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WALK COUNTS AND THE SPECTRAL RADIUS OF GRAPHS

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A b s t r a c t. *We develop a new method that uses walk counts for comparing spectral radii of graphs similar in a precisely defined fashion. The method is applied to the cases where a path-like or a star-like structure is coalesced to a graph, in order to prove weak inequality in the conjectured inequality of Belardo, Li Marzi and Simić, and to resolve the Brualdi-Solheid problem for the classes of graphs consisting of rooted products with the same rooted graph.*

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1. *Introduction*

Study of the spectral radius of adjacency matrix of graphs has been a central research theme in spectral graph theory since its inception in the 1950s [3] to this day. Numerous results on the spectral radius have been surveyed by Cvetković and Rowlinson [6] in 1990 and in a recent research monograph of the author [14].

Graphs mostly considered in the literature are simple graphs, due to the fact that their adjacency matrix is real and symmetric, so that its eigenvectors can be chosen to provide an orthonormal basis for $\mathbb{R}^n$ [8]. A simple graph $G = (V, E)$ consists of the vertex set V with $n = |V|$ vertices and the edge set $E \subseteq \binom{V}{2}$ with $m = |E|$ edges.

The adjacency matrix $A(G)$ of the simple graph G is the $n \times n$ matrix, indexed by V, defined by

$$A(G)_{uv} = \begin{cases} 1, & \text{if } uv \in E, \\ 0, & \text{if } uv \notin E. \end{cases}$$

Let us denote the eigenvalues of $A(G)$ by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G),$$

and the corresponding orthonormal eigenvectors by

$$x_1(G),\ x_2(G),\ \ldots,\ x_n(G),$$

so that

$$A(G)x_i(G) = \lambda_i(G)x_i(G), \qquad i = 1, \ldots, n. \tag{1.1}$$

and for $i, j = 1, \ldots, n$,

$$x_i^T(G)x_j(G) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{1.2}$$

In the sequel we will drop the parameter G when the graph is clear from the context.

The eigenvalues and the orthonormality of eigenvectors provide spectral decomposition of the adjacency matrix [14]:

$$A = \sum_{i=1}^{n} \lambda_i x_i x_i^T. \tag{1.3}$$

The eigenvalues of A are also the roots of its characteristic polynomial

$$P_G(\lambda) = \det(\lambda I - A). \tag{1.4}$$

By the Perron-Frobenius theorem [8, Chap. XIII], when the graph G is connected, its adjacency matrix A is irreducible, so that its largest eigenvalue λ_1 is also the spectral radius of A. In addition, λ_1 is a simple eigenvalue with a positive eigenvector x_1.

Most of the research on the spectral radius of graphs deals with the Brualdi-Solheid's general question [2] that asks to characterize graphs with extremal values of the spectral radius in a given class of graphs (where *extremal* usually means *maximal*). The basic ingredient in tackling such extremal problems is the ability to compare spectral radii of different candidate graphs. Two well-developed techniques are mostly used in the literature for such comparisons.

The first technique relies on the classical characterization of the largest eigenvalue λ_1 in terms of the Rayleigh quotient of A [8]:

$$\lambda_1 = \max_{y \neq 0} \frac{y^T A y}{y^T y} = \frac{2 \sum_{uv \in E} y_u y_v}{\sum_{u \in V} y_u^2}, \tag{1.5}$$

with the maximum attained for and only for $y = x_1$. From here it is easy to compare spectral radii of two graphs, where one of them is obtained by a small modification of the other one:

a) If $pq \notin E$ then

$$\begin{aligned} \lambda_1(G+pq) &\geq \frac{x_1^T A(G+pq) x_1}{x_1^T x_1} \\ &= \frac{x_1^T A x_1}{x_1^T x_1} + \frac{2x_{1,p}x_{1,q}}{x_1^T x_1} > \lambda_1, \end{aligned}$$

due to positivity of x_1 (and hence of $x_{1,p}x_{1,q}$).

b) If $pq \in E$, $pr \notin E$ and $x_{1,q} \leq x_{1,r}$, then [13]

$$\begin{aligned} \lambda_1(G-pq+pr) &\geq \frac{x_1^T A(G-pq+pr) x_1}{x_1^T x_1} \\ &= \frac{x_1^T A x_1}{x_1^T x_1} + \frac{2x_{1,p}(x_{1,r}-x_{1,q})}{x_1^T x_1} > \lambda_1. \end{aligned}$$

The equality cannot hold above as in such case one would have that x_1 is also the principal eigenvector of $G - pq + pr$ and that $x_{1,q} = x_{1,r}$, which would then imply contradictory statement $x_{1,r} = 0$, by considering the eigenvalue equation (1.1) in both G and $G - pq + pr$ at the vertex s.

c) If $pq, rs \in E$, $pr, qs \notin S$ and $(x_{1,p} - x_{1,s})(x_{1,r} - x_{1,q}) \geq 0$, then [7]

$$\begin{aligned} \lambda_1(G-pq-rs+pr+qs) &\geq \frac{x_1^T A(G-pq-rs+pr+qs) x_1}{x_1^T x_1} \\ &= \frac{x_1^T A x_1}{x_1^T x_1} + \frac{(x_{1,p}-x_{1,s})(x_{1,r}-x_{1,q})}{x_1^T x_1} \\ &\geq \lambda_1. \end{aligned}$$

The second technique relies on the fact that the value of the characteristic polynomial $P_G(y)$ is positive whenever $y > \lambda_1$. Thus, if one can show that for two graphs G and H holds

$$(\forall y > \lambda_1(G)) P_G(y) < P_H(y)$$

then $P_H(y)$ cannot have real roots that are greater than or equal to $\lambda_1(G)$, so that it must hold $\lambda_1(G) > \lambda_1(H)$.

Illustrative examples of the use of the first technique may be found in [10], and those of the use of the second technique both in [10] and [1].

Our goal here is to propose yet another technique for comparing spectral radii of two graphs, based on the comparisons of closed walk counts in these graphs. We have used comparisons of closed walk counts earlier to compare the Estrada indices of trees [11]. The technique presented in Section 2. is a comprehensive upgrade of the approach used in [11], applied to the spectral radius instead of the Estrada index. In Section 3. we show that the vertices of a path, in the rooted product of a path and another graph, have unimodal closed walk counts. This result helps to showcase fruitfulness of the walk count technique in Section 4., where we give new proofs of the well-known 1979 lemmas of Li and Feng [12], and prove weak inequality in the conjectured inequality of Belardo, Li Marzi and Simić [1].

