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HYPERSPACES OF 0-DIMENSIONAL SPACES REVISITED

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A b s t r a c t. An immediate reason for revisiting hyperspaces of 0*-dimensional spaces is the paper of Sh. Oka* [*Topology Appl.* **149** (2005), no. 1-3, 227–237]*, where the results of M. M. Marjanović* [*Publ. Inst. Math.* (*Beograd*) (*N.S.*) **14 (28)** (1972), 97–109] *have been reproved. Aside from that, we take this opportunity to highlight the concepts of accumulation order and accumulation spectrum as a system of topological invariants which can be used efficiently in some situations of determining topological types in this class of spaces.*

In particular the Cartesian multiplication of the accumulation orders is an operation with respect to which the set of natural numbers $\mathbb{N}$ *becomes a semi-group that can be used to reduce some subtle topological problems (for example, the existence of non-homeomorphic spaces with homeomorphic squares) to simple arithmetic verifications. The paper summarizes central ideas and details of the main constructions and may serve as an overview and introduction to this area of mathematics.*

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1. *Introduction*

When new results are considered to be particularly interesting, it is an academic custom to present them a number of times (at some conferences, visiting some universities, etc.). When it is done again after some forty years, then there must exist an exceptional reason as a justification. And in the case of this note a good reason for taking a look back to hyperspaces of 0-dimensional spaces is the coincidence of results in [Ma2] and [O1].

All spaces that we consider here are supposed to be compact Hausdorff and all mappings continuous. For a space X, let $\exp(X)$ be the set of all non-empty closed subsets of X and for a sequence $U_1, U_2, \ldots, U_n$ of open sets in X, let

$$\langle U_1, U_2, \ldots, U_n \rangle = \big\{ F \in \exp(X) \mid (\forall i)\, F \cap U_i \neq \emptyset \text{ and } F \subset U_1 \cup U_2 \cup \ldots \cup U_n \big\}. \quad (1.1)$$

When $U_1, U_2, \ldots, U_n$ runs over all finite sequences of open sets in X, the sets $\langle U_1, U_2, \ldots, U_n \rangle$ constitute the basis for a topology on $\exp(X)$ which is called the *Vietoris topology*. For a mapping $f : X \to Y$, let $\exp(f) : \exp(X) \to \exp(Y)$ be the induced map defined by $\exp(f)(F) = f(F)$, and then the correspondence

$$\begin{array}{ccc} X & & \exp(X) \\ \downarrow f & \longrightarrow & \downarrow \exp(f) \\ Y & & \exp(Y) \end{array} \quad (1.2)$$

is a *covariant functor* from the category of all compact Hausdorff spaces and continuous mappings into itself.

Iterating the functor $\exp : \exp(\exp(X)) = \exp^2(X)$ and for $n > 2$, by putting $\exp(\exp^{n-1}(X)) = \exp^n(X)$, *hyperspaces of higher order* are obtained. Let $u : \exp^2(X) \to \exp(X)$ be the union mapping. By putting $u^{(1)} = u$ and for $n > 1, u^{(n)} = \exp(u^{(n-1)})$, an inverse system is obtained

$$\cdots \longrightarrow \exp^3(X) \xrightarrow{u^{(2)}} \exp^2(X) \xrightarrow{u^{(1)}} \exp(X) \quad (1.3)$$

whose limit $\exp^{(\omega)}(X)$ contains X and $\exp(\exp^{(\omega)}(X)) = \exp^{(\omega)}(X)$ ([Ma1]). We call the space $\exp^{(\omega)}(X)$ the *hyperspace of X of the rank ω*.

We call a space X which is homeomorphic to its hyperspace *exponentially complete*. Evidently we were somewhat fascinated with this property of such spaces that we used the phrase "exponentially complete spaces", entitling four of our papers.

2. *Focus on hyperspaces of Peano continua*

In the early 1920s, L. Vietoris ([V1] and [V2]) and T. Wazewski ([W]) proved the following result:

- *If X is a non-degenerate Peano continuum then* $\exp(X)$ *is a Peano continuum.*

This apparently nice result was dramatically surpassed several times in the course of a number of decades. Namely, in addition to be a Peano continuum, it has been proved that:

- $\exp(X)$ *has a subspace homeomorphic to the Hilbert cube*, (K. Borsuk and S. Mazurkiewicz, in [B–M]).
- $\exp(X)$ *is an absolute retract and a contractible space* (M. Wojdyslawski, in [Wo]).

