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BOCHNER-FLAT KÄHLER MANIFOLDS AND RIEMANNIAN COMPATIBILITY OF THE RICCI TENSOR

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A b s t r a c t. In this paper we investigate Riemannian compatibility of Ricci tensor of a Bochner-flat Kähler manifold, and specially of a such manifolds which is of quasi-constant holomorphic sectional curvature. Also, we extend our consideration to manifolds without Bochner-flat condition. In all cases we found necessary and sufficient conditions on the Ricci tensor of considered manifolds to be Riemannian compatible.

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1. *Introduction*

Let (M, g) be a Riemannian manifold. We denote by R, ρ and κ the Riemannian curvature tensor, the Ricci tensor and the scalar curvature respectively.

In [2], [3] and [4] it was introduced the algebraic notion of Riemannian compatible tensors as follows.

Definition 1.1. A symmetric tensor b_{ij} is compatible with Riemannian curvature tensor if

$$b_{ia}R^a_{hjk} + b_{ja}R^a_{hki} + b_{ka}R^a_{hij} = 0. \tag{1.1}$$

The metric tensor is trivially Riemannian compatible. If (M, g) is an Einstein manifold, then

$$\rho_{ia}R^a_{hjk} + \rho_{ja}R^a_{hki} + \rho_{ka}R^a_{hij} = 0, \tag{1.2}$$

that is, the Ricci tensor is Riemannian compatible. The relation (1.2) is also satisfied if (M, g) is conformally flat, i.e. if

$$\begin{aligned} R_{ihjk} &= \frac{1}{n-2}\Big(g_{ik}\rho_{hj} + g_{jh}\rho_{ik} - g_{ij}\rho_{hk} - g_{hk}\rho_{ij}\Big) \\ &\quad - \frac{\kappa}{(n-1)(n-2)}\Big(g_{ik}g_{hj} - g_{ij}g_{hk}\Big), \quad n = \dim M. \end{aligned}$$

It is not the same for the Kähler manifolds. In general, the Ricci tensor of a Bochner-flat Kähler manifold does not satisfy the condition (1.2). The aim of the present paper is to find the necessary and the sufficient conditions for such a compatibility. This is done in Section 2. In Section 3, to find an example, we discuss this problem in the case of Kähler manifolds of quasi-constant holomorphic sectional curvature.

2. *Compatibility of Ricci tensor for a Bochner-flat Kähler manifold*

Kähler manifold (M, g, J) is a differentiable manifold M, $\dim M = 2n$, endowed with Hermitian metric g and the parallel complex structure J. This means that, with respect to the local coordinates $(x^1, x^2, \ldots, x^{2n})$ we have

$$J^i_a\, J^a_j = -\delta^i_j, \quad J^a_i\, J^b_j\, g_{ab} = g_{ij}, \quad \nabla_k J^i_j = 0,$$

where ∇ is the operator of the covariant derivative with respect to the Levi-Civita connection. It follows that the $(0, 2)$ tensor

$$F_{ij} = J^a_i\, g_{aj}$$

satisfies the conditions

$$F_{ij} = -F_{ji}, \quad \nabla_k F_{ij} = 0.$$

The condition $\nabla_k J^i_j = 0$ implies,

$$J^a_h\, J^b_k\, R_{ijab} = R_{ijhk},$$

from which it follows

$$J^a_i\, \rho_{aj} = -J^a_j\, \rho_{ai}, \quad J^a_i\, \overset{2}{\rho}_{aj} = -J^a_j\, \overset{2}{\rho}_{ai}, \tag{2.1}$$

where

$$\overset{2}{\rho}_{ij} = \rho_{ia}\,\rho_i^a.$$

The Bochner curvature tensor is

$$B_{ijhk} = R_{ijhk} - \frac{1}{2(n+2)}\theta_{ijhk} + \frac{\kappa}{4(n+1)(n+2)}G_{ijhk},$$

where

$$\begin{aligned}\theta_{ijhk} = {} & g_{ik}\,\rho_{jh} + g_{jh}\,\rho_{ik} - g_{ih}\,\rho_{jk} - g_{jk}\,\rho_{ih} \\ & + F_{ik}\,J_j^a\,\rho_{ah} + F_{jh}\,J_i^a\,\rho_{ak} - F_{ih}J_j^a\,\rho_{ak} - F_{jk}\,J_i^a\,\rho_{ah} \\ & - 2F_{ij}\,J_h^a\,\rho_{ak} - 2F_{hk}\,J_i^a\,\rho_{aj},\end{aligned} \tag{2.2}$$

and

$$G_{ijhk} = g_{ik}g_{jh} - g_{ih}g_{jk} + F_{ik}F_{jh} - F_{ih}F_{jk} - 2F_{ij}F_{hk}. \tag{2.3}$$

