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LINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS II

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A b s t r a c t. *We continue study of the existence and analytic form of solutions to linear fractional differential equations in which coefficients are bounded functions on a bounded interval, as we did in Part I. Now in Part II we elaborated some special classes of equations and applied obtained results to the fractional damped Mathieu's equation.*

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1. *Introduction*

In the first part (cf. [3]) we proved the existence of a solution to equation (1.1) with fractional derivatives

$$\left({}_0D_t^{\alpha_m} - \lambda \sum_{i=0}^{m-1} A_i(t)\,{}_0D_t^{\alpha_i}\right) y(t) = f(t), \quad 0 < t \le b, \tag{1.1}$$

supposing: 1) $\alpha_m \geq 2$; 2) $\alpha_m \neq [\alpha_m]$; 3) $A_i,\ i = 0, 1, \ldots, m-1$, are bounded functions on $[0, b]$; 4) $\alpha_i > \alpha_{i-1},\ i = 1, \ldots, m$.

In this second part some special cases of equation (1.1) will be elaborated as well as application of the proved results to the damped Mathieu's equation. The second part is self-contained and can be read independently from the first part.

It is important to state precisely what properties the solution that we are looking for must have, and what problems appear in obtaining such solutions. In the analysis that follows, the function $y(t)$ is a solution to equation (1.1), where ${}_0D_t^{\alpha_i}$ is the Riemann-Liouville derivative, if: ${}_0D_t^{\alpha_i}y,\ i = 0, 1, \ldots, m$, belong to a space $\mathcal{J}_0$ defined on $[0, b]$: ${}_0D_t^{\alpha_i}y,\ i = 0, 1, \ldots, m$, satisfy equation (1.1) and $y(t)$ can be extend on the whole $[0, b]$ continuously. The initial conditions are given by

$$\left({}_0I_t^{1-\gamma_m}y\right)^{(j)}(0^+), \quad j = 0, \ldots, [\alpha_m]. \tag{1.2}$$

Using the Abel integral equation and (1.1), (1.2) we can obtain initial conditions given by $y^{(j)}(0^+)$; the values of $y^{(j)}(0^+)$, for some j may not be defined.

In solving equations like equation (1.1) one often supposes that the solution y belongs to $AC^{n_m}([0, b])$. This is restrictive, because in this case one supposes that all derivatives $y^{(i)}(t),\ i = 0, 1, \ldots, [\alpha_m]$, are continuous functions on $[0, b]$. However it proves to be useful if ${}_0D_t^{\alpha_i}y \in \mathcal{J}_0$ only.

If we improve the conditions for the solutions y we have to make it very carefully, because to improve the common space $\mathcal{J}_0$ we have to introduce new limitations (cf. Lemma 2.1).

It turns out that the assumptions on ${}_0I_t^{1-\gamma_i}y$ are better to use then assumptions on y, in solving equation (1.1) (cf. Theorem 3.1 and [16]).

The case when $\alpha_i = [\alpha_i],\ i = m,\ m - i, \ldots, i_0$, (1.1) has specific properties that are specially analyzed.

We call reader's attention to papers: [12], [11], [18], [10], [13], [20], [15].

2. *Preliminaries*

To discuss existence of solution and to explicitly solve equation (1.1) we use the spaces $\mathcal{J}_n([0, b]), n \in \mathbb{N}_0$ (cf. [4]) instead of the space $AC^n([0, b])$ (cf. [11]) or instead of spaces $C_\alpha,\ \alpha \in \mathbb{R}$ (cf. [11] and [14]), for example. For the existence of $D^{\alpha_i}y$ in (1.1) and for the construction of a solution to (1.1), it turned out that the assumption that $({}_0I_t^{1-\gamma_i}y) \in \mathcal{J}_{[\alpha_i]+1}([0,b])$, as it is done in our paper, is better then to suppose that $y \in AC^n([0,b])$. See also [4], [3] and Theorem 3.1 C in this paper.

The spaces $\mathcal{J}_n([0,b])$, $n \in \mathbb{N}_0$, defined in [4] as: $\mathcal{J}_0([0,b]) = \{f \in L([0,b])|,\ f$ locally bounded on $(0,b]\}$; $\mathcal{J}_n([0,b]) = \{f,\ f^{(n)} \in \mathcal{J}_0([0,b])\}$, $n \in \mathbb{N}$.