In this paper of Wojdyslawski, the famous hypothesis:

- *If X is a non-degenerate Peano continuum, then* $\exp(X)$ *is homeomorphic to the Hilbert cube,*

was launched for the first time and according to the spoken reports of K. Kuratowski, the hypothesis was already known to Polish mathematicians in 1920s.

According to some results contributed to the hyperspace theory, it could be guessed that this hypothesis has been absorbing the interest of a number of outstanding mathematicians at least for some time. But finally, in 1972, R. M. Schori and J. E. West proved:

- $\exp(I)$*, where I is the unit interval is homeomorphic to the Hilbert cube Q*, (see [S–W1], proof with all details in [S–W2]).

And in 1974, D. W. Curtis and R. M. Schori proved:

- *X is a non-degenerate Peano continuum if and only if $exp(X)$ is homeomorphic to the Hilbert cube Q* ([C–S1], proof with all details in [C–S2]).

Without any doubt, confirmation of this hypothesis is the most significant result on hyperspaces and it has motivated significant further research, e.g. characterizations of the Hilbert cube manifolds, etc. ([T] et al.).

3. *A long period without concrete examples*

Leaving out the trivial case of finite spaces having $2n - 1$ points, no other concrete example of hyperspaces was known until 1948, when G. Choquet proved that the Cantor set C is homeomorphic to its hyperspace $\exp(C)$, ([Ch]). In a situation which was lacking in concrete examples, it was quite natural to raise the question of existence of two non-homeomorphic spaces having their hyperspaces homeomorphic (V. Ponomarev, in [Po]). The answer came from A. Pelczynski ([Pe]), who proved that:

- *When X is compact metric 0-dimensional space having the set of its isolated points everywhere dense, then $\exp(X)$ is homeomorphic to the space $T(C)$, being the Cantor set with a point interpolated in each of its removed intervals.*

Pelczynski also quotes a result of A. Mostowski according to which there are continuum many different topological types of compact metric 0-dimensional spaces having the sets of their isolated points everywhere dense, but they still have their hyperspaces homeomorphic to $T(C)$.

4. *A search for more concrete examples*

The class **Z** of compact metric 0-dimensional spaces is a natural framework to look for some other concrete examples of hyperspaces and to try to fix their topological types. This led us to consider a classification of points of spaces in **Z** that reveals some of their interesting topological properties. We give here a somewhat less formally tight description of this classification.

Let $X \in \mathbf{Z}$ and let X_0 be the set of isolated points of X. If $X = X_0$, the set X_0 is finite and we will also write $s(X) = 0$. Let X_1 be the set of points of X, having a neighborhood without isolated points. When $X = X_1$, X_1 is homeomorphic to the Cantor set C and then we write $s(X) = 1$. When $X = X_0 \cup X_1$, X_0 is a finite set and $X_1 \approx C$ and then we write $s(X) = 0, 1$. Let $X_{(0)}$ be the set of accumulation points of X_0, then $X = X_0 \cup X_1 \cup X_{(0)}$. Let X_2 be the subset of those points of $X_{(0)}$ which are not accumulation points of X_1. When $X_1 = \emptyset$, then $X = X_0 \cup X_2$ and we also write $s(X) = 0, 2$. When $X_1 \neq \emptyset$, then $X = X_0 \cup X_1 \cup X_{(0)(1)}$.

Let $n > 1$ and suppose that the sequence $X_0, \dots, X_{n-1}, X_{(0)\dots(n-2)}$ of disjoint subsets of X has been defined. Let X_n be the set of points of $X_{(0)\dots(n-2)}$ which are accumulation points of X_{n-2} and not of X_{n-1} and $X_{(0)\dots(n-2)(n-1)}$ those points which are accumulation points of X_{n-1}. Then

$$X = X_0 \cup \ldots \cup X_{n-1} \cup X_n \cup X_{(0)\dots(n-2)(n-1)}.$$

As soon as $X_{n-1} = \emptyset$, then $X_{(0)\dots(n-2)(n-1)} = \emptyset$ and $X = X_0 \cup \cdots \cup X_{n-2} \cup X_n$. In this case we write $s(X) = 0, \dots, n-2, n$.

When $X_{n-1} \neq \emptyset$ and $X_{(0)\dots(n-2)(n-1)} = \emptyset$, then $X = X_0 \cup \cdots \cup X_{n-1} \cup X_n$ and we write $s(X) = 0, \dots, n-2, n-1, n$. When for each n, $X_{(0)\dots(n-2)(n-1)} \neq \emptyset$, the intersection X_ω of this descending sequence of compact sets is non-empty. In this case

$$X = X_0 \cup \cdots \cup X_n \cup \cdots \cup X_\omega$$

and we write $s(X) = 0, \dots, n, \dots, \omega$.