Thus, if the Kähler manifold is Bochner-flat, then

$$R_{ijhk} = \frac{1}{2(n+2)}\theta_{ijhk} - \frac{\kappa}{4(n+1)(n+2)}G_{ijhk}, \tag{2.4}$$

and therefore

$$\begin{aligned}\rho_{ia}R^a_{hjk} + \rho_{ja}R^a_{hki} + \rho_{ka}R^a_{hij} = {} & \frac{1}{2(n+2)}\Big(\rho_{ia}\theta^a_{hjk} + \rho_{ja}\theta^a_{hki} + \rho_{ka}\theta^a_{hij}\Big) \\ & - \frac{\kappa}{4(n+1)(n+2)}\Big(\rho_{ia}G^a_{hjk} + \rho_{ja}G^a_{hki} + \rho_{ka}G^a_{hij}\Big).\end{aligned}$$

But, according (2.2) and (2.3), we have

$$\begin{aligned}\rho_{ia}\theta^a_{hjk} + \rho_{ja}\theta^a_{hki} + \rho_{ka}\theta^a_{hij} = 2\Big[& - F_{hi}J_j^a\,\overset{2}{\rho}_{ak} - F_{hj}J_k^a\,\overset{2}{\rho}_{ai} - F_{hk}J_i^a\,\overset{2}{\rho}_{aj} \\ & + J_h^a\Big(F_{ij}\,\overset{2}{\rho}_{ak} + F_{jk}\,\overset{2}{\rho}_{ai}\;F_{ki}\,\overset{2}{\rho}_{aj}\Big)\Big]\end{aligned}$$

and

$$\begin{aligned}\rho_{ia}G^a_{hjk} + \rho_{ja}G^a_{hki} + \rho_{ka}G^a_{hij} = 2\Big[& - F_{hi}J_j^a\rho_{ak} - F_{hj}J_k^a\rho_{ai} - F_{hk}J_i^a\rho_{aj} \\ & + J_h^a\Big(F_{ij}\rho_{ak} + F_{jk}\rho_{ai} + F_{ki}\rho_{aj}\Big)\Big],\end{aligned}$$

such that the Ricci tensor is Riemannian compatible if and only if

$$
\begin{aligned}
-F_{hi}J^a_j\left(\overset{2}{\rho}_{ak}-\frac{\kappa}{2(n+1)}\rho_{ak}\right)-F_{hj}J^a_k\left(\overset{2}{\rho}_{ai}-\frac{\kappa}{2(n+1)}\rho_{ai}\right)\\
-F_{hk}J^a_i\left(\overset{2}{\rho}_{aj}-\frac{\kappa}{2(n+1)}\rho_{aj}\right)\\
+J^a_h\Bigg[F_{ij}\left(\overset{2}{\rho}_{ak}-\frac{\kappa}{2(n+1)}\rho_{ak}\right)+F_{jk}\left(\overset{2}{\rho}_{ai}-\frac{\kappa}{2(n+1)}\rho_{ai}\right)\\
+F_{ki}\left(\overset{2}{\rho}_{aj}-\frac{\kappa}{2(n+1)}\rho_{aj}\right)\Bigg]=0. \qquad (2.5)
\end{aligned}
$$

Transvecting (2.5) with J^h_t and then contracting with g^{tj}, we find

$$\overset{2}{\rho}_{ij}-\frac{\kappa}{2(n+1)}\rho_{ij}=fg_{ij}, \qquad (2.6)$$

where f is a scalar function.

Conversely, if the relation (2.6) holds, then

$$
\begin{aligned}
\rho_{ia}R^a_{hjk}+\rho_{ja}R^a_{hki}+\rho_{ka}R^a_{hij}=\frac{f}{2(n+2)}\Big(-F_{hi}F_{jk}-F_{hj}F_{ki}-F_{hk}F_{ij}\\
+F_{ij}F_{hk}+F_{jk}F_{hi}+F_{ki}F_{hj}\Big)=0,
\end{aligned}
$$

and the Ricci tensor is Riemannian compatible.

Thus, we can state the following theorem:

Theorem 2.1. *The Ricci tensor of a Bochner-flat Kähler manifold is Riemannian compatible if and only if it satisfies the condition* (2.6).

