y\in Y$.

Example 2.1 (A two-valued group structure on $\mathbb{Z}_+$, [10])**.** Let us consider the set of nonnegative integers $\mathbb{Z}_+$ and define a mapping

$$m : \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2,$$
$$m(x, y) = [x + y, |x - y|].$$

This mapping provides a structure of a two-valued group on $\mathbb{Z}_+$ with the unit $e = 0$ and the inverse equal to the identity $\mathrm{inv}(x) = x$.

In [10], the algebraic action of this group on $\mathbb{CP}^1$ was studied and it was shown that in the irreducible case all such actions are generated by the Euler-Chasles correspondences.

There is another 2-valued group and its action on $\mathbb{CP}^1$ which is also closely related to the Euler-Chasles correspondence and to the Great Poncelet Theorem. This action is intimately related to the Kowalevski fundamental equation and to the Kowalevski change of variables as well.

Let us consider one more simple example.

Example 2.2. Two-valued group p_2 is defined by the relation

$$\begin{aligned} m_2 : \mathbb{C} \times \mathbb{C} \to (\mathbb{C})^2, \\ x *_2 y = [(\sqrt{x} + \sqrt{y})^2, (\sqrt{x} - \sqrt{y})^2] \end{aligned} \tag{2.4}$$

The product $x *_2 y$ is given by the solutions of the polynomial equation

$$p_2(z, x, y) = 0,$$

in z, where

$$p_2(z, x, y) = (x + y + z)^2 - 4(xy + yz + zx).$$

The polynomial $p_2(z, x, y)$ is discriminantly separable:

$$\mathcal{D}_z(p_2)(x, y) = P(x)P(y), \;\; \mathcal{D}_x(p_2)(y, z) = P(y)P(z), \;\; D_y(p_2)(x, z) = P(x)P(z),$$

where $P(x) = 2x$.

The polynomial p_2 as a discriminantly separable, generates a case of generalized Kowalevski system of differential equations from [14].

2.2. 2-valued group on $\mathbb{CP}^1$ and the Kowalevski top

It appears that the general equation of pencil of conics corresponds to an action of a two valued group. We use this correspondence to provide a novel interpretation of 'the mysterious Kowalevski change of variables' (the adjective being borrowed from

[4]). This line of thoughts may be seen as a further development of the ideas of Weil and Jurdjevic (see [38], [29], [30]). It turned out that the associativity condition for this action is equivalent to the Great Poncelet Theorem for a triangle, see [14].

The general pencil equation $\mathcal{F}(s, x_1, x_2) = 0$ is related to two elliptic curves $\tilde{\Gamma}_1 : y^2 = P(x)$, $\tilde{\Gamma}_2 : t^2 = J(s)$, where the polynomials P, J are of degree four and three respectively. These two elliptic curves are isomorphic. Rewrite the cubic one $\tilde{\Gamma}_2$ in the canonical form $\tilde{\Gamma}_2 : t^2 = J'(s) = 4s^3 - g_2 s - g_3$. Let $\psi : \tilde{\Gamma}_2 \to \tilde{\Gamma}_1$ be a birational morphism between the curves induced by a fractional-linear transformation $\hat{\psi}$ which maps the three zeros of J' and ∞ to the four zeros of the polynomial P.

The curve $\tilde{\Gamma}_2$ as a cubic has a group structure with the neutral element at infinity. With the subgroup $\mathbb{Z}_2$, it defines the standard two-valued group structure on $\mathbb{CP}^1$ (see [8]):

$$s_1 *_c s_2 = \left[-s_1 - s_2 + \left(\frac{t_1 - t_2}{2(s_1 - s_2)} \right)^2, -s_1 - s_2 + \left(\frac{t_1 + t_2}{2(s_1 - s_2)} \right)^2 \right], \quad (2.5)$$

where $t_i = J'(s_i)$, $i = 1, 2$.

Theorem 2.1 ([14])**.** *The general pencil equation after fractional-linear transformations*

$$\mathcal{F}(s, \hat{\psi}^{-1}(x_1), \hat{\psi}^{-1}(x_2)) = 0$$

induces the two valued coset group structure $(\tilde{\Gamma}_2, \mathbb{Z}_2)$ *defined by the relation* (2.5).

A proof is given in [14].

2.3. Review of the fundamental steps of the Kowalevski integration procedure

The Kowalevski top [33] is a celebrated example of a heavy rigid body which rotates about a fixed point, under the conditions $I_1 = I_2 = 2I_3$, $I_3 = 1$, $Y_0 = Z_0 = 0$ (see subsection 2.1). More about the theory of motion of heavy rigid-bodies one may find in, for example, [3], [28], [5], [18], [27], [17]. Denote by $c = mgX_0$, where m is the mass of the top, and denote by (p, q, r) the vector of angular velocity $\vec{\Omega}$. Then the equations of motion take the following form, see [33], [28], [35], [34], [26]:

$$\begin{aligned} 2\dot{p} &= qr, & \dot{\Gamma}_1 &= r\Gamma_2 - q\Gamma_3, \\ 2\dot{q} &= -pr - c\Gamma_3, & \dot{\Gamma}_2 &= p\Gamma_3 - r\Gamma_1, \\ \dot{r} &= c\Gamma_2, & \dot{\Gamma}_3 &= q\Gamma_1 - p\Gamma_2. \end{aligned} \quad (2.6)$$

