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FIXED POINT OF MAPPINGS OF PEROV TYPE FOR w-CONE DISTANCE

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A b s t r a c t. In this paper, we investigate fixed points of mappings of Perov type for w-cone distance. Many results of Ćiric, Lakzian and Rakočević, Suzuki and Takahashi, Abbas and Rhoades, Pathak and Shahzad and Raja and Veazpour are generalized. Our results could not be obtained by Du's scalarization method because, in our case, contractive constant is replaced by an operator.

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1. *Introduction and preliminaries*

Kada, Suzuki and Takahashi [12] introduced w-distance in 1996. and indicated that it is more general concept than metric. They gave examples of w-distance and improved Caristi's fixed point theorem [3], Eklands variationals principle [8] and the nonconvex minimization theorem according to Takahashi [18].

Definition 1.1 ([12]). Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:
(1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;

(2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semi-continuous;

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \le \delta$ and $p(z, y) \le \delta$ imply $d(x, y) \le \varepsilon$.

Example 1.1. If $(X, \|\cdot\|)$ is a normed space, then $w(x, y) = \|x\| + \|y\|$, $x, y \in X$, is an w-distance on X. Obviously, every metric d is an w-distance.

Many papers considering different fixed point conditions with w-distance were recently published.

Kurepa [13] in 1934 has initiated the idea of more general concept of metric space that was later introduced by Zabreiko [20] as K-metric space and by Huang and Zhang [10] as cone metric space. There were published many results concerning fixed point theorems on both normal and non-normal cone metric spaces in the sense of Huang and Zhang (see e.g., [1, 11, 20]).

Du [7] has noticed that some fixed point theorems on cone metric spaces could be easily derived for similar results on metric space by scalarization. His scalarization method establishes many equivalences between fixed point results on metric and cone metric spaces, but it is important to mention that there exists many exceptions where the scalarization method isn't applicable. The results presented in this article could not be derived from analogous results on metric spaces by Du's scalarization method since the contractive condition contains an operator instead of a constant.

In 2013 Ćirić, Lakzian and Rakočević [5] generalized w-distance concept of Kada et al. to the tvs-cone metric space where the underlying cone is in topological vector space instead of Banach space as in [10]. Therefore, they generalized many results including [1], [9], [15] and [19] and established some unsolved problems.

Let us remark that Perov [16] studied Banach contraction principle on a generalized metric space. He replaced the contractive constant with a matrix with nonnegative entries and spectral radius less than 1. His generalized metric space is a special case of a normal cone metric space. In this article, we investigate fixed points of mappings of Perov type for w-cone distance but we include a bounded linear operator instead of a contractive constant.

For the convenience of the reader, we give some basic definitions and properties related to cone metric spaces and w-cone distance that are presented in [1, 5, 10, 12].

Let E be a real Banach space and $\theta \in E$ the zero vector. A subset P of E is called a cone if:

(i) P is closed, nonempty and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \ge 0$, and $x, y \in P$ imply $ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subseteq E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ denotes $y - x \in \operatorname{int} P$ (interior of P).

The cone P in a real Banach space E is called normal if

$$\inf\big\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\big\} > 0 \tag{1.1}$$

or, equivalently, if there is a number $K > 0$ such that for all $x, y \in P$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\,\|y\|\,. \tag{1.2}$$

The least positive number satisfying (1.2) is called the normal constant of P.

The cone P is called solid if int$(P) \neq \emptyset$. Further on, in the case that P is a non-normal, we will assume that P is a solid cone.

Definition 1.2 ([10])**.** Let X be a nonempty set, and let P be a cone on a real ordered Banach space E. Suppose that the mapping $d : X \times X \mapsto E$ satisfies:

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 1.2. Let $E = C^1[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ on } [0, 1]\}$. This cone is not normal. Consider, for example,

$$x_n(t) = \frac{1 - \sin nt}{n + 2} \quad \text{and} \quad y_n(t) = \frac{1 + \sin nt}{n + 2}.$$

Since, $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| = \frac{2}{n+2} \to 0$, it follows by (1.1) that P is a non-normal cone.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every c in E with $\theta \ll c$, there is n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$, then it is said that $\{x_n\}$ converges to x, and we denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$, $n \to \infty$. If for every c in E with $\theta \ll c$, there is n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

The following properties are often used (particulary when dealing with cone metric spaces in which the cone need not to be normal):

(p_1) If $u \preceq v$ and $v \ll w$ then $u \ll w$.