3. *Kähler manifold of quasi-constant holomorphic sectional curvature*

The Kähler manifold is said to be of *quasi-constant holomorphic sectional curvature if*

$$R=\mathcal{L}_0\,G+\mathcal{L}_1\,\phi+\mathcal{L}_2\,\psi, \qquad (3.1)$$

where $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are some scalar functions and the tensors ϕ and ψ are defined as follows

$$\begin{aligned}\phi_{ijhk} &= g_{ik}V_{jh} + g_{jh}V_{ik} - g_{ih}V_{jk} - g_{jk}V_{ih} + F_{ik}J_i^a\,V_{ah} + F_{jh}J_i^a\,V_{ak}\\ &\quad -F_{ih}J_j^a\,V_{ak} - F_{jk}J_i^a\,V_{ah} - 2F_{ij}J_h^a\,V_{ak} - 2F_{hk}J_i^a\,V_{aj},\\ \psi_{ijhk} &= J_i^a\,V_{hj}\,J_h^b\,V_{bk},\\ V_{ij} &= v_iv_j + J_i^aJ_j^bv_av_b,\end{aligned}$$

where v is a vector field. Without loss of generality, in behalf of the functions $\mathcal{L}_1$ and $\mathcal{L}_2$ we can suppose thet v is unit vector field. Then

$$V_{ia}V_{jb}\,g^{ab} = V_{ij}, \qquad V_{ab}\,g^{ab} = 2, \tag{3.2}$$

and because $J_i^av_a \perp v_i$, we also have

$$J_i^aJ_j^b\,V_{ab} = V_{ij}. \tag{3.3}$$

The class of Kähler manifolds of quasi-constant holomorphic sectional curvature is analogous to the class of Riemannian manifolds of quasi-constant sectional curvature. It was appeared first in the papers [5], [6] and [7] dedicated to the holomorphically subprojective Kähler manifolds. In [1] the authors considered the Kähler manifolds having the following property: for any $p \in M$, and any angle $\alpha \in [0, \pi/2]$, all holomorphic planes in the tangent vector space $T_p(M)$ making the angle α with given vector $v \in T_p(M)$, have the same holomorphic sectional curvature. They proved, among others, that the Riemannian curvature tensor of such manifolds has the form (3.1). If $\mathcal{L}_2 = 0$, that is if

$$R = \mathcal{L}_0G + \mathcal{L}_1\,\phi \tag{3.4}$$

the manifold is Bochner-flat. The Ricci tensor, corresponding to the tensor (3.4), is

$$\rho_{ji} = a\,g_{ij} + b\,V_{ij}, \tag{3.5}$$

where

$$a = 2\big[(n+1)\mathcal{L}_0 + \mathcal{L}_1\big], \qquad b = 2(n+2)\mathcal{L}_1, \tag{3.6}$$

such that, in view of (3.2) and (3.3), we have

$$\overset{2}{\rho}_{ij} = a^2g_{ij} + b(2a+b)V_{ij}.$$

Also,

$$\kappa = 2(na+b).$$

Thus,

$$\overset{2}{\rho}_{ij} - \frac{\kappa}{2(n+1)}\,\rho_{ij} = \frac{a(a-b)}{n+1} g_{ij} + \frac{b}{n+1}\Big[(n+1)a + nb\Big] V_{ij}.$$

The right hand side is of the form $f g_{ij}$ if $b=0$, or $(n+2)a + nb = 0$.

According (3.6), the condition $b = 0$ means that $\mathcal{L}_1 = 0$. But then (3.4) reduces to $R = \mathcal{L}_0 G$, and the manifold is of constant holomorphic sectional curvature. As for the condition $(n+2)a+nb=0$, it is $\mathcal{L}_0 + \mathcal{L}_1 = 0$ in view of (3.6). Thus, and in view of Theorem 2.1, we can state the following result:

Theorem 3.1. *The Ricci tensor of Bochner-flat Kähler manifold of quasi-constant holomorphic sectional curvature is Riemannian compatible if and only if* $\mathcal{L}_1 = -\mathcal{L}_0$.

Theorem 3.1 gives an example of Bochner-flat Kähler manifold satisfying the condition (2.5), i.e. satisfying Riemannian compatibility of the Ricci tensor. But,if $\mathcal{L}_1 = -\mathcal{L}_0$, the Ricci tensor is Riemannian compatible even if $\mathcal{L}_2 \neq 0$, thet is even the manifold is not Bochner-flat. Namely, in the case (3.1), the Ricci tensor is

$$\rho_{ij} = a\, g_{ij} + b_1\, V_{ij}, \tag{3.7}$$

where

$$b_1 = 2(n+2)\mathcal{L}_1 - \mathcal{L}_2 = b - \mathcal{L}_2.$$

Thus, taking into account that the tensors G, ϕ and ψ satisfy the first Bianchi identity, we have