The system (2.6) as any other system of equations of a heavy rigid body, has three well known first integrals of motion. In this particular case there is also an additional, fourth, first integral, discovered by Kowalevski

$$\begin{aligned} &2(p^2+q^2)+r^2=2c\Gamma_1+6l_1,\\ &2(p\Gamma_1+q\Gamma_2)+r\Gamma_3=2l,\\ &\Gamma_1^2+\Gamma_2^2+\Gamma_3^2=1,\\ &\big((p+iq)^2+\Gamma_1+i\Gamma_2\big)\big((p-iq)^2+\Gamma_1-i\Gamma_2\big)=k^2. \end{aligned} \tag{2.7}$$

A significance of the Kowalevski top is that the additional first integral is of fourth degree in momenta.

By using the change of variables

$$\begin{aligned} x_1&=p+iq, \quad e_1=x_1^2+c(\Gamma_1+i\Gamma_2),\\ x_2&=p-iq, \quad e_2=x_2^2+c(\Gamma_1-i\Gamma_2), \end{aligned} \tag{2.8}$$

the first integrals (2.7) transform into

$$\begin{aligned} r^2&=E+e_1+e_2,\\ rc\Gamma_3&=G-x_2e_1-x_1e_2,\\ c^2\Gamma_3^2&=F+x_2^2e_1+x_1^2e_2,\\ e_1e_2&=k^2, \end{aligned} \tag{2.9}$$

with $E=6l_1-(x_1+x_2)^2$, $F=2cl+x_1x_2(x_1+x_2)$, $G=c^2-k^2-x_1^2x_2^2$. One easily gets

$$(E+e_1+e_2)(F+x_2^2e_1+x_1^2e_2)-(G-x_2e_1-x_1e_2)^2=0,$$

which has an equivalent form

$$e_1P(x_2)+e_2P(x_1)+R_1(x_1,x_2)+k^2(x_1-x_2)^2=0, \tag{2.10}$$

where the polynomial P is

$$P(x_i)=x_i^2E+2x_1F+G=-x_i^4+6l_1x_i^2+4lcx_i+c^2-k^2, \quad i=1,2,$$

and

$$\begin{aligned} &R_1(x_1,x_2)=EG-F^2\\ &=-6l_1x_1^2x_2^2-(c^2-k^2)(x_1+x_2)^2-4lc(x_1+x_2)x_1x_2+6l_1(c^2-k^2)-4l^2c^2. \end{aligned}$$

A remarkable and not obvious property of P is its dependence on only one variable. Let

$$R(x_1,x_2)=Ex_1x_2+F(x_1+x_2)+G.$$

From (2.10), following Kowalevski, one gets

$$(\sqrt{P(x_1)e_2} \pm \sqrt{P(x_2)e_1})^2 = -(x_1 - x_2)^2k^2 \pm 2k\sqrt{P(x_1)P(x_2)} - R_1(x_1, x_2). \quad (2.11)$$

After a few transformations, (2.11) can be written in the form

$$\left[\sqrt{e_1}\frac{\sqrt{P(x_2)}}{x_1 - x_2} \pm \sqrt{e_2}\frac{\sqrt{P(x_1)}}{x_1 - x_2}\right]^2 = (w_1 \pm k)(w_2 \mp k), \quad (2.12)$$

where w_1, w_2 are the solutions of an equation, quadratic in s:

$$Q(s, x_1, x_2) = (x_1 - x_2)^2s^2 - 2R(x_1, x_2)s - R_1(x_1, x_2) = 0. \quad (2.13)$$

The quadratic equation (2.13) is known as *the Kowalevski fundamental equation.* As it has been observed in [14], the discriminant separability condition for $Q(s, x_1, x_2)$ is satisfied

$$\mathcal{D}_s(Q)(x_1, x_2) = 4P(x_1)P(x_2),$$

$$\mathcal{D}_{x_1}(Q)(s, x_2) = -8J(s)P(x_2),\ \mathcal{D}_{x_2}(Q)(s, x_1) = -8J(s)P(x_1),$$

with

$$J(s) = s^3 + 3l_1s^2 + s(c^2 - k^2) + 3l_1(c^2 - k^2) - 2l^2c^2.$$

The equations of motion (2.6) in new variables $(x_1, x_2, e_1, e_2, r, \Gamma_3)$ take the form:

$$\begin{aligned} 2\dot{x}_1 &= -if_1, \quad \dot{e}_1 = -me_1, \\ 2\dot{x}_2 &= if_2, \qquad \dot{e}_2 = me_2. \end{aligned} \quad (2.14)$$

Two additional differential equations for $\dot{r}$ and $\dot{\Gamma}_3$ can be easily derived. Here $m = ir$ and $f_1 = rx_1 + c\Gamma_3,\ f_2 = rx_2 + c\Gamma_3$. The following formulas hold:

$$f_1^2 = P(x_1) + e_1(x_1 - x_2)^2, \quad f_2^2 = P(x_2) + e_2(x_1 - x_2)^2. \quad (2.15)$$

Further steps of the integration procedure are presented in [33], see for example [23].