A necessary and sufficient condition that ${}_0D_t^{\alpha}f \in \mathcal{J}_0([0,b])$ is that

$${}_0I_t^{1-\gamma}f \in \mathcal{J}_n([0,b]),$$

where $\alpha = n-1+\gamma$, $n \in \mathbb{N}$, $\gamma \in (0,1)$.

For the properties of the space $\mathcal{J}_n([0,b])$ see ([10]).

The reason to introduce $\alpha_m \geq 2$ is given in the following.

Remark 2.1. If the function $f \in \mathcal{J}_n([0,b])$, i.e., $f^{(n)} \in \mathcal{J}_0([0,b])$, then there exist

$$\lim_{t\to 0} f^{(i)}(t) = f^{(i)}(0^+), \quad i = 0,1,\dots,n-1.$$

If $n \geq 2$, then $f^{(i)}(t)$, $i = 0,1,\dots,n-1$ can be extended as continuous function on $[0,b]$. In case $n=1$ we can take $f(t)$ only as a bounded function. (cf. [8], Satz 2, p.100).

We quote some properties of operators ${}_0I_t^{\alpha}$ and ${}_0D_t^{\alpha}$, that we use.

1) If $\alpha > 0$ and $\beta > 0$, then

$${}_0I_t^{\alpha}\,{}_0I_t^{\beta}f(x) = {}_0I_t^{\alpha+\beta}f(x), \quad \text{a.e. for } f \in L^1(0,b).$$

2) If $\alpha > \beta > 0$, then

$${}_0D_t^{\beta}\,{}_0I_t^{\alpha}f(x) = {}_0I_t^{\alpha-\beta}f(x), \quad \text{a.e. for } f \in L^1(0,b)$$

(cf. [11], p. 73–74). In particular, when $\alpha = \beta$, we have

$${}_0D_t^{\beta}\,{}_0I_t^{\beta}f(x) = f(x), \text{ a.e.}$$

In [4] we proved Lemma 3.5 which gives sufficient properties that a function f has all fractional derivatives ${}_0D_t^{\alpha_i}f$, $i = 0,1,\dots,m$, which we need in treating equation (1.1). Here we present a more precise result given in the next lemma.

Lemma 2.1. *Let* $[\alpha_2]-[\alpha_1]=q,\ q\geq 1$. *If* ${}_0I_t^{1-\gamma_2}f\in\mathcal{J}_{[\alpha_2]+1}([0,b])$ *and* $({}_0I_t^{1-\gamma_2}f)^{(j)}(0^+)=0,\ j=0,\ldots,j_0,$ $(j_0=[\alpha_2]-q$ *if* $\gamma_1\geq\gamma_2$, *or* $j_0=[\alpha_2]-q-1$, *if* $\gamma_2>\gamma_1)$, *then* ${}_0I_t^{1-\gamma_1}f\in\mathcal{J}_{[\alpha_1]+1}$. *If* $q=0$, *then we have* $\gamma_2>\gamma_1$ *and* $j_0=[\alpha_i]-1$.

Proof. We use the notation $[\alpha_i]+1=n_i,\ i=1,2$. Then we have for $({}_0I_t^{1-\alpha_1}f)^{(n_1)}$:

$$\begin{aligned}({}_0I_t^{1-\gamma_1}f)^{(n_1)}(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n_1}{}_0I_t^{1-\gamma_1}f(t)=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n_1+q}{}_0I_t^{q+1-\gamma_1}f(t)\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n_2}{}_0I_t^{q+1+\gamma_2-\gamma_2-\gamma_1}f(t)\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n_2}{}_0I_t^{\gamma_2-\gamma_1+q}\,{}_0I_t^{1-\gamma_2}f(t)\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n_2}\frac{1}{\Gamma(\gamma_2-\gamma_1+q)}\int_0^t\frac{({}_0I_t^{1-\gamma_2}f)(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\gamma_1-\gamma_2+1-q}}\\ &= \frac{1}{\Gamma(\gamma_2-\gamma_1+q)}\int_0^t\frac{({}_0I_t^{1-\gamma_2}f)^{(n_2)}(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\gamma_1-\gamma_2+1-q}}\\ &+ \sum_{j=0}^{n_2-1}({}_0I_t^{1-\gamma_2}f)^{(j)}(0^+)\left(\frac{t^{\gamma_2-\gamma_1-1+q}}{\Gamma(\gamma_2-\gamma_1+q)}\right)^{(n_2-1-j)}.\end{aligned}\tag{2.1}$$

The first addend in (2.1) is a composition of the function $({}_0I_t^{1-\gamma_2}f)^{(n_2)}\in\mathcal{J}_0([0,b])$ and the function $\dfrac{t^{\gamma_2-\gamma_1-1+q}}{\Gamma(\gamma_2-\gamma_1+q)}\in\mathcal{J}_0([0,b])$. Then the composition belongs to $\mathcal{J}_0([0,b])$.