2. *A walk count technique*

Let $G = (V, E)$ be a simple, connected graph with the adjacency matrix A, the eigenvalues $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ and the orthonormal eigenvectors $x_1, x_2, \ldots, x_n$. We assume that G is nontrivial, i.e., that it contains at least one edge. A sequence $W\colon u = u_0, u_1, \ldots, u_k = v$ of vertices from V such that $u_i u_{i+1} \in E$ is called a walk between u and v in G of length k. A walk W is closed if $u = v$. The following classical result relates the adjacency matrix of a graph to its walk counts:

Theorem 2.1 ([14])**.** *The number of walks of length k, $k \geq 0$, between the vertices u and v in G is equal to $(A^k)_{u,v}$.*

From the spectral decomposition (1.3) and the orthonormality of eigenvectors (1.2) we now have

$$A^k = \sum_{i=1}^{n} \lambda_i^k x_i x_i^T. \tag{2.6}$$

For $k \geq 0$, let N_k denote the number of all walks of length k in G, and let M_k denote the number of all closed walks of length k in G. From (2.6) we have

$$N_k = \sum_{u\in V}\sum_{v\in V}(A^k)_{u,v} = \sum_{i=1}^{n}\lambda_i^k\left(\sum_{u\in V}x_{i,u}\right)^2, \tag{2.7}$$

$$M_k = \sum_{u\in V}(A^k)_{u,u} = \sum_{i=1}^{n}\lambda_i^k\left(\sum_{u\in V}x_{i,u}^2\right) = \sum_{i=1}^{n}\lambda_i^k. \tag{2.8}$$

Lemma 2.1. *For a connected graph G we have*

$$\lambda_1 = \lim_{k\to\infty}\sqrt[k]{N_k}. \tag{2.9}$$

If G is not bipartite, then also

$$\lambda_1 = \lim_{k\to\infty}\sqrt[k]{M_k}, \tag{2.10}$$

while if G is bipartite, then

$$\lambda_1 = \lim_{k\to\infty}\sqrt[2k]{M_{2k}}. \tag{2.11}$$

The first equality above is taken from [4].

PROOF. All three equalities rely on the Perron-Frobenius theorem [8, Chapter XIII], which implies that $\lambda_1 \geq |\lambda_i|$ for each $i = 2, \dots, n$, and that the entries of x_1 in a connected graph G with at least one edge are strictly positive.

The distinction between bipartite and nonbipartite graphs stems from the fact that if G is bipartite, then the spectrum of G is symmetric with respect to zero [4]. In such case, $\lambda_n = -\lambda_1$ is also a simple eigenvalue of G, and if $V = V' \cup V''$, $V' \cap V'' = \emptyset$, represents a bipartition of G, then the eigenvector corresponding to λ_n satisfies

$$x_{n,u} = \begin{cases} x_{1,u}, & \text{if } u \in V', \\ -x_{1,u}, & \text{if } u \in V''. \end{cases}$$

Therefore,

$$\begin{aligned}\sqrt[2k'+1]{N_{2k'+1}} &= \lambda_1 \sqrt[2k'+1]{\left(\sum_{u\in V}x_{1,u}\right)^2 - \left(\sum_{u\in V}x_{n,u}\right)^2 + \sum_{i=2}^{n-1}\left(\frac{\lambda_i}{\lambda_1}\right)^{2k'+1}\left(\sum_{u\in V}x_{i,u}\right)^2} \\ &= \lambda_1 \sqrt[2k'+1]{2\left(\sum_{u\in V'}x_{1,u}\right)\left(\sum_{u\in V''}x_{1,u}\right) + \sum_{i=2}^{n-1}\left(\frac{\lambda_i}{\lambda_1}\right)^{2k'+1}\left(\sum_{u\in V}x_{i,u}\right)^2},\end{aligned}$$

$$\begin{aligned}\sqrt[2k']{N_{2k'}} &= \lambda_1 \sqrt[2k']{\left(\sum_{u\in V} x_{1,u}\right)^2 + \left(\sum_{u\in V} x_{n,u}\right)^2 + \sum_{i=2}^{n-1}\left(\frac{\lambda_i}{\lambda_1}\right)^{2k'}\left(\sum_{u\in V} x_{i,u}\right)^2} \\ &= \lambda_1 \sqrt[2k']{2\left(\sum_{u\in V'} x_{1,u}\right)^2 + 2\left(\sum_{u\in V''} x_{1,u}\right)^2 + \sum_{i=2}^{n-1}\left(\frac{\lambda_i}{\lambda_1}\right)^{2k'}\left(\sum_{u\in V} x_{i,u}\right)^2}.\end{aligned}$$

Eq. (2.9) follows from here, as both

$$\left(\sum_{u\in V'} x_{1,u}\right)\left(\sum_{u\in V''} x_{1,u}\right) \quad \text{and} \quad \left(\sum_{u\in V'} x_{1,u}\right)^2 + \left(\sum_{u\in V''} x_{1,u}\right)^2$$

are positive constants, and for each $i = 2, \ldots, n-1$ holds $|\lambda_i/\lambda_1| < 1$, while the term $\left(\sum_{u\in V} x_{i,u}\right)^2$ does not depend on k.

For the closed walks we have $M_{2k'+1} = 0$ for $k' \geq 0$, while

$$\sqrt[2k']{M_{2k'}} = \lambda_1 \sqrt[2k']{2 + \sum_{i=2}^{n-1}\left(\frac{\lambda_i}{\lambda_1}\right)^{2k'}},$$

from where (2.11) follows, due to $|\lambda_i/\lambda_1| < 1$ for each $i = 2, \ldots, n-1$.

On the other hand, if G is not bipartite, then $\lambda_n > -\lambda_1$, so that

$$\begin{aligned}\sqrt[k]{N_k} &= \lambda_1 \sqrt[k]{\left(\sum_{u\in V} x_{1,u}\right)^2 + \sum_{i=2}^{n}\left(\frac{\lambda_i}{\lambda_1}\right)^k \left(\sum_{u\in V} x_{i,u}\right)^2}, \\ \sqrt[k]{M_k} &= \lambda_1 \sqrt[k]{1 + \sum_{i=2}^{n}\left(\frac{\lambda_i}{\lambda_1}\right)^k}.\end{aligned}$$

From here both (2.9) and (2.10) follow, since $\left(\sum_{u\in V} x_{1,u}\right)^2$ is a positive constant and for each $i = 2, \ldots, n$, $|\lambda_i/\lambda_1| < 1$, while the term $\left(\sum_{u\in V} x_{i,u}\right)^2$ does not depend on k.

Our first new result is a simple lemma stating that a connected graph with more walks of arbitrarily large lengths also has the larger spectral radius.

Lemma 2.2. *Let G_1 and G_2 be connected graphs such that for an infinite sequence of indices $k_0 < k_1 < \ldots$ holds*

$$(\forall i \geq 0) \quad N_{k_i}(G_1) \geq N_{k_i}(G_2). \tag{2.12}$$

Then $\lambda_1(G_1) \geq \lambda_1(G_2)$.

PROOF. From Lemma 2.1 we get

$$\lim_{k\to\infty} \frac{\lambda_1(G_1)}{\lambda_1(G_2)} \sqrt[k]{\frac{N_k(G_2)}{N_k(G_1)}} = 1,$$

which implies

$$(\forall \varepsilon > 0)(\exists k_0)(\forall k \geq k_0) \quad \frac{\lambda_1(G_1)}{\lambda_1(G_2)} > (1-\varepsilon) \sqrt[k]{\frac{N_k(G_1)}{N_k(G_2)}}.$$

The condition (2.12), with i_0 taken to be the smallest index such that $k_{i_0} \geq k_0$, now implies

$$(\forall \varepsilon > 0)(\exists i_0)(\forall i \geq i_0) \quad \frac{\lambda_1(G_1)}{\lambda_1(G_2)} > 1-\varepsilon.$$

However, since $\lambda_1(G_1)$ and $\lambda_1(G_2)$ are the constants that do not depend on i, the previous expression actually means that

$$(\forall \varepsilon > 0) \quad \frac{\lambda_1(G_1)}{\lambda_1(G_2)} > 1-\varepsilon,$$

which is equivalent to $\lambda_1(G_1) \geq \lambda_1(G_2)$.