Theorem 2.2 ([14])**.** *The Kowalevski fundamental equation coincides with the point pencil equation generated by the conics given by their tangential equations*

$$\begin{aligned} &\hat{C}_1 :\ -2w_1^2 + 3l_1w_2^2 + 2(c^2 - k^2)w_3^2 - 4clw_2w_3 = 0; \\ &C_2 : w_2^2 - 4w_1w_3 = 0. \end{aligned} \quad (2.16)$$

The Kowalevski variables w, x_1, x_2 get a novel geometric interpretation in this settings: they are the pencil parameter, and the Darboux coordinates with respect to the conic C_2, respectively.

The Kowalevski case is extracted from the general case of pencil of conics by the conditions $a_1 = 0,\ a_5 = 0,\ a_0 = -2$. The last relation is nothing but a normalization condition, provided $a_0 \neq 0$. The Kowalevski parameters l_1, l, c can be expressed by the formulas

$$l_1 = \frac{a_2}{3}, \quad l = \pm\frac{1}{2}\sqrt{-a_4 + \sqrt{a_4 + 4a_3^2}}, \quad c = \mp\frac{a_3}{\sqrt{-a_4 + \sqrt{a_4 + 4a_3^2}}},$$

with an additional condition that l and c are real. For the sake of historic clarity, we observe that Kowalevski in [33], didn't use the relation (2.13), but an equivalent one. The equivalence is obtained by putting $w = 2s - l_1$.

The success of the mechanism of the Kowalevski change of variables is based on the following consequence of the discriminant separability property of the polynomial $\mathcal{F} = Q$:

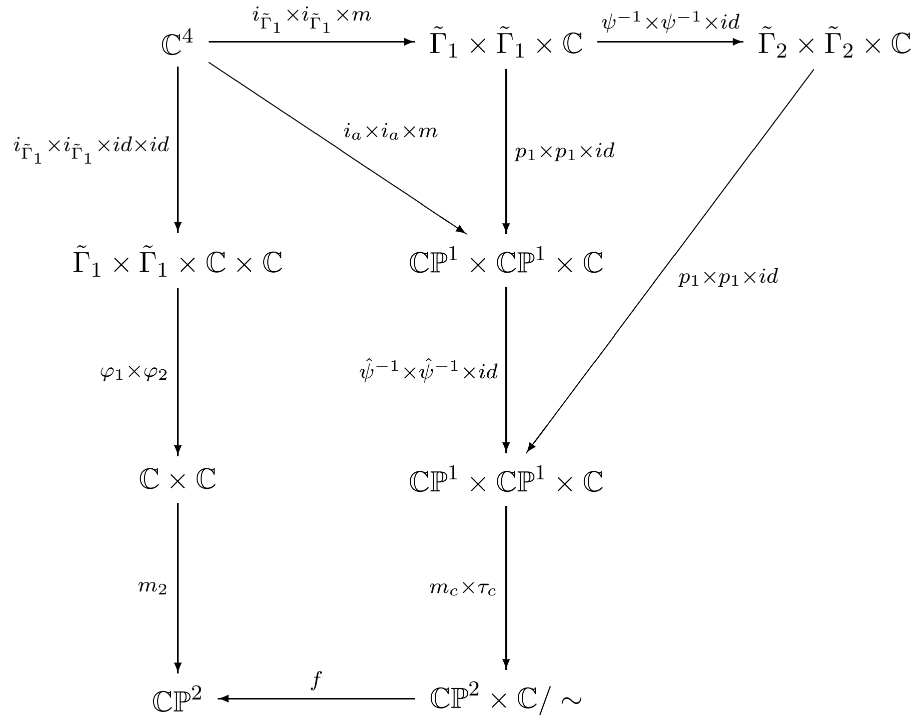
$$\begin{aligned} \frac{\mathrm{d}x_1}{\sqrt{P(x_1)}} + \frac{\mathrm{d}x_2}{\sqrt{P(x_2)}} &= \frac{\mathrm{d}w_1}{\sqrt{J(w_1)}}, \\ \frac{\mathrm{d}x_1}{\sqrt{P(x_1)}} - \frac{\mathrm{d}x_2}{\sqrt{P(x_2)}} &= \frac{\mathrm{d}w_2}{\sqrt{J(w_2)}}. \end{aligned} \tag{2.17}$$

The Kowalevski change of variables (see equations (2.17)) can be seen as an infinitesimal of the correspondence which maps a pair of points (M_1, M_2) to a pair of points (S_1, S_2). Both pairs belong to a $\mathbb{P}^1$ as a factor of an appropriate elliptic curve. A geometric interpretation of this mapping is the correspondence which maps *two tangents to the conic C to the pair of conics from the pencil which contain the intersection point of the two lines.*

Theorem 2.3 ([14])**.** *The Kowalevski change of variables is equivalent to an infinitesimal of the action of the two valued coset group $(\tilde{\Gamma}_2, \mathbb{Z}_2)$ on $\mathbb{P}^1$ as a factor of the elliptic curve. Up to a fractional-linear transformation, it is equivalent to the operation of the two valued group $(\tilde{\Gamma}_2, \mathbb{Z}_2)$.*

Now, the Kötter trick (see [32], [14]) can be applied to the following commutative diagram.