$$\begin{aligned} \rho_{ia} R^a_{hjk} + \rho_{ja} R^a_{hki} + \rho_{ka} R^a_{hij} = (b - \mathcal{L}_2)\Big\{ & \mathcal{L}_0\big(V_{ia} G^a_{hjk} + V_{ja} G^a_{hki} + V_{ka} G^a_{hij}\big) \\ & + \mathcal{L}_1\big(V_{ia}\phi^a_{hjk} + V_{ja}\phi^a_{hki} + V_{ka}\phi^a_{hij}\big) \\ & + \mathcal{L}_2\big(V_{ia}\,\psi^a_{hjk} + V_{ja}\,\psi^a_{hki} + V_{ka}\,\psi^a_{hij}\big)\Big\}. \end{aligned} \tag{3.8}$$

In view of (3.2 and (3.3), we have

$$\begin{aligned} V_{ia} G^a_{hjk} + V_{ja} G^a_{hki} + V_{ka} G^a_{hij} &= V_{ia}\,\phi^a_{hjk} + V_{ja}\,\phi^a_{hki} + V_{ka}\,\phi^a_{hij} \\ &= 2\big(- F_{hj} J^a_k\, V_{ai} - F_{hk} J^a_i\, V_{aj} - F_{hi} J^a_j\, V_{ak} \\ &\qquad + F_{jk} J^a_h\, V_{ai} + F_{ki} J^a_h\, V_{aj} + F_{ij} J^a_h\, V_{ak}\big), \end{aligned}$$

while

$$V_{ia}\,\psi^a_{hjk} + V_{ja}\,\psi^a_{hki} + V_{ka}\,\psi^a_{hij} = 0,$$

such that the relation (3.8) becomes

$$\rho_{ia}R^a_{hjk} + \rho_{ja}R^a_{hki} + \rho_{ka}R^a_{hij} = 2(b-\mathcal{L}_2)(\mathcal{L}_0+\mathcal{L}_1)\Big[-F_{hj}J^a_k\,V_{ai} - F_{hk}J^a_i\,V_{aj} - F_{hi}J^a_j\,V_{ak} + F_{jk}J^a_h\,V_{ai} + F_{ki}J^a_h\,V_{aj} + F_{ij}J^a_h\,V_{ak}\Big].$$

Thus, the Ricci tensor is Riemannian compatible if $\mathcal{L}_1 = -\mathcal{L}_0$, or $b_1 \equiv b - \mathcal{L}_2 = 0$, or

$$-F_{hj}J^a_kV_{ai} - F_{hk}J^a_iV_{aj} - F_{hi}J^a_jV_{ak} + J^a_h\big(F_{jk}V_{ai} + F_{ki}V_{aj} + F_{ij}V_{ak}\big) = 0. \quad (3.9)$$

If (3.9) holds, proceeding like as in the case of the condition (2.4), we get

$$V_{ij} = \frac{1}{n}\,g_{ij}, \quad \text{i.e.,} \quad v_iv_j + J^a_iJ^b_jv_av_b = \frac{1}{n}\,g_{ij},$$

from which, contracting with v^j, we get $v_i = \frac{1}{n}\,v_i$. Thus, if $\dim M > 2$, the relation (3.9) can not hold.

If $b_1 = 0$, the relation (3.7) reduces to $\rho_{ij} = a\,g_{ij}$, (M, g, J) is the Einstein manifold and the Ricci tensor is trivially Riemannian compatible.

Thus, we can state the following result:

Theorem 3.2. *The Ricci tensor of the Kähler manifold of quasi-constant holomorphic sectional curvature is Riemannian compatible if $\mathcal{L}_1 = -\mathcal{L}_0$ or if the manifold is Einstein one.*

We get from (3.7)

$$\overset{2}{\rho}_{ij} = a_2\,g_{ij} + b_2\,V_{ij},$$

where

$$a_2 = (a_1)^2 = a^2, \qquad b_2 = b_1(2a_1 + b_1).$$

In general,

$$\overset{p}{\rho}_{ij} = \overset{p-1}{\rho}_{ia}\,\rho^a_j = a_p\,g_{ij} + b_p\,V_{ij},$$

where a_p and b_p are some functions of $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$. Proceeding in the same way as in the case of the condition (3.7), we can state the following theorem:

Theorem 3.3. *The tensor $\overset{p}{\rho}_{ij}$ of the Kähler manifold of quasi-constant holomorphic sectional curvature, is Riemannian compatible if $\mathcal{L}_1 = -\mathcal{L}_0$ or if $b_p = 0$. In the last case it is trivially Riemannan compatible.*

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