Remark 2.1. In order for previous lemma to imply that $\lambda_1(G_1)$ is strictly larger than $\lambda_1(G_2)$, instead of (2.12) one would need to prove that

$$(\exists \varepsilon > 0)(\forall i_0)(\exists i \geq i_0) \quad N_{k_i}(G_1) \geq \left(1 + \frac{\varepsilon}{1-\varepsilon}\right)^{k_i} N_{k_i}(G_2),$$

which is not always feasible.

We will, thus, allow our forthcoming results to include equality as a feasible case. When applied to graphs in a certain class, this essentially means that, while these lemmas provide characterization of the extremal value of the spectral radius of graphs in that class, they cannot provide characterization of all graphs with the extremal spectral radius. Instead, the lemmas will provide just one example of such extremal graph. In many classes the extremal graph is unique, so that the lemmas will necessarily pinpoint it, but they cannot be used to prove that there are no other extremal graphs.

It is obvious from Lemma 2.1 that the previous result can be stated in the terms of closed walk counts as well. We restrict ourselves here to closed walks of even length simply to avoid the trouble of considering whether the graphs in question are bipartite or not.

Lemma 2.3. *Let G_1 and G_2 be connected graphs such that for an infinite sequence of indices $k_0 < k_1 < \cdots$ holds*

$$(\forall i \geq 0) \quad M_{2k_i}(G_1) \geq M_{2k_i}(G_2). \tag{2.13}$$

Then $\lambda_1(G_1) \geq \lambda_1(G_2)$.

Let us now define a graph operation that will be the basis for our comparison technique.

Definition 2.1. Let F and G be the graphs with disjoint vertex sets $V(F)$ and $V(G)$. For $p \in \mathbb{N}$, let $u_1, \dots, u_p$ be distinct vertices from $V(F)$, and let $v_1, \dots, v_p$ be distinct vertices from $V(G)$. Assume, in addition, that there is no pair (i, j), $i \neq j$, such that both $u_i u_j$ is an edge of F and $v_i v_j$ is an edge of G. The multiple coalescence of F and G with respect to the vertex lists $u_1, \dots, u_p$ and $v_1, \dots, v_p$, denoted by

$$F(u_1 = v_1, \dots, u_p = v_p)G,$$

is the graph obtained from the union of F and G by identifying the vertices u_i and v_i for each $i = 1, \dots, p$.

The multiple coalescence is a generalization of the standard coalescence of two vertex-disjoint graphs, which is obtained by identifying a single pair of vertices, one from each graph [5]. Fig. 1 shows an example of multiple coalescence of the graphs F and G, with respect to the selected vertices u_1, u_2, u_3 and v_1, v_2, v_3.

The above assumption that for any $i \neq j$ it is not allowed that both $u_i u_j$ is an edge of F and $v_i v_j$ is an edge of G, serves to prevent the creation of multiple edges in the multiple coalescence. This assumption is needed later, as our goal will be to have each walk in the multiple coalescence clearly separated in smaller parts whose all edges will belong to only one of its constituents. In such setting, the vertices $v_1, \dots, v_p$ may be considered as the *entrance points* for a walk coming from F to enter G (and vice versa).

Our main tool is the following lemma.

Lemma 2.4. *Let F and G be graphs with disjoint vertex sets $V(F)$ and $V(G)$. For $p \in \mathbb{N}$, choose distinct vertices $u_1, \dots, u_p \in V(F)$, and make two separate choices of distinct vertices $v_1, \dots, v_p \in V(G)$ and $w_1, \dots, w_p \in V(G)$. Let G^v and G^w be the multiple coalescences*

$$\begin{aligned} G^v &= F(u_1 = v_1, \dots, u_p = v_p)G, \\ G^w &= F(u_1 = w_1, \dots, u_p = w_p)G, \end{aligned}$$

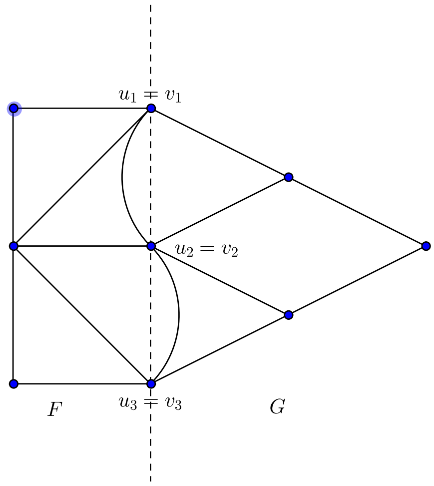

Figure 1. An example of multiple coalescence of two graphs

such that both G^v and G^w are connected.

Let A be the adjacency matrix of G. If for each $1 \leq i, j \leq p$ (including the case $i = j$) and for each $k \geq 1$ holds

$$(A^k)_{v_i,v_j} \geq (A^k)_{w_i,w_j}, \tag{2.14}$$

then

$$\lambda_1(G^v) \geq \lambda_1(G^w).$$

Note that in the above lemma, while we request that $v_i \neq v_j$ and $w_i \neq w_j$ for all $i \neq j$, the possibility that $v_i = w_j$ for some i and j is allowed.

PROOF. Let us first count the closed walks of length $2k$ in G^v. From the fact that F and G, as constituents of G^v, do not have common edges, we see that the number of closed walks in G^v, whose all edges belong to the same constituent, is equal to $M_{2k}(F) + M_{2k}(G)$.

The remaining closed walks in G^v contain edges from both F and G.

$$W : W_0, W_1, \ldots, W_{2l-1},$$

for some $l \in \mathbb{N}$, such that the edges of the even-indexed subwalks $W_0, \ldots, W_{2l-2}$ all belong to F, while the edges of the odd-indexed subwalks $W_1, \ldots, W_{2l-1}$ all belong to G. As a walk can enter from F to G only through one of the entrance points, we also see that the endpoints of the even-indexed subwalks belong to $\{u_1, \dots, u_p\}$, while the endpoints of the odd-indexed subwalks belong to $\{v_1, \dots, v_p\}$. Thus, let $(i_0, \dots, i_{2l-1})$ denote the $2l$-tuple of indices such that

the walk W_{2j} goes from $u_{i_{2j}}$ to $u_{i_{2j+1}} (= v_{i_{2j+1}})$ in F, while

the walk W_{2j+1} goes from $v_{i_{2j+1}}$ to $v_{i_{2j+2}} (= u_{i_{2j+2}})$ in G,

for $j = 0, \dots, l-1$. (The addition above is modulo $2l$, so that $i_{2l} = i_0$.)

In addition, let k_j denote the length of the walk W_j for $j = 0, \dots, 2l-1$. The $4l$-tuple

$$(i_0, \dots, i_{2l-1}; k_0, \dots, k_{2l-1})$$

is called the *signature* of the closed walk W. Due to the fact that the walk W is closed, its signatures are rotationally equivalent in the sense that the above signature is identical to the signature

$$(i_{2p}, \dots, i_{2l-1}, i_0, \dots, i_{2p-1}; k_{2p}, \dots, k_{2l-1}, k_0, \dots, k_{2p-1})$$

for each $p = 1, \dots, l-1$. In order to assign a unique signature to W, we may assume its signature is chosen to be lexicographically minimal among all rotationally equivalent signatures.