Proposition 2.1 ([14])**.** *The Kowalevski integration procedure may be coded in* *the following commutative diagram:*



The mappings are defined as follows

$$
\begin{aligned}
i_{\tilde\Gamma_1} &: x \mapsto (x, \sqrt{P(x)}),\\
m &: (x,y) \mapsto x\cdot y,\\
i_a &: x \mapsto (x,1),\\
p_1 &: (x,y) \mapsto x,\\
m_c &: (x,y) \mapsto x *_c y,\\
\tau_c &: x \mapsto (\sqrt{x}, -\sqrt{x}),\\
\varphi_1 &: (x_1,x_2,e_1,e_2) \mapsto \sqrt{e_1}\frac{\sqrt{P(x_2)}}{x_1-x_2},\\
\varphi_2 &: (x_1,x_2,e_1,e_2) \mapsto \sqrt{e_2}\frac{\sqrt{P(x_1)}}{x_1-x_2},\\
f &: ((s_1,s_2,1),(k,-k)) \mapsto [(\gamma^{-1}(s_1)+k)(\gamma^{-1}(s_2)-k), (\gamma^{-1}(s_2)+k)(\gamma^{-1}(s_1)-k)].
\end{aligned}
$$

3. *Systems of the Kowalevski type. Definition*

Following [21, 23, 24], we present a class of integrable systems, which generalize the Kowalevski top. Instead of the Kowalevski fundamental equation (see formula (2.13)), the starting point here would be an arbitrary discriminantly separable polynomial of degree two in each of three variables.

Given a discriminantly separable polynomial of the second degree in each of three variables

$$\mathcal{F}(x_1, x_2, s) := A(x_1, x_2)s^2 + B(x_1, x_2)s + C(x_1, x_2), \tag{3.1}$$

such that

$$\mathcal{D}_s(\mathcal{F})(x_1, x_2) = B^2 - 4AC = 4P(x_1)P(x_2),$$

and

$$\mathcal{D}_{x_1}(\mathcal{F})(s, x_2) = 4P(x_2)J(s),$$
$$\mathcal{D}_{x_2}(\mathcal{F})(s, x_1) = 4P(x_1)J(s).$$

Suppose, that a given system in variables x_1, x_2, e_1, e_2, r, γ_3, after some transformations reduces to

$$\begin{aligned} 2\dot{x}_1 &= -if_1, \quad \dot{e}_1 = -me_1,\\ 2\dot{x}_2 &= if_2, \qquad \dot{e}_2 = me_2, \end{aligned} \tag{3.2}$$

where

$$f_1^2 = P(x_1) + e_1A(x_1, x_2), \quad f_2^2 = P(x_2) + e_2A(x_1, x_2). \tag{3.3}$$

Suppose additionally, that the first integrals of the initial system reduce to a relation

$$P(x_2)e_1 + P(x_1)e_2 = C(x_1, x_2) - e_1e_2A(x_1, x_2). \tag{3.4}$$

The equations for $\dot{r}$ and $\dot{\Gamma}_3$ are not specified for the moment and m is a function of system's variables.

If a system satisfies the above assumptions we will call it *a system of the Kowalevski type*. As it has been pointed out in the previous subsection, see formulae (2.10, 2.13, 2.14,2.15), the Kowalevski top is an example of the systems of the Kowalevski type.

The following theorem is quite general, and concerns all the systems of the Kowalevski type. It explains in full a subtle mechanism of a quite miraculous jump in genus, from one to two, in integration procedure, which has been observed in the Kowalevski top, and now it is going to be established as a characteristic property of the whole new class of systems.

Theorem 3.1. *Given a system which reduces to* (3.2, 3.3, 3.4). *Then the system is linearized on the Jacobian of the curve*

$$y^2 = J(z)(z-k)(z+k),$$

where J is a polynomial factor of the discriminant of $\mathcal{F}$ as a polynomial in x_1 and k is a constant such that

$$e_1 e_2 = k^2.$$

The last Theorem basically formalizes the original considerations of Kowalevski, in a slightly more general context of the discriminantly separable polynomials. A proof is presented in [24].

3.1. An example of systems of the Kowalevski type

In this subsection we present the Sokolov system given in [37] as an example of systems of the Kowalevski type, see [23], [24]. Consider [37] the Hamiltonian

$$\hat{H} = M_1^2 + M_2^2 + 2M_3^2 + 2c_1\gamma_1 + 2c_2(\gamma_2 M_3 - \gamma_3 M_2) \tag{3.5}$$

on $e(3)$ with the Lie-Poisson brackets

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = \epsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0, \tag{3.6}$$

where ϵ_{ijk} is the totally skew-symmetric tensor. In [31], an explicit map between the integrable system on $e(3)$ with the Hamiltonian (3.5) and the Kowalevski top on $so(4)$ has been found. The separation of variables for the system (3.5) was performed. The aim of this section is to show that the system can be seen as an element of the class of the systems of the Kowalevski type, [23], [24].