(p_2) If $a \preceq b + c$ for each $c \in \operatorname{int} P$ then $a \preceq b$.

(p_3) If $\theta \preceq x_n \preceq y_n$ for each $n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$, then $\theta \preceq x \preceq y$.

(p_4) If $\theta \preceq d(x_n, x) \preceq b_n$ and $b_n \to \theta$, then $x_n \to x$.

(p_5) If $c \in \operatorname{int} P$, $\theta \preceq a_n$ and $a_n \to \theta$, $n \to \infty$, then there exists n_0 such that for any $n > n_0$ we have $a_n \ll c$.

From the property (p_5) it follows that the sequence $\{x_n\}$ converges to some $x \in X$ if $d(x_n, x) \to \theta$ as $n \to \infty$ and $\{x_n\}$ is a Cauchy sequence if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$. In the situation with a non-normal cone, we have only one part of Lemmas 1 and 4 from [10]. Also, in this case the fact that $d(x_n, y_n) \to d(x, y)$, if $x_n \to x$ and $y_n \to y$, is not applicable.

In the definition of w-cone distance, Ćirić *at al.* assumed that E is a real Hausdorff topological vector space (*tvs* for short) but instead we will, as for cone metric space, assume that E is a Banach space.

A mapping $T : X \mapsto X$ is a continuous mapping on X if for any $x \in X$ and a sequence $\{x_n\} \subseteq X$ such that $\lim\limits_{n\to\infty} x_n = x$, it follows $\lim\limits_{n\to\infty} Tx_n = Tx$.

Function $G : X \to P$ is lower semi-continuous at $x \in X$ if for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that

$$G(x) \preceq G(x_n) + \varepsilon, \quad \text{for all} \quad n \geq n_0, \tag{1.3}$$

whenever $\{x_n\}$ is a sequence in X and $x_n \to x$, $n \to \infty$.

Definition 1.3 ([5])**.** Let (X, d) be a cone metric space. Then a function $p : X \times X \to P$ is called a w-cone distance on X if the following conditions are satisfied:

(w_1) $p(x, z) \preceq p(x, y) + p(y, z)$, for any $x, y, z \in X$;

(w_2) For any $x \in X$, $p(x, \cdot) : X \to P$ is lower semi-continuous;

(w_3) For any ε in E with $\theta \ll \varepsilon$, there is δ in E with $\theta \ll \delta$, such that $p(z, x) \ll \delta$ and $p(z, y) \ll \delta$ imply $d(x, y) \ll \varepsilon$.

It is important to mention that every cone metric is w-cone distance and there exist w-cone distances such that underlying cone is not normal.

Lemma 1.1 ([5])**.** *Let (X, d) be a tvs-cone metric space and let p be a w-cone distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ with $\theta \preceq \alpha_n$, and $\{\beta_n\}$ with $\theta \preceq \beta_n$, be sequences in E converging to θ, and $x, y, z \in X$. Then:*

(i) *If $p(x_n, y) \preceq \alpha_n$ and $p(x_n, z) \preceq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = \theta$ and $p(x, z) = \theta$, then $y = z$.*

(ii) *If $p(x_n, y_n) \preceq \alpha_n$ and $p(x_n, z) \preceq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.*

(iii) *If $p(x_n, x_m) \preceq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

(iv) *If $p(y, x_n) \preceq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

We write $\mathscr{B}(E)$ for the set of all bounded linear operators on E and $\mathscr{L}(E)$ for the set of all linear operators on E. $\mathscr{B}(E)$ is a Banach algebra, and if $\mathcal{A} \in \mathscr{B}(E)$ let

$$r(\mathcal{A}) = \lim_{n\to\infty} \|\mathcal{A}^n\|^{1/n} = \inf_n \|\mathcal{A}^n\|^{1/n}$$

be the spectral radius of $\mathcal{A}$. Let us remark that if $r(\mathcal{A}) < 1$, then the series $\sum\limits_{n=0}^{\infty} \mathcal{A}^n$ is absolutely convergent, $\mathcal{I} - \mathcal{A}$ is invertible in $\mathscr{B}(E)$ and

$$\sum_{n=0}^{\infty} \mathcal{A}^n = (I - \mathcal{A})^{-1}.$$

Furthermore, if $\|\mathcal{A}\| < 1$, then $\mathcal{I} - \mathcal{A}$ is invertible and

$$\|(\mathcal{I} - \mathcal{A})^{-1}\| \le \frac{1}{1 - \|\mathcal{A}\|}.$$

If $\mathcal{A}, \mathcal{B} \in \mathscr{B}(E)$ and $\mathcal{AB} = \mathcal{BA}$, then $r(\mathcal{A} + \mathcal{B}) \le r(\mathcal{A}) + r(\mathcal{B})$ and $r(\mathcal{AB}) \le r(\mathcal{A})r(\mathcal{B})$.