Now, let B be the adjacency matrix of F. Then for any feasible signature

$$(i_0, \dots, i_{2l-1}; k_0, \dots, k_{2l-1})$$

the number of closed walks in G^v with that signature is equal to

$$\prod_{j=0}^{l-1} (B^{k_{i_{2j}}})_{u_{i_{2j}}, u_{i_{2j+1}}} \prod_{j=0}^{l-1} (A^{k_{i_{2j+1}}})_{v_{i_{2j+1}}, v_{i_{2j+2}}}.$$

The argument is identical for closed walks of length $2k$ in G^w: the number of closed walks, whose all edges belong to the same constituent of G^w, is equal to

$$M_{2k}(F) + M_{2k}(G),$$

while the number of closed walks with the feasible signature $(i_0, \dots, i_{2l-1}; k_0, \dots, k_{2l-1})$ is equal to

$$\prod_{j=0}^{l-1} (B^{k_{i_{2j}}})_{u_{i_{2j}}, u_{i_{2j+1}}} \prod_{j=0}^{l-1} (A^{k_{i_{2j+1}}})_{w_{i_{2j+1}}, w_{i_{2j+2}}}.$$

From the condition (2.14) we now see that for any feasible signature the number of closed walks with that signature in G^v is larger than or equal to the number of such closed walks in G^w. Summing over all feasible signatures we obtain that

$$M_{2k}(G^v) \geq M_{2k}(G^w),$$

and, thus, from Lemma 2.3 we conclude that $\lambda_1(G^v) \geq \lambda_1(G^w)$.

The usefulness of the above lemma is clearly visible: in order to obtain an inequality between the spectral radii of the multiple coalescences G^v and G^w it is enough to count just the walks in the G-part of the coalescences–the walk counts in the F-part have no influence, since the entrance points to F are the same in both G^v and G^w.

Remark 2.2. Let $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ and $x_1, x_2, \ldots, x_n$ denote the eigenvalues and the corresponding orthonormal eigenvectors of the adjacency matrix A of a connected graph G. Recall that

$$\begin{aligned} (A^k)_{v_i,v_j} &= \sum_{p=1}^{n} \lambda_p^k x_{p,v_i} x_{p,v_j}, \\ (A^k)_{w_i,w_j} &= \sum_{p=1}^{n} \lambda_p^k x_{p,w_i} x_{p,w_j}. \end{aligned}$$

Since λ_1 has the largest absolute value among all eigenvalues and a positive eigenvector, the most important summands in the above expressions, especially for larger values of k, become $\lambda_1^k x_{1,v_i} x_{1,v_j}$ and $\lambda_1^k x_{1,w_i} x_{1,w_j}$. It is, thus, tempting to think that the condition (2.14) in Lemma 2.4 might be replaced by a simpler condition

$$x_{1,v_i} x_{1,v_j} \geq x_{1,w_i} x_{1,w_j}.$$

This, however, cannot be done, as shown by the following example. Let u be an arbitrary vertex of the complete graph K_{50}, and let G be the graph shown in Fig. 2. Although

$$0.41712 \approx x_{1,a} < x_{1,b} \approx 0.45699,$$

we still have that

$$49.00123 \approx \lambda_1(K_{50}(u=a)G) > \lambda_1(K_{50}(u=b)G) \approx 49.00083.$$

The reason for such behavior lies simply in the fact that the degree of a is larger than the degree of b. Note that the degree of a vertex represents, at the same time, also

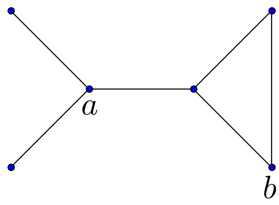

Figure 2. The vertex a has smaller principal eigenvector component than the vertex b, but there are more closed walks of even lengths up to 12 that start at a than at b

the number of closed walks of length two starting from that vertex. When coalesced with K_{50}, which has substantially more closed walks than G, the spectral radius of the coalescence is roughly determined by the spectral radius of the larger K_{50}, but tends to be fine tuned by the shorter (i.e., the shortest) closed walks in G, of which there are more that start at a than those that start at b.

3. *On closed walk counts in rooted products of paths and stars*

In order to be able to apply Lemma 2.4 we need to exhibit sufficiently many graphs satisfying (2.14). Paths are among the simplest such graphs. The following lemma appeared in the authors' earlier paper with Ilić:

Lemma 3.1 ([11]). *Let A be the adjacency matrix of the path P_n on vertices $1,\dots,n$. Then for every $k \geq 0$ holds*

$$(A^k)_{1,1} \leq (A^k)_{2,2} \leq \cdots \leq (A^k)_{\lceil n/2\rceil,\lceil n/2\rceil} \tag{3.15}$$

and

$$(A^k)_{1,2} \leq (A^k)_{2,3} \leq \cdots \leq (A^k)_{\lfloor n/2\rfloor,\lfloor n/2\rfloor+1}. \tag{3.16}$$

We reprint here the proof of this lemma from [11], as it serves as the basis for the proof of a more general lemma that follows.

PROOF. We prove slightly more than stated in (3.15) and (3.16): that each diagonal of A^k, parallel to the main diagonal, is unimodal. Due to the automorphism of the path P_n given by $\alpha\colon i \to n+1-i$ for $i = 1,\dots,n$, it is enough to prove that each of these diagonals is nondecreasing up to its middle entry.

We proceed by induction on k and prove that for all $2 \leq i,j \leq n$ such that $i+j \leq n+1$ holds

$$(A^k)_{i-1,j-1} \leq (A^k)_{i,j}. \tag{3.17}$$

This is trivial for $k = 0$ and $k = 1$, as each diagonal of $A^0 = I$ and $A^1 = A$ is either all-zero or all-one. Suppose now that (3.17) has been proved for some $k \geq 1$. The expression $A^{k+1} = A^k \cdot A$ then yields

$$\begin{aligned}(A^{k+1})_{i-1,j-1} &= (A^k)_{i-1,j-2} + (A^k)_{i-1,j},\\ (A^{k+1})_{i,j} &= (A^k)_{i,j-1} + (A^k)_{i,j+1}.\end{aligned}$$

(To avoid dealing separately with the endpoints 1 and n of the path P_n, we simply assume that $(A^k)_{i-1,0} = 0$ and $(A^k)_{i,n+1} = 0$ in the above equations.) We have

$$(A^k)_{i-1,j-2} \leq (A^k)_{i,j-1}$$

from the inductive hypothesis (and the nonnegativity of $(A^k)_{i,j-1}$). If $i + j + 1 \leq n + 1$, then

$$(A^k)_{i-1,j} \leq (A^k)_{i,j+1}$$

also follows from the inductive hypothesis. For $i+j+1 = n+2$, from the automorphism $\alpha\colon i \to n + 1 - i$ and the symmetry of A^k we have

$$(A^k)_{i-1,j} = (A^k)_{n+1-j,n+2-i} = (A^k)_{i,j+1}.$$

This proves (3.17).

We will now extend this lemma to the rooted products of a path by another graph.

Definition 3.1 ([9])**.** Let H be a labeled graph on n vertices, and let $G_1, \dots, G_n$ be a sequence of n rooted graphs. The rooted product of H by $G_1, \dots, G_n$, denoted as $H[G_1, \dots, G_n]$, is the graph obtained by identifying the root of G_i with the i-th vertex of H for $i = 1, \dots, n$. In the case when all the rooted graphs G_i, $i = 1, \dots, n$, are isomorphic to a rooted graph G, we denote $H[\underbrace{G, \dots, G}_{n}]$ simply as $H[G, n]$.