The Lie-Poisson brackets (3.6) have two Casimirs:

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = a,$$
$$\gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M3 = b.$$

As in [31] and [33], one can introduce new variables $z_1 = M_1 + iM_2$, $z_2 = M_1 - iM_2$ and

$$e_1 = z_1^2 - 2c_1(\gamma_1 + i\gamma_2) - c_2^2 a - c_2(2\gamma_2 M_3 - 2\gamma_3 M_2 + 2i(\gamma_3 M_1 - \gamma_1 M_3)),$$
$$e_2 = z_2^2 - 2c_1(\gamma_1 - i\gamma_2) - c_2^2 a - c_2(2\gamma_2 M_3 - 2\gamma_3 M_2 + 2i(\gamma_1 M_3 - \gamma_3 M_1)).$$

The second first integral of motion of the system (3.5) can be written as

$$e_1 e_2 = k^2. \tag{3.7}$$

The equations of motion for new variables z_i, e_i can be written in the form of (3.2) and (3.3). This is in a full accordance with our definition of the systems of Kowalevski type. It is easy to prove that:

$$\dot{e}_1 = -4iM_3e_1, \qquad \dot{e}_2 = 4iM_3e_2,$$

and

$$\begin{aligned} -\dot{z_1}^2 &= P(z_1) + e_1(z_1 - z_2)^2, \\ -\dot{z_2}^2 &= P(z_2) + e_2(z_1 - z_2)^2, \end{aligned} \tag{3.8}$$

where P is a polynomial of degree four:

$$P(z) = -z^4 + 2Hz^2 - 8c_1bz - k^2 + 4ac_1^2 - 2c_2^2(2b^2 - Ha) + c_2^4a. \tag{3.9}$$

The biquadratic form and the separated variables were defined [31]:

$$\begin{aligned} F(z_1, z_2) &= -\frac{1}{2}\left(P(z_1) + P(z_2) + (z_1^2 - z_2^2)^2\right), \\ s_{1,2} &= \frac{F(z_1, z_2) \pm \sqrt{P(z_1)P(z_2)}}{2(z_1 - z_2)^2}, \end{aligned} \tag{3.10}$$

such that

$$\dot{s_1} = \frac{\sqrt{P_5(s_1)}}{s_1 - s_2}, \quad \dot{s_2} = \frac{\sqrt{P_5(s_2)}}{s_2 - s_1}, \quad P_5(s) = P_3(s)P_2(s),$$

with

$$\begin{aligned} P_3(s) &= s(4s^2 + 4sH + H^2 - k^2 + 4c_1^2a + 2c_2^2(Ha - 2b^2) + c_2^4a^2) + 4c_1^2b^2, \\ P_2(s) &= 4s^2 + 4(H + c_2^2a)s + H^2 - k^2 + 2c_2^2ha + c_2^4a^2. \end{aligned}$$

To verify that, we still need to show that a relation of the form of (3.4) is satisfied and to relate it with a corresponding discriminantly separable polynomial in the form of (3.1). Starting from the equations

$$\dot{z_1} = -2M_3(M_1 - iM_2) + 2c_2(\gamma_1M_2 - \gamma_2M_1) + 2c_1\gamma_3$$

and

$$\dot{z_2} = -2M_3(M_1 + iM_2) + 2c_2(\gamma_1M_2 - \gamma_2M_1) + 2c_1\gamma_3,$$

one can prove that

$$\dot{z_1} \cdot \dot{z_2} = -\left(F(z_1, z_2) + (H + c_2^2a(z_1 - z_2)^2)\right),$$

where $F(z_1, z_2)$ is given by (3.10). After equating the square of $\dot{z_1}\dot{z_2}$ from previous relation and $\dot{z_1}^2 \cdot \dot{z_2}^2$ with $\dot{z_i}^2$ given by (3.8) we get

$$\begin{aligned}(z_1 - z_2)^2[2F(z_1, z_2)(H + c_2^2 a) + (z_1 - z_2^2)^4(H + c_2^2 a)^2 - P(z_1)e_2 \\ - P(z_2)e_1 - e_1 e_2 (z_1 - z_2)^2] + F^2(z_1, z_2) - P(z_1)P(z_2) = 0.\end{aligned} \tag{3.11}$$

Denote by $C(z_1, z_2)$ a biquadratic polynomial such that $F^2(z_1, z_2) - P(z_1)P(z_2) = (z_1 - z_2)^2 C(z_1, z_2)$. Then we can rewrite relation (3.11) in the form of (3.4):

$$P(z_1)e_2 + P(z_2)e_1 = \tilde{C}(z_1, z_2) - e_1 e_2 (z_1 - z_2)^2, \tag{3.12}$$

with

$$\tilde{C}(z_1, z_2) = C(z_1, z_2) + 2F(z_1, z_2)(H + c_2^2 a) + (H + c_2^2 a)^2 (z_1 - z_2)^2. \tag{3.13}$$

Further integration procedure follows Theorem 3.1. The discriminantly separable polynomial of three variables of degree two in each variable "plays role" of the Kowalevski fundamental equation in this case: i

$$\tilde{F}(z_1, z_2, s) = (z_1 - z_2)^2 s^2 + \tilde{B}(z_1, z_2)s + \tilde{C}(z_1, z_2), \tag{3.14}$$

with

$$\tilde{B}(z_1, z_2) = F(z_1, z_2) + (H + c_2^2 a)(z_1 - z_2)^2.$$

The discriminants of (3.14) as polynomials in s and in z_i, for $i = 1, 2$ are

$$\mathcal{D}_s(\tilde{F})(z_1, z_2) = P(z_1)P(z_2),$$

$$\mathcal{D}_{z_1}(\tilde{F})(s, z_2) = 8J(s)P(z_2),\ \mathcal{D}_{z_2}(Q)(s, z_1) = 8J(s)P(z_1),$$

where J is a polynomial of degree three

$$\begin{aligned}J = s^3 &+ (H + 3ac_2^2)s^2 + (-4c_2^2 b^2 - 2k^2 + 4ac_1^2 + 4c_2^4 a^2 + 4c_2^2 Ha)s \\ &- 8c_1^2 b^2 - 4c_2^4 ab^2 + 4c_1^2 a^2 c_2^2 - k^2 c_2^2 a - Hk^2 + 2aH^2 c_2^2 - 4Hb^2 c_2^2 \\ &+ 4Hc_1^2 a + 4c_2^4 Ha^2 + 2c_2^6 a^3.\end{aligned}$$