2. *Main results*

We start with two auxiliary results.

Lemma 2.1. *Let E be a Banach space and $\mathcal{A} \in \mathscr{B}(E)$ a bounded linear operator. If $r(\mathcal{A}) < 1$, then*

$$r\left((\mathcal{I} - \mathcal{A})^{-1}\right) \le \frac{1}{1 - r(\mathcal{A})}.$$

PROOF. If $r(\mathcal{A}) < 1$, then, as previously stated, $(\mathcal{I} - \mathcal{A})^{-1} = \sum\limits_{n=0}^{\infty} \mathcal{A}^n$, and

$$r\left((\mathcal{I} - \mathcal{A})^{-1}\right) = r\left(\mathcal{I} + \mathcal{A}(\mathcal{I} - \mathcal{A})^{-1}\right) \le 1 + r(\mathcal{A})r\left((\mathcal{I} - \mathcal{A})^{-1}\right).$$

Therefore,

$$r\left((\mathcal{I} - \mathcal{A})^{-1}\right) \le \frac{1}{1 - r(\mathcal{A})}. \qquad \square$$

Lemma 2.2 ([4])**.** *Let E be Banach space, $P \subseteq E$ cone in E and $A : E \mapsto E$ a linear operator. The following conditions are equivalent:*

(i) *$\mathcal{A}$ is increasing, i.e., $x \le y$ implies $\mathcal{A}(x) \le \mathcal{A}(y)$.*

(ii) *$\mathcal{A}$ is positive, i.e., $\mathcal{A}(P) \subseteq P$.*

PROOF. If $\mathcal{A}$ is a monotonically increasing and $p \in P$, it follows $p \succeq \theta$ and $\mathcal{A}(p) \succeq \mathcal{A}(\theta) = \theta$. Thus, $\mathcal{A}(p) \in P$, and $\mathcal{A}(P) \subseteq P$.

To prove the other implication, let us assume that $\mathcal{A}(P) \subseteq P$ and $x, y \in E$ are such that $x \preceq y$. Now, $y - x \in P$, and so $\mathcal{A}(y - x) \in P$.

Thus, finally $\mathcal{A}(x) \preceq \mathcal{A}(y)$. $\square$

In the following theorem, which extends and improves Theorem 2 of [12] and Theorem 1 of [17], we give an estimation for a $w-$cone distance $p(x_n, z)$ of an approximate value x_n and a fixed point z.

Theorem 2.1. *Let (X, d) be a complete cone metric space with w-cone distance p on X. Suppose that for some increasing operator $\mathcal{A} \in \mathscr{B}(E)$, $r(\mathcal{A}) < 1$, a mapping $T : X \to X$ satisfies the following condition:*

$$p(Tx, T^2x) \preceq \mathcal{A}\left(p(x, Tx)\right), \quad \text{for all } x \in X. \tag{2.4}$$

Assume that either of the following holds:

(i) *If $y \neq Ty$, there exists $c \in \operatorname{int}(P)$, $c \neq \theta$, such that*

$$c \ll p(x, y) + p(x, Tx), \quad \text{for all } x \in X;$$

(ii) *T is continuous.*

Then, there exists $z \in X$, such that $z = Tz$ and

$$p(T^n x, z) \preceq \mathcal{A}^n \left(I - \mathcal{A}\right)^{-1} \left(p(x, Tx)\right), \quad for \quad n \in \mathbb{N}, \tag{2.5}$$

where $z = \lim\limits_{n\to\infty} T^n x$.

Moreover, if $y = Ty$ for some $y \in X$, then $p(y, y) = \theta$.