Lemma 3.2. *Let n be a positive integer and let G be an arbitrary rooted graph. Denote by $G_1, \dots, G_n$ the copies of G, and for any vertex u of G, denote by u_i the corresponding vertex in the copy G_i, $i = 1, \dots, n$. If A is the adjacency matrix of the rooted product $P_n[G, n]$, then for any two* (*not necessarily different*) *vertices u and v of G and for every $k \geq 0$ holds*

$$(A^k)_{u_1,v_1} \leq (A^k)_{u_2,v_2} \leq \cdots \leq (A^k)_{u_{\lceil n/2\rceil},v_{\lceil n/2\rceil}} \tag{3.18}$$

and

$$(A^k)_{u_1,v_2} \leq (A^k)_{u_2,v_3} \leq \cdots \leq (A^k)_{u_{\lfloor N/2\rfloor},v_{\lfloor N/2\rfloor+1}}. \tag{3.19}$$

PROOF. Let r denote the root vertex of G, so that $r_1, \dots, r_n$ then also denote the vertices of P_n in the rooted product $P_n[G, n]$.

The number of k-walks between u_i and v_i whose edges fully belong to G_i is, obviously, equal to the number of k-walks between u and v in G. If a k-walk W between u_i and v_i contains other edges of $P_n[G, n]$, then let W' denote longest subwalk of W such that W' is a closed walk that starts and ends at r_i: simply, the first edge of W' is the first edge of W that does not belong to G_i, and the last edge of W' is the last edge of W that does not belong to G_i. It is easy to see then that the number of k-walks between u_i and v_i in $P_n[G, n]$ is governed by the numbers of walks between u and v in G, and the numbers of closed walks (of lengths k and less) that start and end at r_i in $P_n[G, n]$. In particular, the chain of inequalities (3.18) follows from

$$(A^k)_{r_1,r_1} \leq (A^k)_{r_2,r_2} \leq \cdots \leq (A^k)_{r_{\lceil n/2 \rceil}, r_{\lceil n/2 \rceil}}. \tag{3.20}$$

Similarly, the number of k-walks between u_i in the copy G_i and v_{i+1} in the copy G_{i+1} is governed by the numbers of walks between u and r in G (that get mapped to walks between u_i and r_i in G_i), the numbers of walks between r and v in G (that get mapped to walks between r_{i+1} and v_{i+1} in G_{i+1}), and the numbers of walks between r_i and r_{i+1} in $P_n[G, n]$. Thus, the chain of inequalities (3.19) follows from

$$(A^k)_{r_1,r_2} \leq (A^k)_{r_2,r_3} \leq \cdots \leq (A^k)_{r_{\lfloor N/2 \rfloor}, r_{\lfloor N/2 \rfloor + 1}}. \tag{3.21}$$

Similarly as in the proof of Lemma 3.1, (3.18) and (3.19) are the special cases of the inequalities

$$(A^k)_{u_{i-1},v_{j-1}} \leq (A^k)_{u_i,v_j}, \qquad 2 \leq i, j \leq n, \quad i + j \leq n + 1, \tag{3.22}$$

which are, from the argument above, corollaries of the inequalities

$$(A^{k'})_{r_{i-1},r_{j-1}} \leq (A^{k'})_{r_i,r_j}, \qquad k' \leq k, \quad 2 \leq i, j \leq n, \quad i + j \leq n + 1. \tag{3.23}$$

We will now prove (3.22) by induction on k. This is trivial for $k = 0$, as $A^0 = I$.

Suppose, therefore, that (3.22) has been proved for all values of k' up to some $k \geq 0$. We will now prove that (3.23) holds for $k' = k+1$, from which the correctness of (3.22) for $k' = k + 1$ follows as well. (Actually, from the above discussion it is easy to see that the correctness of (3.22) for $k' = k + 1$ follows already from the inductive hypothesis if at least one of u, v is not r. Therefore, one only needs to prove (3.23) for $k' = k + 1$.)

Let $N(r)$ denote the set of neighbors of the root r in the graph G. Then

$$\begin{aligned}
(A^{k+1})_{r_{i-1},r_{j-1}} &= (A^k)_{r_{i-1},r_{j-2}} + (A^k)_{r_{i-1},r_j} + \sum_{u \in N(r)} (A^k)_{r_{i-1},u_{j-1}}, \\
(A^{k+1})_{r_i,r_j} &= (A^k)_{r_i,r_{j-1}} + (A^k)_{r_i,r_{j+1}} + \sum_{u \in N(r)} (A^k)_{r_i,u_j}.
\end{aligned}$$

The inequalities

$$(A^k)_{r_{i-1},r_{j-2}} \leq (A^k)_{r_i,r_{j-1}}$$

and

$$(A^k)_{r_{i-1},u_{j-1}} \leq (A^k)_{r_i,u_j}$$

hold by the inductive hypothesis. (We now again assume that $(A^k)_{r_{i-1},r_0} = 0$ and $(A^k)_{r_i,r_{n+1}} = 0$ to avoid dealing separately with the end vertices of the path P_n.)

If $i + j + 1 \leq n + 1$, then

$$(A^k)_{r_{i-1},r_j} \leq (A^k)_{r_i,r_{j+1}}$$

also holds by the inductive hypothesis. For $i+j+1 = n+2$, from the automorphism $\beta\colon r_i \to r_{n+1-i}$ of $P_n[G,n]$ and the symmetry of A^k we have

$$(A^k)_{r_{i-1},r_j} = (A^k)_{r_{n+1-j},r_{n+2-i}} = (A^k)_{r_i,r_{j+1}}.$$

This proves (3.23), and consequently (3.22).

In order to be able to prove the conjecture of Belardo, Li Marzi and Simić [1], we need to consider a slight extension of the previous lemma as well.

Lemma 3.3. *Let n be a positive integer and let G be an arbitrary rooted graph with the root r. Let $P_n^+[G,n]$ denote the graph obtained from the rooted product $P_n[G,n]$ by adding two new pendant vertices r_0 and r_{n+1} and the edges r_0r_1 and r_nr_{n+1} to it. If A is the adjacency matrix of $P_n^+[G,n]$, then for any two (not necessarily different) vertices u and v of G and for every $k \geq 0$ holds*

$$(A^k)_{u_1,v_1} \leq (A^k)_{u_2,v_2} \leq \cdots \leq (A^k)_{u_{\lceil n/2\rceil},v_{\lceil n/2\rceil}} \tag{3.24}$$

and

$$(A^k)_{u_1,v_2} \leq (A^k)_{u_2,v_3} \leq \cdots \leq (A^k)_{u_{\lfloor N/2\rfloor},v_{\lfloor N/2\rfloor+1}}. \tag{3.25}$$

PROOF. The proof of this lemma is fully analogous to the proof of Lemma 3.2, with the difference that now the terms $(A^k)_{r_{i-1},r_0}$ and $(A^k)_{r_i,r_{n+1}}$ are no longer considered to be identically equal to 0. We will, therefore, indicate here only the differences that the introduction of the pendant vertices r_0 and r_{n+1} produces in the proof.

In the proof of (3.22) and (3.23) by induction on k, the basis remains trivial and can be extended to both $k = 0$ and $k = 1$, as the values $A_{u_{i-1},v_{j-1}}$ and A_{u_i,v_j} are nonzero (and equal to 1) if and only if either $i = j$ and u and v are adjacent in G or $|i - j| = 1$ and both u and v are equal to r.