4. *Classification of strongly discriminantly separable polynomials of degree two in three variables*

In this section we present a classification from [22] of the strongly discriminantly separable polynomials $\mathcal{F}(x_1, x_2, x_3) \in \mathbb{C}[x_1, x_2, x_3]$ which are of degree two in each of three variables. This classification is done modulo the group of the Möbius transformations

$$x_1 \mapsto \frac{ax_1+b}{cx_1+d}, \quad x_2 \mapsto \frac{ax_2+b}{cx_2+d}, \quad x_3 \mapsto \frac{ax_3+b}{cx_3+d}. \tag{4.1}$$

Denote by

$$\mathcal{F}(x_1,x_2,x_3) = \sum_{i,j,k=0}^{2} a_{ijk} x_1^i x_2^j x_3^k \tag{4.2}$$

a strongly discriminantly separable polynomial with

$$\mathcal{D}_{x_i}\mathcal{F}(x_j,x_k) = P(x_j)P(x_k), \quad (i,j,k) = c.p.(1,2,3). \tag{4.3}$$

One gets a system of 75 equations of degree two with 27 unknowns a_{ijk}, by plugging (4.2) into (4.3) for a given polynomial $P(x) = Ax^4+Bx^3+Cx^2+Dx+E$.

Theorem 4.1. *Given a nonzero polynomial $P(x)$. The strongly discriminantly separable polynomials $\mathcal{F}(x_1,x_2,x_3)$ of degree two in each of the three variables which satisfy* (4.3)*, are exhausted modulo Möbius transformations, by the following list coded by the structure of the roots of the polynomial $P(x)$:*

(A) *If P has four simple zeros, it can be transformed to a canonical form $P_A(x) = (k^2x^2-1)(x^2-1)$, and*

$$\begin{aligned}\mathcal{F}_A =& \frac{1}{2}(-k^2x_1^2 - k^2x_2^2 + 1 + k^2x_1^2x_2^2)x_3^2 + (1-k^2)x_1x_2x_3 \\ &+ \frac{1}{2}(x_1^2 + x_2^2 - k^2x_1^2x_2^2 - 1),\end{aligned}$$

(B) *if P has two simple zeros and one double, it can be transformed to a canonical form $P_B(x) = x^2 - e^2$, $e \neq 0$, and*

$$\mathcal{F}_B = x_1x_2x_3 + \frac{e}{2}(x_1^2 + x_2^2 + x_3^2 - e^2),$$

(C) *If P has two double zeros, and the canonical form $P_C(x) = x^2$, then*

$$\begin{aligned}\mathcal{F}_{C_1} &= \lambda x_1^2x_2^2 + \mu x_1x_2x_3 + \nu x_3^2, \quad \mu^2 - 4\lambda\nu = 1, \\ \mathcal{F}_{C_2} &= \lambda x_1^2x_3^2 + \mu x_1x_2x_3 + \nu x_2^2, \quad \mu^2 - 4\lambda\nu = 1, \\ \mathcal{F}_{C_3} &= \lambda x_2^2x_3^2 + \mu x_1x_2x_3 + \nu x_1^2, \quad \mu^2 - 4\lambda\nu = 1, \\ \mathcal{F}_{C_4} &= \lambda x_1^2x_2^2x_3^2 + \mu x_1x_2x_3 + \nu, \quad \mu^2 - 4\lambda\nu = 1,\end{aligned}$$

(D) *if P has one simple and one triple zero, then the canonical form is $P_D(x) = x$, and,*

$$\mathcal{F}_D = -\frac{1}{2}(x_1x_2 + x_2x_3 + x_1x_3) + \frac{1}{4}(x_1^2 + x_2^2 + x_3^2),$$

(E) *if P has one quadruple zero, then the canonical form is $P_E(x) = 1$, and*

$$\mathcal{F}_{E_1} = \lambda(x_1 + x_2 + x_3)^2 + \mu(x_1 + x_2 + x_3) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\mathcal{F}_{E_2} = \lambda(x_2 + x_3 - x_1)^2 + \mu(x_2 + x_3 - x_1) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\mathcal{F}_{E_3} = \lambda(x_1 + x_3 - x_2)^2 + \mu(x_1 + x_3 - x_2) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\mathcal{F}_{E_4} = \lambda(x_1 + x_2 - x_3)^2 + \mu(x_1 + x_2 - x_3) + \nu, \quad \mu^2 - 4\lambda\nu = 1.$$

The proof from [22] is performed by a straightforward calculation and solving the system of equations (4.3) for the canonical representatives of the polynomials P. The correspondence between this classification and pencil of conics in the case (A) is as follows: In the case of a general position, the conics of a pencil intersect in four distinct points, and we code such situation with $(1, 1, 1, 1)$. It corresponds to the case above where the polynomial P has four simple zeros. In this case, the family of strongly discriminantly separable polynomials corresponds to the equations of the families constructed above of general pencils of conics. These families were indicated in [14]. A corresponding pencil of conics is presented on Fig. 1.