PROOF. Let $x \in X$ be arbitrary and define a sequence $\{x_n\}$ by $x_0 = x$, $x_n = T^n x$, for any $n \in \mathbb{N}$. Then from (2.4) we have, for any $n \in \mathbb{N}$,

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\preceq \mathcal{A}(p(x_{n-1}, x_n)) \preceq \cdots \preceq \mathcal{A}^n(p(x, Tx)), \end{aligned} \tag{2.6}$$

since $\mathcal{A}$ is an increasing operator. Thus, if $m > n$, then from (w_1) of Definition 1.3

and (2.6),

$$\begin{aligned} p(x_n, x_m) &\preceq \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \\ &\preceq \sum_{i=n}^{m-1} \mathcal{A}^i(p(x, Tx)) \\ &\preceq \sum_{i=n}^{\infty} \mathcal{A}^i(p(x, Tx)) \\ &= \mathcal{A}^n(\mathcal{I} - \mathcal{A})^{-1}(p(x, Tx)). \end{aligned} \tag{2.7}$$

However $\mathcal{A}^n(\mathcal{I} - \mathcal{A})^{-1}(p(x, Tx)) \to \theta$, $n \to \infty$, so $\{x_n\}$ is a Cauchy sequence in X by Lemma 1.1 and, because X is a complete, $\{x_n\}$ converges to some $z \in X$.

We will prove that z is a fixed point of T by estimating $p(x_n, z)$. Since $x_n \to z$, as $n \to \infty$, from the lower semi-continuity of w distance, we have that for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that for any $m \geq n_0$

$$p(x_n, z) \preceq p(x_n, x_m) + \varepsilon.$$

Therefore, for an arbitrary $n \in \mathbb{N}$, if we choose $m > \max\{n, n_0\}$, then, from (2.7), it follows that the inequality

$$p(x_n, z) \preceq \mathcal{A}^n(\mathcal{I} - \mathcal{A})^{-1}(p(x, Tx)) + \varepsilon$$

holds for any $\varepsilon \gg \theta$, i.e., (2.5) holds for any $n \in \mathbb{N}$.

Let us assume that (i) is satisfied and that $Tz \neq z$. Then, there exists $c \gg \theta$, $c \neq \theta$, such that

$$c \ll p(x, z) + p(x, Tx), \quad \text{for all} \quad x \in X. \tag{2.8}$$

Obviously, $\mathcal{A}^n(\mathcal{I} - \mathcal{A})^{-1}(p(x, Tx)) \to \theta$ and $\mathcal{A}^n(p(x, Tx)) \to \theta$ as $n \to \infty$, so, from the definition of convergence and (p_5), there exists $n_1 \in \mathbb{N}$ such that

$$\mathcal{A}^n(\mathcal{I} - \mathcal{A})^{-1}(p(x, Tx)) \ll \frac{c}{3} \quad \text{and} \quad \mathcal{A}^n(p(x, Tx)) \ll \frac{c}{3},$$

for any $n \geq n_1$. The last observation contradicts to (2.8) since, for any $n \geq n_1$ inequalities

$$\begin{aligned} c &\ll p(x_n, z) + p(x_n, x_{n+1}) \\ &\preceq \mathcal{A}^n(\mathcal{I} - \mathcal{A})^{-1}(p(x, Tx)) + \mathcal{A}^n(p(x, Tx)) \\ &\ll \frac{2c}{3}, \end{aligned}$$

imply that $-c/3 \gg 0$, i.e., $c = \theta$. But, we have already assumed that $c \neq \theta$, hence $Tz = z$ in this case.

Otherwise, if T is a continuous, then, since $x_{n+1} = Tx_n \to Tz$, $n \to \infty$, by (i) of Lemma 1.1, we may conclude that $Tz = z$.

It remains to prove that if $Ty = y$, then $p(y,y) = \theta$. Obviously,

$$p(y,y) = p(Ty, T^2y) \preceq \mathcal{A}(p(y,Ty)).$$

The operator $(\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \mathcal{A}^n$ is an increasing linear operator and the last inequality gives us $p(y,y) \preceq (\mathcal{I} - \mathcal{A})^{-1}(\theta) = \theta$, i.e., $p(y,y) = \theta$. □

Example 2.1. Let $X = E$, where E and P are defined as in Example 1.2. Let us define cone metric $d : X \times X \mapsto E$ for any $f, g \in X$ by

$$d(f,g) = \begin{cases} f+g, & f \neq g, \\ 0, & f = g. \end{cases}$$

If $T : X \mapsto X$ is defined by $T(f) = f/2$, $f \in X$, then

$$d(Tf, T^2f) \preceq \mathcal{A}(d(f,Tf)), \quad f \in X,$$

where $\mathcal{A} : E \mapsto E$, is a bounded linear operator defined by $\mathcal{A}(f) = f/2$, $f \in E$.

Clearly, $r(\mathcal{A}) = \|\mathcal{A}\| = 1/2$ and T is a continuous, thus all the assumptions from Theorem 2.1 are satisfied. Hence, T has a fixed point $f = 0 \in X$ and it is evidently an unique fixed point of T.