Since we assume $2 \leq i, j \leq n$, $i+j \leq n+1$ in (3.22), the only difference in the proof of the inductive step lies in encountering the case $j=2$, where we need to additionally prove that

$$(A^k)_{r_{i-1},r_0} \leq (A^k)_{r_i,r_1}.$$

This, however, follows immediately from the fact that r_1r_0 has to be the last edge in any walk from r_{i-1} to r_0, so that

$$(A^k)_{r_{i-1},r_0} = (A^{k-1})_{r_{i-1},r_1}$$

and the inequality

$$(A^k)_{r_i,r_1} = (A^{k-1})_{r_{i-1},r_1} + (A^{k-1})_{r_{i+1},r_1} + \sum_{u\in N(r)} (A^{k-1})_{u_i,r_1} \geq (A^{k-1})_{r_{i-1},r_1}.$$

Hence (3.23), and consequently (3.22), holds again, which implies the chains of inequalities (3.24) and (3.25).

Stars form another, even simpler class of graphs that satisfy (2.14). Let c be the center, and $l_1, \ldots, l_{n-1}$ the leaves of the star S_n, $n \geq 2$. The inequality

$$(A^k)_{l_i,l_i} \leq (A^k)_{c,c} \tag{3.26}$$

for $i \in \{1, \ldots, n-1\}$ follows easily by induction on k. For $k=0$ we have $(A^0)_{l_i,l_i} = (A^0)_{c,c} = 1$. Assuming that the inequality (3.26) has been proved up to some $k \geq 0$, we then have

$$(A^{k+1})_{l_i,l_i} = (A^k)_{c,l_i} \leq \sum_{j=1}^{n-1} (A^k)_{c,l_j} = (A^{k+1})_{c,c},$$

simply by observing that any walk that stars at l_i must use the edge l_ic first.

Inequality (3.26) can also be extended to the rooted products of a star by another graph.

Lemma 3.4. *For $n \geq 2$, let c be the center and l an arbitrary leaf of the star S_n. Let G be an arbitrary rooted graph. Denote by G_c the copy of G in $S_n[G,n]$ whose root is identified with c, and by G_l the copy of G in $S_n[G,n]$ whose root is identified with l. For any vertex u of G, let u_c and u_l denote the corresponding vertices in G_c and G_l, respectively. If A is the adjacency matrix of the rooted product $S_n[G,n]$, then for any two (not necessarily different) vertices u and v of G and for every $k \geq 0$ holds*

$$(A^k)_{u_l,v_l} \leq (A^k)_{u_c,v_c}. \tag{3.27}$$

PROOF. Let r denote the root vertex of G, so that r_c and r_l become identified with c and l, respectively, in $S_n[G,n]$. Following the argument from the proof of Lemma 3.2, inequality (3.27) for arbitrary u and v will follow from

$$(A^k)_{r_l,r_l} \leq (A^k)_{r_c,r_c}. \tag{3.28}$$

We prove (3.27) by induction on k. This is trivial for $k=0$, as $A^0 = I$.

Suppose, therefore, that (3.27) has been proved for all values of k' up to some $k \geq 0$. We prove that (3.28) holds for $k' = k+1$, from which the correctness of (3.27) for $k' = k+1$ follows as well. Let $N(r)$ denote the set of neighbors of the root r in the graph G. Then

$$\begin{aligned} (A^{k+1})_{r_l,r_l} &= (A^k)_{r_c,r_l} + \sum_{u \in N(r)} (A^k)_{u_l,r_l}, \\ (A^{k+1})_{r_c,r_c} &\geq (A^k)_{r_l,r_c} + \sum_{u \in N(r)} (A^k)_{u_c,r_c}, \end{aligned}$$

where in the second expression we have deliberately disregarded k-walks between roots of other copies of G and r_c. From the inductive hypothesis we have

$$(A^k)_{u_l,r_l} \leq (A^k)_{u_c,r_c}$$

for any vertex $u \in N(r)$. Together with the fact that A^k is symmetric, this proves (3.28), and consequently (3.27).

4. *Spectral radii of certain multiple coalescences*

As our simplest examples of the use of Lemma 2.4 and the walk count lemmas from the previous section, we first provide new proofs for the useful and well-cited 1979 lemmas of Li and Feng [12]. Note, however, that the original lemmas claim the strict inequality between the spectral radii, and that we actually prove the weak inequality here, due to reasons explained in Remark 2.1 on page 39.

Lemma 4.1 ([12]). *Let u be a vertex of a connected graph G and for positive integers p and q, let $G^u_{p,q}$ denote the graph obtained from G by adding two pendants paths of lengths p and q at u. If $p \geq q \geq 1$, then*

$$\lambda_1(G^u_{p,q}) \geq \lambda_1(G^u_{p+1,q-1}).$$

Lemma 4.2 ([12]). *Let u and v be two adjacent vertices of a connected graph G and for positive integers p and q, let $G^{u,v}_{p,q}$ denote the graph obtained from G by adding pendant paths of length p at u and q at v. If $p \geq q \geq 1$, then*

$$\lambda_1(G^{u,v}_{p,q}) \geq \lambda_1(G^{u,v}_{p+1,q-1}).$$

The proof of Lemma 4.1 follows by observing that $G^u_{p,q}$ is a coalescence of the graph G and the path P_{p+q+1} by identifying the vertex u from G and the vertex $q+1$ of P_{p+q+1}. (Here the vertices of P_{p+q+1} are enumerated with $1,\dots,q+1,\dots,p+q+1$, starting from the endpoint of P_{q+1} toward u, and then continuing from u toward the endpoint of P_{p+1}.) The inequality

$$\lambda_1(G^u_{p,q}) = \lambda_1(G(u=q+1)P_{p+q+1}) \geq \lambda_1(G(u=q)P_{p+q+1}) = \lambda_1(G^u_{p+1,q-1})$$

then follows from (3.15) and Lemma 2.4.

The proof of Lemma 4.2 further follows by observing that the graphs $G^{u,v}_{p,q}$ and $G^{u,v}_{p+1,q-1}$ are multiple coalescences of the edge-deleted graph $G-uv$ and the path P_{p+q+2}:

$$\begin{aligned} G^{u,v}_{p,q} &\cong G-uv(u=q+2, v=q+1)P_{p+q+2}, \\ G^{u,v}_{p+1,q-1} &\cong G-uv(u=q+1, v=q)P_{p+q+2}. \end{aligned}$$

Lemma 2.4 requires that for $k\geq 1$

$$\begin{aligned} (A^k)_{q+2,q+2} &\geq (A^k)_{q+1,q+1}, \\ (A^k)_{q+1,q+1} &\geq (A^k)_{q,q}, \\ (A^k)_{q+2,q+1} &\geq (A^k)_{q+1,q}, \end{aligned}$$

which are the special cases of (3.15) and (3.16), with

$$(A^k)_{q+2,q+2} = (A^k)_{q+1,q+1}$$

in the case $p=q$ due to the automorphism of the path P_{2q+2}.

Next, we improve these lemmas by showing their analogs when, instead of a path, the rooted product of a path gets attached to the basis graph.