Definition 4.1. For $x \in X_n$, $(0 \le n \le \omega)$, the number n is called the *accumulation order* of x and denoted by $n = ord(x)$. The sequence $s(X)$ that is assigned to X is called the *accumulation spectrum* of X.

Examining the hyperspaces of spaces in **Z**, first we tried to see how the set $(\exp(X))_n$ of elements of order n in $\exp(X)$ depends on the sets X_n, $n = 0, 1, \ldots$, hoping to discover a regularity in this dependence. A great surprise arose when we found that for each X in **Z**, $(\exp(X))_6 = \emptyset$, which meant that the spaces $\exp(X)$ had no elements of order higher than 7. Namely, we established the following formulae:

$$\begin{aligned}
&(\exp(X))_0 = \langle X_0 \rangle \\
&(\exp(X))_1 = \rangle X_1 \langle \qquad := \{F \in \exp(X) \mid F \cap X_1 \neq \emptyset\} \\
&(\exp(X))_2 = \langle X_0 \cup X_2, X_2 \rangle \\
&(\exp(X))_3 = \langle X_0 \cup X_3, X_3 \rangle \\
&(\exp(X))_4 = \langle X_0 \cup X_2 \cup X_4, X_4 \rangle \\
&(\exp(X))_5 = \langle X_0 \cup X_2 \cup X_3 \cup X_5, X_2 \cup X_5, X_3 \cup X_5 \rangle \\
&(\exp(X))_6 = \emptyset
\end{aligned}$$

This unexpected discovery turned an easily going consideration into a serious search for topological types of hyperspaces in **Z**.

Let us call a space X in **Z** *full* if for each $n > 0$, whenever the sets X_n are non-empty they contain no isolated point. Then, we proved:

- *For each X, $\exp(X)$ is a full space.*

As a variation on Brouwer's topological characterization of the Cantor set, we also proved:

- *When X and Y are full spaces having finite spectra, then $s(X) = s(Y)$ implies $X \approx Y$.*

At the end, a sequence of full spaces was constructed, starting with $C_{-1} = \emptyset$, $C_0 = \{1\}$ (one point space), and $C_1 = C$ (the Cantor set). Assuming that the sequence $C_{-1}, C_0, C_1, \ldots, C_n$, $(n > 0)$ has been constructed, then C_{n+1} is the space obtained from the Cantor set C when a (small enough) copy of $C_{n-2} \oplus C_{n-1}$ is interpolated in each of its removed intervals. For $n > 1$

$$s(C_n) = 0, \ldots, n-2, n; \quad s(C_{n-1} \oplus C_n) = 0, 1, \ldots, n-1, n$$

(and for the Cantor set C_1, $s(C_1) = 1$). Excluding the cases of finite spaces having more than one point, for some n, each other full space with finite spectrum is homeomorphic to either C_n or $C_{n-1} \oplus C_n$. Let us note that somewhat more complicated construction of full spaces was given in [Ma2] and the one that is presented here was announced in [Ma3].

Thus, our search for hyperspaces in $\mathbf{Z}$ culminated in the following statements:

- *The only exponentially complete spaces in* $\mathbf{Z}$ *are*

$$C_0,\, C_1,\, C_0 \oplus C_1,\, C_2,\, C_1 \oplus C_2,\, C_3,\, C_4,\, C_5,\, C_7.$$

- *Excluding the trivial case of spaces with a finite number of isolated points, the only hyperspaces in* $\mathbf{Z}$ *are*

$$C_1,\, C_2,\, C_1 \oplus C_2,\, C_3,\, C_4,\, C_5,\, C_7.$$

The fact that, excluding the trivial cases, there exists only a finite number of topological types of hyperspaces in $\mathbf{Z}$, added some spectacularity to these results. Published in 1972, under the title “Exponentially complete spaces III” ([Ma2]) and communicated several times in some lectures delivered by this author (Seminar Kuratowski–Engelking at Polish Academy of Sciences, Chair of General Topology at the Moscow State University, several international topological conferences, etc.) these results were widely known to the specialists in this area of mathematics.

A. N. Vybornov (and his mentor V. Ponomarev) followed the way of researching from [Ma2] to investigate the hyperspaces of 0-dimensional Polish spaces ([Vy]). With some extra elegance, S. Todorčević also presents these results in his Springer’s monograph [To].

Somewhat surprisingly the results from [Ma2] were reproved by Sh. Oka in his paper [O1], without any referring to our paper. Thanks to J. van Mill and J. Vaughan, Oka wrote a corrigendum ([O2]) giving credit for my results.