Corollary 2.1. *Let (X,d) be a complete cone metric space with $w-$cone distance p on X and $\mathcal{A} \in \mathscr{B}(E)$ an increasing operator with spectral radius less than $1/2$. Suppose that the mapping $T : X \to X$ satisfies either* (i) *or* (ii) *of Theorem* 2.1 *and*

$$p(Tx, T^2x) \preceq \mathcal{A}(p(x, T^2x)), \quad \textit{for all } x \in X.$$

Then, there exists $z \in X$, such that $z = Tz$ and if $y = Ty$, then $p(y,y) = \theta$.

PROOF. If $x \in X$ is arbitrary, then

$$p(Tx, T^2x) \preceq \mathcal{A}(p(x, T^2x)) \preceq \mathcal{A}(p(x,Tx) + p(Tx, T^2x)).$$

Hence,

$$p(Tx, T^2x) \preceq \mathcal{A}(\mathcal{I} - \mathcal{A})^{-1}(p(x,Tx)).$$

Observe that

$$\begin{aligned} r(\mathcal{A}(\mathcal{I}-\mathcal{A})^{-1}) &\leq r(\mathcal{A})r\left((\mathcal{I}-\mathcal{A})^{-1}\right) \\ &\leq \frac{r(\mathcal{A})}{1-r(\mathcal{A})} < 1, \end{aligned}$$

and the condition (2.4) is satisfied. All the conclusions of this corollary follows directly from Theorem 2.1. □

If $T : X \to X$ and $F(T)$ is a set of all fixed points of T, then T has a property P if $F(T) = F(T^n)$ for each $n \in \mathbb{N}$. The following theorem extends and improves Theorem 2 of [1] and Theorem 12 of [5] for cone metric space.

Theorem 2.2. *Let (X, d) be a complete cone metric space with w-cone distance p on X. Suppose $T : X \to X$ satisfies the condition (2.4) for an increasing operator $\mathcal{A} \in \mathscr{B}(E)$. If $r(\mathcal{A}) < 1$, then T has property P.*

PROOF. Obviously, $F(T) \subseteq F(T^n)$, $n \in \mathbb{N}$, so it remains to show that $Tz = z$ for any $z \in F(T^n)$ and arbitrary $n > 1$.

Remark that

$$p(T^i z, T^{i+1} z) = p(T^{kn+i} z, T^{kn+i+1} z) \preceq \mathcal{A}^{kn+i}(p(z, Tz)), \quad k, i \in \mathbb{N},$$

allows us to determine that, because $\mathcal{A}^{kn+i}(p(z, Tz)) \to 0$, as $k \to \infty$, when $r(\mathcal{A}) < 1$, $p(T^i z, T^{i+1} z) = \theta$, $i \in \mathbb{N}$, and, furthermore, $Tz = T^n z = z$. □

Instead of observing contractive conditions on X, we observe only T-orbit $O(x, \infty)$ of an arbitrary element $x \in X$ where $O(x, \infty) = \{T^n x \mid n \in \mathbb{N}_0\}$.
Function $G : X \to P$ is a T-orbitally lower semi-continuous at x if for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that (1.3) holds whenever $\{x_n\} \subseteq O(x; \infty)$ and $x_n \to x$, $n \to \infty$.

The following theorems implies some results of [9], [15], [10] and [19].

Theorem 2.3. *Let (X, d) be a complete cone metric space with w-cone distance p on X and $\mathcal{A} \in \mathscr{B}(E)$ an increasing operator with spectral radius less than 1. Suppose that $T : X \to X$ and there exists an $x \in X$ such that*

$$p(Ty, T^2 y) \preceq \mathcal{A}(p(y, Ty)), \quad \text{for all } y \in O(x, \infty).$$

Then,

(i) $\lim_{n\to\infty} T^n x = z$ *exists and*

$$p(T^n x, z) \preceq \mathcal{A}^n (\mathcal{I} - \mathcal{A})^{-1} \left(p(x, Tx)\right), \quad n \in \mathbb{N};$$

(ii) $p(z,Tz) = \theta$ *if and only if* $G(x) = p(x,Tx)$ *is* T*-orbitally lower semi-continuous at* z*.*

PROOF. (i) First observation easily follows from the proof of Theorem 2.1.