Lemma 4.3. *Let G be a rooted graph, H a connected graph, and p and q two positive integers. For a vertex u of H, suppose that H contains a rooted subgraph G', with u as its root, that is isomorphic to the rooted graph G.*

Let $H^{u,G}_{p,q}$ denote the graph obtained from H by identifying the rooted subgraph G' with the $(q+1)$-st copy of G in the rooted product $P_{p+q+1}[G,p+q+1]$ (see Fig. 3). If $p\geq q\geq 1$, then

$$\lambda_1(H^{u,G}_{p,q}) \geq \lambda_1(H^{u,G}_{p+1,q-1}).$$

Lemma 4.4. *Let G be a rooted graph, H a connected graph, and p and q two positive integers. For two adjacent vertices u and v of H, suppose that H contains* *two vertex-disjoint rooted subgraphs G', with a root u, and G'', with a root v, both isomorphic to the rooted graph G.*

Let $H^{u,v,G}_{p,q}$ denote the graph obtained from H by identifying the rooted subgraph G' with the $(q+2)$-nd copy of G and the rooted subgraph G'' with the $(q+1)$-st copy of G in the rooted product $P_{p+q+2}[G, p+q+2]$ (see Fig. 3). If $p \geq q \geq 1$, then

$$\lambda_1(H^{u,v,G}_{p,q}) \geq \lambda_1(H^{u,v,G}_{p+1,q-1}).$$

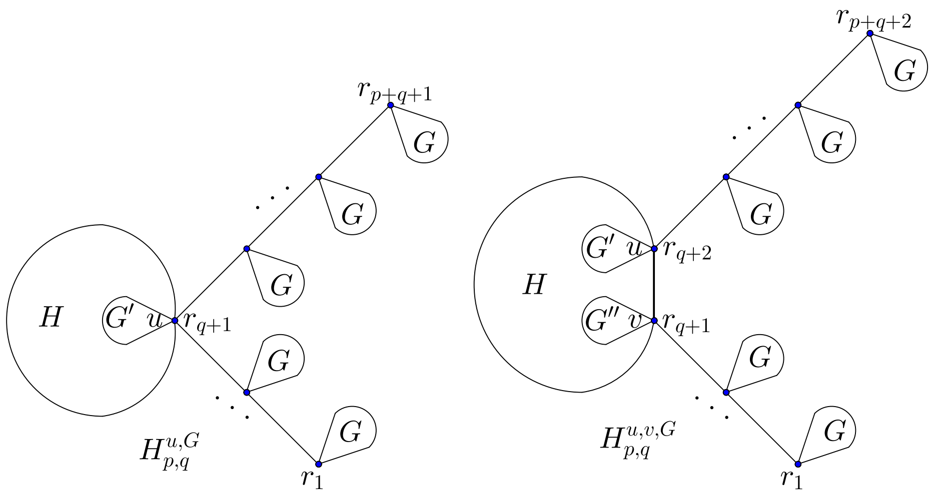

Figure 3. The graphs $H^{u,G}_{p,q}$ and $H^{u,v,G}_{p,q}$

Both of these lemmas follow directly from Lemmas 2.4 and 3.2 by observing that both $H^{u,G}_{p,q}$ and $H^{u,v,G}_{p,q}$ are multiple coalescences.

If H' is the graph obtained from H by deleting the edges of G', then $H^{u,G}_{p,q}$ is the multiple coalescence of H' and $P_{p+q+1}[G, p+q+1]$, obtained by identifying the corresponding vertices of G' in H' and the $(q+1)$-st copy of G in $P_{p+q+1}[G, p+q+1]$.

If H'' is the graph obtained from H by deleting the edges of G' and G'', then $H^{u,v,G}_{p,q}$ is the multiple coalescence of H'' and $P_{p+q+2}[G, p+q+2]$, obtained by identifying the corresponding vertices of G' and the $(q+2)$-nd copy of G in

$$P_{p+q+2}[G, p+q+2],$$

and by identifying the corresponding vertices of G'' and the $(q+1)$-st copy of G in $P_{p+q+2}[G, p+q+2]$.

In addition, note that the conditions that H has to contain rooted subgraphs isomorphic to G can be easily removed from the last two lemmas: if H does not contain a complete copy of G rooted at u as its subgraph, then we can form H' from H by adding the necessary number of isolated vertices, and then apply Lemmas 2.4 and 3.2 to the multiple coalescence of H' and $P_{p+q+1}[G, p+q+1]$, where the new isolated vertices are identified with the vertices of the $(q+1)$-st copy of G that do not originally appear in H. Similar argument holds in the case of adjacent vertices u and v and the two vertex-disjoint copies of G needed in H. In the extreme case, we can just identify the vertex u (or u and v) of H with the root(s) of the copies of G in the rooted product, and apply Lemmas 2.4 and 3.2 to obtain the following two lemmas:

Lemma 4.5. *Let G be a rooted graph with the root r, p and q two positive integers, and let r_q and r_{q+1} denote the roots of the q-th and the $(q+1)$-st copies of G, respectively, in the rooted product $P_{p+q+1}[G, p+q+1]$.*

If $p \geq q \geq 1$, then for any connected graph H and any vertex u of H holds

$$\lambda_1(H(u = r_{q+1})P_{p+q+1}[G, p+q+1]) \geq \lambda_1(H(u = r_q)P_{p+q+1}[G, p+q+1]).$$

Lemma 4.6. *Let G be a rooted graph with the root r, p and q two positive integers, and let r_{q-1}, r_q and r_{q+1} denote the roots of the $(q-1)$-st, q-th and the $(q+1)$-st copies of G, respectively, in the rooted product $P_{p+q+2}[G, p+q+2]$.*

If $p \geq q \geq 1$, then for any connected graph H and any two adjacent vertices u and v of H holds

$$\lambda_1(H(u = r_{q+2}, v = r_{q+1})P_{p+q+2}[G, p+q+2]) \geq \lambda_1(H(u = r_{q+1}, v = r_q)P_{p+q+2}[G, p+q+2]).$$

The use of Lemma 3.3 instead of Lemma 3.2 further allows us to state Lemmas 4.3–4.6 in terms of multiple coalescences with $P^+_{p+q+1}[G, p+q+1]$ and $P^+_{p+q+2}[G, p+q+2]$ as well. This leads to the observation that the weak inequality in the 2009 conjecture of Belardo, Li Marzi and Simić [1] becomes merely a corollary of Lemmas 2.4 and 3.3:

Conjecture 4.1 ([1]). *Let G be a rooted graph having r as its root, with $\deg(r) \geq \Delta - 2$. Denote by $G_\Delta(l, m)$, with $l, m \geq 0$, the graph obtained from G by identifying r with two pendant vertices of $P^+[K_{1,\Delta-2}, l]$ and $P^+[K_{1,\Delta-2}, m]$ (see Fig. 4). If G is not the star $K_{1,\Delta-2}$ and $l \geq m \geq 1$ then*

$$\lambda_1(G_\Delta(l, m)) > \lambda_1(G_\Delta(l+1, m-1)). \qquad (4.29)$$

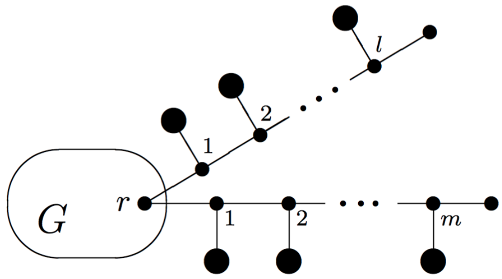

Figure 4. The graph $G_\Delta(l,m)$ (reprinted from [1]). Large black vertices denote co-cliques of order $\Delta-2$, so that the degrees of the vertices $1,\dots,m,\dots,l$ are all equal to Δ

The key here is to observe that the graph $G_\Delta(l,m)$ is another instance of a multiple coalescence. Let $s_1,\dots,s_{\Delta-2}$ be distinct neighbors of r in G, and let G^* be the edge-deleted subgraph

$$G^* = G - rs_1 - \cdots - rs_{\Delta-2}.$$

Next, let u_{m+1} be the root of the $(m+1)$-st copy of $K_{1,\Delta-2}$ in the graph

$$P^+[K_{1,\Delta-2}, l+m+1]$$

(counting the copies of $K_{1,\Delta-2}$ backwards from the m-end in Fig. 4), and let $t_{m+1,1},\dots,t_{m+1,\Delta-2}$ denote the leaves adjacent to u_{m+1} in $P^+[K_{1,\Delta-2}, l+m+1]$.