5. *The other cases of the use of accumulation orders*

In a number of other cases the accumulation orders were used as an efficient system of invariants. In [Ma4], we announced that for $X \in \mathbf{Z}$, $\exp^{\omega}(X)$ is only one of the spaces

$$C_0,\, C_1,\, C_0 \oplus C_1,\, C_2,\, C_1 \oplus C_2,\, C_4$$

and a complete proof of this fact was given in [Ma5].

For the Cartesian product of spaces, there exists a corresponding Cartesian product of accumulation orders. Namely, let $x \in X$, $y \in Y$ and $\operatorname{ord}(x) = m$, $\operatorname{ord}(y) = n$. Then $m \times n$ is defined as $\operatorname{ord}((x, y))$, $(x, y) \in X \times Y$. The product $m \times n$ does not depend on the choice of the spaces X and Y and is commutative and associative and for each n, $n \times 1 = 1$. Up to 7, this multiplication is given by the following multiplication table,

$\times$	0	2	3	4	5	7
0	0	2	3	4	5	7
2	2	2	5	4	5	7
3	3	5	3	7	5	7
4	4	4	7	4	7	7
5	5	5	5	7	5	7
7	7	7	7	7	7	7

As each $n \in N$ can be written in the form $n = 6k + r$, where $k = 0, 1, 2, \dots$ and $r = 0, 2, 3, 4, 5, 7$, the Cartesian product of $m = 6k_1 + r_1$ and $n = 6k_2 + r_2$ is given by the following formula,

$$m \times n = 6(k_1 + k_2) + r_1 \times r_2.$$

For example $8 \times 13 = (6 \cdot 1 + 2) \times (6 \cdot 1 + 7) = 6(1 + 1) + 2 \times 7 = 12 + 7 = 19.$

The semi-group $(N, \times)$ was defined and its properties established in [Ma3]. Based on the fact that in the case of two full spaces, equality of their accumulation spectra implies their homeomorphism, it was a matter of simple verification to see that:

- The pairs of non-homeomorphic spaces,
$$C_{6k+2} \oplus C_{6k+3}, \quad C_{6k+5}, \quad k = 0, 1, 2, \dots$$
have homeomorphic squares C_{12k+5}.

The existence of such pairs of spaces in **Z** solves a problem of P. R. Halmos, posed in the Boolean algebra terms in his book [H].

In full analogy with the case of spaces in **Z**, the class $\mathbf{Z}_0$ of countable metric 0-dimensional spaces was considered in [Ma-Vu]. In $\mathbf{Z}_0$, the role of the Cantor set in **Z**, is reserved for the space Q_1 of rational numbers. In this class of spaces, a variation on Sierpinski's topological characterization of the space Q_1 of rationals is somewhat more demanding statement than its analogue in **Z**.

Let us note that Q_1 can be taken, in a topologically equivalent way, to be the set of end points of the removed intervals of the Cantor set C. Starting with $Q_{-1} = \emptyset$, $Q_0 = \{1\}$ (one point space), Q_1 the set of rationals and proceeding inductively, Q_n is taken to be Q_1 together with a (small enough) copy of $Q_{n-3} \oplus Q_{n-2}$ interpolated in each removed interval of C. Then, it is easy to verify that, for example,

- $Q_2 \oplus Q_3$, Q_5 *is a pair of non-homeomorphic spaces having homeomorphic squares.*

The problem of existence of such pairs of spaces in $\mathbf{Z}_0$ was posed in [Ko–Tr] and [Tr].

6. *Some other properties related to hyperspaces*

Though not specific for 0-dimensional spaces, we include here some properties related to the construction of the hyperspaces of rank ω. In [M-V-Z] the following results have been proved:

- *The union mapping* $u : \exp^2(X) \to \exp(X)$ *is open.*
- *The union mapping* $u : \exp^2(I) \to \exp(I)$*, where* $I = [0, 1]$ *is universal in the sense that each continuous mapping of a compact metric space is a restriction of the union mapping u.*
- *For* X *non-degenerate Peano continuum,* $\exp^\omega(X)$ *is not locally connected.*

In light of the fact that $\exp(\exp^\omega(X)) \approx \exp^\omega(X)$, an alluring idea that $\exp^\omega(X)$ is unique for all compact connected metric spaces X easily crosses the mind but, due to the complexity of this construction has never been investigated.

At the end let us add that when throughout this whole construction closed sets are replaced with closed connected sets, then:

- *For a non-degenerate Peano continuum* X*, the hyperspace of rank* ω *of close connected subsets is homeomorphic to the Hilbert cube* ([Ma–Vr]).

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