(ii) If $p(z,Tz) = \theta$ then G is obviously T-orbitally lower semi-continuous at z. Otherwise, choose $\varepsilon \gg \theta$ arbitrary. There exists $n_1 \in \mathbb{N}$ such that

$$\mathcal{A}^n\left(p(x,Tx)\right) \ll \frac{\varepsilon}{2}$$

for any $n \geq n_1$, and $n_2 \in \mathbb{N}$ such that

$$G(z) \preceq G(T^n x) + \frac{\varepsilon}{2}, \quad n \geq n_2.$$

Then, for $n \geq \max\{n_1, n_2\}$,

$$\begin{aligned} p(z,Tz) &\preceq p(T^n x, T^{n+1}x) + \frac{\varepsilon}{2} \\ &\preceq \mathcal{A}^n(p(z,Tx)) + \frac{\varepsilon}{2} \ll \varepsilon. \end{aligned}$$

The last inequality holds for any $\varepsilon \gg \theta$, and by (p_2), $p(z,Tz) = \theta$. $\square$

Theorem 2.4. *Let* (X,d) *be a complete cone metric space with* w*-cone distance* p *on* X *and* $\mathcal{A} \in \mathscr{B}(E)$ *an increasing operator with spectral radius less than* 1*. Suppose that* $T : X \to X$ *is a p-contractive mapping of Perov type, i.e.,*

$$p(Tx,Ty) \preceq \mathcal{A}\left(p(x,y)\right), \quad \textit{for all } x,y \in X.$$

Then, T *has a unique fixed point* $z \in X$*, and* $p(z,z) = \theta$*.*

PROOF. ¿From the proof of Theorem 2.1 we get that $T^n x \to z$ as $n \to \infty$, $Tz = z$ and $p(z,z) = \theta$.

If $Ty = y$, then

$$p(y,z) = p(Ty,Tz) \preceq \mathcal{A}(p(y,z)) \implies p(y,z) \preceq (\mathcal{I} - \mathcal{A})^{-1}(\theta) = \theta,$$

thus $p(y,z) = \theta$ and $p(z,z) = \theta$ imply, by (i) of Lemma 1.1, that $y = z$. $\square$

Example 2.2. Let X be $C[0,1]$, set of real continuous functions on a closed interval $[0,1]$ with a norm $\|x\| = \max_{t \in =[0,1]} |x(t)|$, $x \in X$, and $P \subseteq X$ a cone defined with

$$x \in P \Leftrightarrow x(t) \geq 0 \ \text{ for all } t \in [0,1].$$

Then, $d(x,y) = |x-y|$, $x,y \in X$, is a cone metric on X, where $|x|(t) = |x(t)|$, $t \in [0,1]$, and $p = d$. If $f \in X$ is chosen arbitrary, then for $0 < L < 1$, define a mapping $T : X \mapsto X$,

$$(Tx)(t) = f(t) + \int_0^t Lx(\sqrt{s})\,\mathrm{d}s, \quad t \in [0,1].$$

Remark that T is a continuous mapping.

Let us define a bounded linear operator $\mathcal{A} \in \mathscr{B}(X)$,

$$(Ax)(t) = \int_0^t Lx(\sqrt{s})\,\mathrm{d}s, \quad t \in [0,1].$$

Zima proved in [21] that the spectral radius of operator $\mathcal{A}$ is $L/2$, thus less than 1 and, evidently, $\mathcal{A}$ is an increasing operator. Then, easily follows,

$$p(Tx, T^2x) \preceq \mathcal{A}(p(x,Tx)), \quad x \in X$$

Hence, we may apply Theorem 2.1 and conclude that there exists $g \in X$ such that $Tg = g$.

Remark 2.1. Theorem 2.1 does not imply uniqueness of fixed point but it is not difficult to show that T has an unique fixed point in X. Let us suppose that $Th = h$. Then

$$d(g,h)(t) = d(Tg,Th)(t) = \left| \int_0^t L(g(\sqrt{s}) - h(\sqrt{s}))\,\mathrm{d}s \right| \leq Ltd(g,h)(t),$$

and $Lt < 1$ implies $d(g,h)(t) = 0$ for any $t \in [0,1]$, i.e., $g = h$.

Remark 2.2. Let us remark that De Pascale and De Pascale [6] used K-normed space to prove that Lou's fixed point theorem [14] in a space of continuous functions is equivalent to the Banach contraction principle with contractive constant replaced by bounded linear operator with spectral radius < 1. Observe that in [6] cone is normal, but we have investigated the case when cone is not normal [4]. It is interesting to to investigate possibility of extending Lou's theorem in the case when cone is not normal.

Let us notice that the condition $\mathcal{A}(P) \subseteq P$ is unnecessary in some cases.