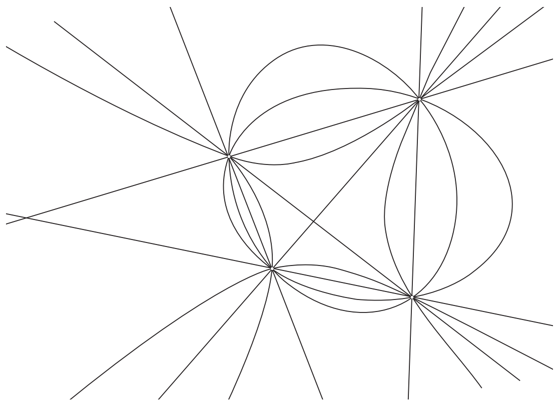


Figure 1. Pencil with four simple points

Without loss of generality, we use C_2 as the conic with respect to which the Darboux coordinates are defined. The obtained families of polynomials in cases (A), (B), and (D) are unique up to Möbius equivalence. Each of them is Möbius-equivalent to a corresponding strongly discriminantly separable polynomial, and they represent the equations of pencils of conics of the types $(\mathrm{A}) = (1,1,1,1)$, $(\mathrm{B}) = (1,1,2)$, and $(\mathrm{D}) = (1,3)$. The pencils $(1,1,2)$ consist of conics sharing two simple points and one double point, while the pencils $(1,3)$ consist of conics having one common simple point and one common triple point. However, the situations in the cases (C) and (E) are significantly different. Not only are uniqueness, up to Möbius equivalence, of the families of the polynomials, is lost, but also such a transparent geometric correlation with pencils of conics disappears. We will skip here the details of the connection with pencils of conics in the cases (B) and (D), see [22], which are analog to (A). We will discuss now the cases (C) and (E) and the lack of relationship to the pencils of types $(2,2)$ and (4) respectively. Former pencils contain conics which share two double points, see Fig. 2 (left), while later describe pencils of conics having one point of order 4 in common, see Fig. 2 (right).

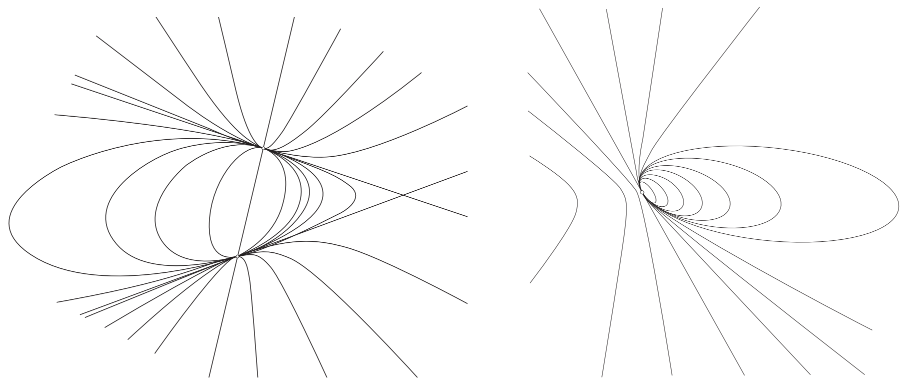


Figure 2. Pencil with two double points (left) and with one quadruple point (right)

This unexpected lack of corresponding pencils of conics in the cases (C) and (E) can be understood better in the light of the following statement:

Proposition 4.1 (Corollary, [36], VIII, Ch. 1)**.** *A symmetric* $(2-2)$ *algebraic correspondence cut on* C *by tangents to* C_2 *splits into a Möbius transformation and its inverse if and only if* C_2 *has double contact with* C*. If* C_2 *coincides with* C*, then the correspondence is the identity taken twice.*

The two points of contact correspond to the fixed points of the Möbius transformation.

Suppose that the two fixed points of the Möbius transformation are the points with the parameters equal to 0 and ∞. The Möbius transformation, denoted by w, is of the form $w(x) = ax$. In a case of the pencil of conics of type $(2, 2)$ with the intersection at two double points, according to the previous Proposition, the polynomial $\hat{F}_s(x_1, x_2) := \mathcal{F}(x_1, x_2, s)$ has to have the form

$$\hat{F}_s(x_1, x_2) = (ax_1 + bx_2)(bx_1 + ax_2). \tag{4.4}$$

For a fixed value of the parameter s, the polynomials $\mathcal{F}_{C_1} - \mathcal{F}_{C_4}$, do not have the form (4.4). Those polynomials do not correspond to the pencils of conics with two double base points. A similar argument when the fixed points of w coincide, explains the case (E).

5. *From discriminantly separable polynomials to integrable quad-equations*

The discriminantly separable polynomials appear to be related to discrete integrable systems. We will show a relationship with integrable quad-equations, from [22]. The theory of quad-graphs and quad-equations emerged in works of Adler, Bobenko, Suris [1], [2], see also [6], [7].