The graph $G_\Delta(l,m)$ from the conjecture above is then a multiple coalescence

$$G_\Delta(l,m) \cong G^*(r=u_{m+1}, s_1=t_{m+1,1},\dots,s_{\Delta-2}=t_{m+1,\Delta-2})H_\Delta(m+1+l),$$

for which the application of Lemmas 2.4 and 3.3 yields the weak inequality in (4.29).

The combination of Lemmas 2.4 and 3.4 yield the following lemmas on multiple coalescence with rooted products of a star by another graph.

Lemma 4.7. *For $n\geq 2$, let c be the center and l an arbitrary leaf of the star S_n. Let G be a rooted graph and let H be a connected graph. For a vertex u of H, suppose that H contains a rooted subgraph G', with u as its root, that is isomorphic to the rooted graph G.*

Let H^l be the multiple coalescence of H and $S_n[G,n]$, obtained by identifying the rooted subgraph G' with a copy of G rooted at l in $S_n[G,n]$, and let H^c be the *multiple coalescence of H and $S_n[G,n]$, obtained by identifying the rooted subgraph G' with a copy of G rooted at c in $S_n[G,n]$. Then*

$$\lambda_1(H^c) \geq \lambda_1(H^l).$$

Lemma 4.8. *For $n \geq 2$, let c be the center and l an arbitrary leaf of the star S_n. Let G be a rooted graph with the root r, and let r_c and r_l denote the roots of copies of G rooted at c and l, respectively, in the rooted product $S_n[G,n]$. Let H be a connected graph and u an arbitrary vertex of H. Then*

$$\lambda_1(H(u = r_c)S_n[G,n]) \geq \lambda_1(H(u = r_l)S_n[G,n]).$$

Lemmas 4.5 and 4.8 enable us to solve the Brualdi-Solheid problem for the classes of graphs consisting of rooted products with the same rooted graph G.

Theorem 4.1. *Let G be an arbitrary rooted graph. If T is a tree on n vertices, then*

$$\lambda_1(P_n[G,n]) \leq \lambda_1(T[G,n]) \leq \lambda_1(S_n[G,n]). \tag{4.30}$$

PROOF. If T is not the path P_n, then let u be a vertex of T with $\deg(u) \geq 3$ and the largest eccentricity (= the maximum distance from u to any other vertex of T). The vertex u cannot lie on a path between any two vertices of degrees at least three, as then one of them would have eccentricity larger than u. This shows all other vertices of T with degree at least three belong to only one of the $\deg(u)$ subtrees of $T - u$. Consequently, the remaining $\deg(u) - 1 \geq 2$ subtrees of $T - u$ represent pendant paths of T attached at u. Let P' and P'' be two such pendant paths of lengths p and q, respectively, and let T^- be the tree obtained by deleting the vertices of these paths (other than u) from T. Let $v_1, \ldots, v_{q+1}$ denote the first $q + 1$ vertices of the path P_{p+q+1} of length $p + q$, counting from one of the endpoints. Tree T can then be represented as a multiple coalescence

$$T \cong T^-(u = v_{q+1})P_{p+q+1},$$

and from Lemma 4.5 we then obtain that

$$\begin{aligned} \lambda_1(T[G,n]) &= \lambda_1(T^-(u = v_{q+1})P_{p+q+1}[G,n]) \\ &\geq \lambda_1(T^-(u = v_q)P_{p+q+1}[G,n]) \\ &\;\vdots \\ &\geq \lambda_1(T^-(u = v_1)P_{p+q+1}[G,n]). \end{aligned}$$

The degree of u in tree $T' = T^-(u = v_1)P_{p+q+1}$ has, however, decreased by one. Repeating the above procedure as long as the tree contains vertices of degree at least three, we eventually obtain that $\lambda_1(T[G,n]) \geq \lambda_1(P_n[G,n])$.

With respect to the right-hand side inequality in (4.30), let u be a vertex of T with $d = \deg(u) \geq 2$ and the largest eccentricity. Let $v_1, \dots, v_d$ be the neighbors of u in T. The vertex u cannot lie on a path between any two other vertices of degree at least two, as one of them would then have eccentricity larger than u. If T is not the star S_n, then exactly one neighbor of u, say v_1, has degree at least two, while the remaining neighbors $v_2, \dots, v_d$ all have degree one. Let

$$T' = T - uv_2 - \dots - uv_d + v_1v_2 + \dots + v_1v_d.$$

Further, let T^- be the tree obtained from T by deleting vertices u, $v_2, \dots, v_d$. If c and l are the center and an arbitrary leaf of the star S_{d+1}, then both T and T' can be represented as multiple coalescences:

$$\begin{aligned} T &\cong T^-(v_1 = l)S_{d+1}, \\ T' &\cong T^-(v_1 = c)S_{d+1}. \end{aligned}$$

From Lemma 4.8 we then obtain that

$$\begin{aligned} \lambda_1(T[G,n]) &= \lambda_1(T^-(v_1 = l)S_{d+1}[G,n]) \\ &\leq \lambda_1(T^-(v_1 = c)S_{d+1}[G,n]) = \lambda_1(T'[G,n]). \end{aligned}$$

The degree of u in T' is, however, equal to one. Repeating the above procedure as long as the tree contains at least two vertices of degree at least two, we eventually obtain that $\lambda_1(T[G,n]) \leq \lambda_1(S_n[G,n])$.

Theorem 4.2. *Let G be an arbitrary rooted graph. If H is a connected graph on n vertices, then*

$$\lambda_1(P_n[G,n]) \leq \lambda_1(H[G,n]) < \lambda_1(K_n[G,n]),$$

where K_n denotes the complete graph on n vertices.

PROOF. From the fact that $K_n[G,n]$ contains $H[G,n]$ as a proper subgraph for any $H \ncong K_n$, we immediately see that $\lambda_1(H[G,n]) < \lambda_1(K_n[G,n])$, as the spectral radius of a connected graph strictly increases with the addition of edges (see item a) on page 35). From the same reason, if T is an arbitrary spanning tree of H, then $\lambda_1(T[G,n]) \leq \lambda_1(H[G,n])$. From the previous theorem, we then have $\lambda_1(P_n[G,n]) \leq \lambda_1(T[G,n]) \leq \lambda_1(H[G,n])$.

5. *Conclusion*

We have developed a new method for comparing spectral radii of adjacency matrices of graphs, that applies to graphs that can be represented as multiple coalescences of the same basis graph with different smaller subgraphs. The method, based on Lemma 2.4, works by comparing walk counts in the smaller subgraphs in order to imply inequality between spectral radii for the whole graphs. We have further developed a number of walk count lemmas for cases when smaller subgraphs are rooted products of paths or stars by another graph. Most of the results in this manuscript are named lemmas, as we expect them to become useful ingredients in the proofs of further results. Examples of such results here include the proof of weak inequality in the 2009 conjecture of Belardo, Li Marzi and Simić [1], and the solution of the Brualdi-Solheid problem for the classes of graphs consisting of rooted products with the same rooted graph.

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