Example 2.3. Let

$$\mathcal{A} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$X = \left\{ \begin{bmatrix} x_1 \\ 1 \\ x_3 \end{bmatrix} \mid x_1, x_3 \in \mathbb{R} \right\}$ and $T : X \mapsto X$ a mapping defined by

$$T\left(\begin{bmatrix} x_1 \\ 1 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{2}(x_1+1) \\ 1 \\ \frac{1}{3}(x_3+2) \end{bmatrix}.$$

Set $\|x\| = \max\{|x_1|, 1, |x_3|\}$, $x \in X$, and $d(x,y) = \begin{bmatrix} |x_1 - y_1| \\ 0 \\ |x_3 - y_3| \end{bmatrix}$, $x, y \in X$, and $p = d$.

It is easy to show that $\|\mathcal{A}\| = \frac{3}{4}$ and, consequently, $r(\mathcal{A}) < 1$. Also,

$$p(Tx, T^2x) \leq \mathcal{A}(p(x, Tx)), \quad x \in X.$$

Clearly, $\mathcal{A}(P) \nsubseteq P$, and $(1,1,1)$ is a fixed point of T in X.

We state the similar results when cone metric space (X, d) is normal by replacing the condition $r(\mathcal{A}) < 1$ and excluding the condition that the operator $\mathcal{A}$ is increasing, i.e., not demanding the condition $\mathcal{A}(P) \subseteq P$.

Theorem 2.5. *Let (X, d) be a complete normal cone metric space with normal constant K and w-cone distance p on X. Suppose that for some operator $\mathcal{A} \in \mathscr{B}(E)$, $K\|\mathcal{A}\| < 1$, a mapping $T : X \to X$ satisfies the following condition:*

$$p(Tx, T^2x) \preceq \mathcal{A}\left(p(x, Tx)\right), \quad \textit{for all } x \in X.$$

Assume that either of the following holds:

(i) *If $y \neq Ty$, there exists $c > 0$, such that*

$$c < \|p(x,y)\| + \|p(x,Tx)\|, \quad \textit{for all } x \in X;$$

(ii) *T is continuous.*

Then, there exists $z \in X$, such that $z = Tz$ and if $y = Ty$ for some $y \in X$, then $p(y, y) = \theta$.

Proof. Let $x \in X$ be an arbitrary and let us define a sequence $\{x_n\}$, $x_0 = x$, $x_n = T^n x$, for any $n \in N$. Then,

$$\|p(x_n, x_{n+1})\| \leq K\|\mathcal{A}\|\|p(x_{n-1}, x_n)\| \leq \cdots \leq (K\|\mathcal{A}\|)^n \|p(x, Tx)\|. \quad (2.9)$$

Thus, if $m > n$, then from (w_1) of Definition 1.3 and (2.6),

$$\begin{aligned} \|p(x_n, x_m)\| &\leq \sum_{i=n}^{m-1} (K\|\mathcal{A})^i \|p(x, Tx)\| \\ &\leq \sum_{i=n}^{\infty} (K\|\mathcal{A}\|)^i \|p(x, Tx)\| \\ &= \frac{(K\|\mathcal{A}\|)^n}{1 - K\|\mathcal{A}\|} \|p(x, Tx)\|. \end{aligned} \tag{2.10}$$

However,

$$\frac{(K\|\mathcal{A}\|)^n}{1 - K\|\mathcal{A}\|} \|p(x, Tx)\| \to 0, \quad n \to \infty,$$

so $\{x_n\}$ is a Cauchy sequence in X and, because X is complete, $\{x_n\}$ converges to some $z \in X$.

¿From the lower semi-continuity of w distance, we have that for any $\varepsilon \gg \theta$, there is $n_0 \in \mathbb{N}$ such that for any $m \geq n_0$

$$p(x_n, z) \preceq p(x_n, x_m) + \varepsilon.$$

Moreover, for arbitrary $n \in \mathbb{N}$, if we choose $m > n$, then from (2.10) it follows that the inequality

$$\|p(x_n, z)\| \leq \frac{(K\|\mathcal{A}\|)^n}{1 - K\|\mathcal{A}\|} \|p(x, Tx)\| + K\|\varepsilon\|$$

holds for any $\varepsilon \gg \theta$, so, for $\varepsilon := \varepsilon/n$, $n \in \mathbb{N}$,

$$\|p(x_n, z)\| \leq \frac{(K\|\mathcal{A}\|)^n}{1 - K\|\mathcal{A}\|} \|p(x, Tx)\|.$$

Let us assume that (i) is satisfied and that $Tz \neq z$. Then, there exists $c > 0$ such that

$$c < \|p(x, z)\| + \|p(x, Tx)\|, \quad \text{for all} \quad x \in X. \tag{2.11}$$

Then,

$$c < \frac{(K\|\mathcal{A}\|)^n}{1 - K\|\mathcal{A}\|} \|p(x, Tx)\| + (K\|\mathcal{A}\|)^n \|p(x, Tx)\|$$

for any $n \in \mathbb{N}$ and that is impossible since (2.11) holds.