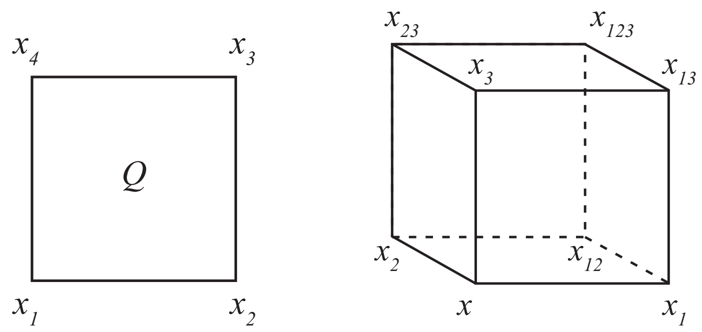


Figure 3. Quad-equation $Q(x_1, x_2, x_3, x_4) = 0$ on an elementary quadrilateral (left) and 3D-consistency (right)

The quad-equations are defined on quadrilaterals and they have the form

$$Q(x_1, x_2, x_3, x_4) = 0. \tag{5.1}$$

Here Q is a polynomial of degree one in each variable. Such a polynomial is said to be multiaffine. So-called field variables x_i are assigned to four vertices of a quadrilateral as in a Figure 3 (left). The polynomial Q depends on the variables $x_1, \dots, x_4 \in \mathbb{C}$, but also depends on two additional parameters $\alpha, \beta \in \mathbb{C}$ that are assigned to the edges of a quadrilateral. The opposite edges carry the same parameter.

The equation (5.1) solved for each variable, gives the solution as a rational function of the other three variables. A solution (x_1, x_2, x_3, x_4) of the equation (5.1) is said to be *singular* with respect to x_i if it also satisfies the equation

$$Q_{x_i}(x_1, x_2, x_3, x_4) = 0.$$

Following [2] we adopt the idea of integrability as *a consistency*, see Figure 3 (right). We assign six quad-equations to the faces of the coordinate cube. The system is *3D-consistent* if the three values for x_{123} obtained from the equations on the right, back, and top faces coincide for arbitrary initial data x, x_1, x_2, x_3.

The discriminant-like operators are introduced in [2]

$$\delta_{x,y}(Q) = Q_x Q_y - Q Q_{xy}, \quad \delta_x(h) = h_x^2 - 2h h_{xx}, \tag{5.2}$$

and one can make a descent from the faces to the edges and then to the vertices of the cube: in that way, from a multiaffine polynomial $Q(x_1, x_2, x_3, x_4)$ we pass to a biquadratic polynomial $h(x_i, x_j) := \delta_{x_k, x_l}(Q(x_i, x_j, x_k, x_l))$ and then, further, to a polynomial $P(x_i) = \delta_{x_j}(h(x_i, x_j))$ of degree up to four. Using the relative invariants of polynomials under fractional linear transformations, the formulae that express Q through the biquadratic polynomials of three edges, were obtained in [2]:

$$\frac{2Q_{x_1}}{Q} = \frac{h^{12}_{x_1} h^{34} - h^{14}_{x_1} h^{23} + h^{23} h^{34}_{x_3} - h^{23}_{x_3} h^{34}}{h^{12} h^{34} - h^{14} h^{23}}. \tag{5.3}$$

A biquadratic polynomial $h(x, y)$ is said to be *nondegenerate* if no polynomial in its equivalence class with respect to the fractional linear transformations, is divisible by a factor of the form $x - c$ or $y - c$, with $c = \text{const}$. A multiaffine function $Q(x_1, x_2, x_3, x_4)$ is said to be of *type* Q if all four of its accompanying biquadratic polynomials h^{jk} are nondegenerate. Otherwise, it is of *type* H. Previous notions were introduced in [2].

Take an arbitrary strongly discriminantly separable polynomial

$$\mathcal{F}(x_1, x_2, \alpha)$$

of degree two in each of the three variables. To relate that polynomial to the corresponding quad-equations, one needs to provide a biquadratic polynomial $h = h(x_1, x_2)$ and a multiaffine polynomial $Q = Q(x_1, x_2, x_3, x_4)$.

The requirement that the discriminants of $h(x_1, x_2)$ are independent on α, see [1], [2], is fulfilled if as a biquadratic polynomials $h(x_1, x_2)$ we select

$$\hat{h}(x_1, x_2) := \frac{\mathcal{F}(x_1, x_2, \alpha)}{\sqrt{P(\alpha)}}.$$

Proposition 5.1 ([22]). *The biquadratic polynomials*

$$\hat{h}_I(x_1, x_2) = \frac{\mathcal{F}_I(x_1, x_2, \alpha)}{\sqrt{P_I(\alpha)}} \tag{5.4}$$

satisfy

$$\delta_{x_1}(\hat{h}) = P_I(x_2), \quad \delta_{x_2}(\hat{h}) = P_I(x_1)$$

for $I = A, B, C, D, E$ *and polynomials* $P_I, \mathcal{F}_I$ *from Theorem* 4.1.

By using the formulae (5.3) and replacing the polynomials h^{ij} by $\hat{h}^{ij}$, one gets the quad-equations which correspond to representatives of discriminantly separable polynomials from Theorem 4.1. These equations are re-parameterizations of the quad-equations of type Q from the list obtained in [2].

For the quad-equations obtained from the biquadratic polynomials $\hat{h}(x_1, x_2)$, that parameter α has a role symmetric to x_1 and x_2.

Another class of discrete integrable systems of a type similar to quad-graphs of a geometric origin has been given in [20].

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