Otherwise, if T is continuous, then, since $x_{n+1} = Tx_n \to Tz$, $n \to \infty$, it follows $Tz = z$.

It remains to prove that if $Ty = y$, then $p(y, y) = \theta$. Obviously,

$$\|p(y, y)\| = \|p(T^n y, T^{n+1} y)\| \leq (K\|\mathcal{A}\|)^n \|p(y, Ty)\|, \quad n \in \mathbb{N},$$

implies $\|p(y, y)\| = 0$. □

Corollary 2.2. *Let (X, d) be a complete normal cone metric space with normal constant K, w-cone distance p on X and $\mathcal{A} \in \mathscr{B}(E)$ an operator, $K\|\mathcal{A}\| < 1/2$. Suppose that the mapping $T : X \to X$ satisfies either* (i) *or* (ii) *of Theorem* 2.1 *and*

$$p(Tx, T^2x) \preceq \mathcal{A}(p(x, T^2x)), \quad \textit{for all } x \in X.$$

Then, there exists $z \in X$, such that $z = Tz$ and if $y = Ty$, then $p(y, y) = \theta$.

PROOF. If $x \in X$ is arbitrary, then

$$\|p(Tx, T^2x)\| \leq K\|\mathcal{A}\|\|p(x, Tx)\| + K\|\mathcal{A}\|\|p(Tx, T^2x)\|.$$

Hence,

$$\|p(Tx, T^2x)\| \leq \frac{K\|\mathcal{A}\|}{1 - K\|\mathcal{A}\|} \, \|p(x, Tx)\, .$$

and, since $K\|\mathcal{A}\|/(1 - K\|\mathcal{A}\|) < 1$ it directly follows by Theorem 2.5. □

Theorem 2.6. *Let (X, d) be a complete normal cone metric space with normal constant K and w-cone distance p on X. Suppose $T : X \to X$ satisfies the condition* (2.4) *for an operator $\mathcal{A} \in \mathscr{B}(E)$. If $K\|\mathcal{A}\| < 1$, then T has property P.*

PROOF. As in the proof of previously stated theorem, it follows that, for any $z \in F(T^n)$ and $i \in \mathbb{N}$,

$$\|p(T^i z, T^{i+1}z)\| \leq (K\|\mathcal{A}\|)^{kn+i}\|p(z, Tz)\| \to 0, \quad k \to \infty,$$

thus $Tz = T^n z = z$. □

The proofs of the following theorems follows similarly as in the case when cone metric space is not normal.

Theorem 2.7. *Let (X, d) be a complete normal cone metric space with normal constant K, w-cone distance p on X, $\mathcal{A} \in \mathscr{B}(E)$ an operator such that $K\|\mathcal{A}\| < 1$. Suppose $T : X \to X$ and there exists an $x \in X$ such that*

$$p(Ty, T^2y) \preceq \mathcal{A}(p(y, Ty)), \quad \textit{for all } y \in O(x, \infty).$$

Then, (i) $\lim_{n\to\infty} T^n x = z$ *exists and*

$$\|p(T^n x, z)\| \leq \frac{(K\|\mathcal{A}\|)^n}{1 - K\|\mathcal{A}\|} \, \|p(x, Tx)\| \quad for\ n \in \mathbb{N};$$

(ii) $p(z, Tz) = \theta$ *if and only if $G(x) = p(x, Tx)$ is T-orbitally lower semi-continuous at z.*

Theorem 2.8. *Let (X, d) be a complete normal cone metric space with normal constant K and w-cone distance p on X and $\mathcal{A} \in \mathscr{B}(E)$ an with such that $K\|\mathcal{A}\| < 1$. Suppose that $T : X \to X$ is a p-contractive mapping of Perov type, i.e.,*

$$p(Tx, Ty) \preceq \mathcal{A}\left(p(x, y)\right), \quad \textit{for all } x, y \in X.$$

Then, T has a unique fixed point $z \in X$, and $p(z, z) = \theta$.

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