The second part of (2.1) is:

$$\sum_{\substack{j=[\alpha_1]+1\ (\gamma_1\geq\gamma_2)\\ j=[\alpha_1]\quad (\gamma_2>\gamma_1)}}^{[\alpha_2]}\left({}_0I_t^{1-\gamma_2}f\right)^{(j)}(0^+)\Big(\frac{t^{\gamma_2-\gamma_1-1+q}}{\Gamma(\gamma_2-\gamma_1+q)}\Big)^{(n_2-1-j)}.\tag{2.2}$$

In the second addend we have a sum of elements. We need that this sum belongs to $\mathcal{J}_0([0,b])$. Then some of elements have to be zero. This is the case when $\gamma_2-\gamma_1-1+q-[\alpha_2]+j\leq -1,\ j\in\mathbb{N}_0$, i.e., $0\leq j\leq\gamma_1-\gamma_2-q+[\alpha_2],\ j\in\mathbb{N}_0$. This happens if $j\leq j_0,\ j\in\mathbb{N}_0$. Consequently $({}_0I_t^{1-\gamma_2}f)^{(j)}(0^+)=0,\ j=0,1,\ldots,j_0$. It remains to find j_0. We start with

$\gamma_2 - \gamma_1 - 1 + q - [\alpha_2] + j \leq -1$. If $\gamma_1 \geq \gamma_2$, j_0 is the biggest number such that $j \leq [\alpha_2] - q + \gamma_1 - \gamma_2$, $j \in \mathbb{N}_0$. Then $j_0 = [\alpha_2] - q$ for $\gamma_1 \geq \gamma_2$. If $\gamma_1 < \gamma_2$, $j \leq [\alpha_2] - q - 1 + (1 - \gamma_2 + \gamma_1)$, $j \in \mathbb{N}_0$, i.e., $j_0 = [\alpha_2] - q - 1$. If $q = 0$, then $j \leq [\alpha_2] - (\gamma_2 - \gamma_1) = [\alpha_2] - 1 + (1 - (\gamma_2 - \gamma_1))$. Hence $j_0 = [\alpha_2] - 1$. This proves Lemma 2.1. □

Remark 2.2. If $\alpha_2 > \alpha_1 > \alpha_0$ and Lemma 2.1 is valid for α_2, α_1 then it is valid also for α_2, α_0. This follows from $[\alpha_2]-[\alpha_0] \geq [\alpha_2]-[\alpha_1] = q \geq 1$.

3. *Solutions to equation* (1.1) *if it satisfies some additional conditions*

3.1. *Case* $\alpha_m \neq [\alpha_m]$, $[\alpha_m - \alpha_{m-1}] = p$, $p \leq \alpha_m - \alpha_{m-1} < p+1$, $p \in \mathbb{N}$

Theorem 3.1. *Suppose that in* (2.1) *we have:*

1) $\alpha_m \neq [\alpha_m]$;
2) $[\alpha_m - \alpha_{m-1}] = p$, $p \leq \alpha_m - \alpha_{m-1} < p+1$ *and* $[\alpha_m] - [\alpha_{m-1}] = q$;
3) $\mathbf{c}$ *is any element of* $\mathbb{R}^p$;
4) $f \in \mathcal{J}_0([0,b])$, *and*
5) *functions* $A_i(t)$, $i = 0, \ldots, m-1$, *are bounded and integrable on* $[0,b]$.

Then

A) *There exist* $u(t,\mathbf{c})$, *bounded solutions to Volterra integral equation,*

$$u(t) - \lambda \int_0^t K(t,\tau) u(\tau)\, \mathrm{d}\tau = F(t,\mathbf{c}), \quad \lambda \in \mathbb{R}, \quad t \in [0,b], \tag{3.1}$$

where

$$K(t,\tau) = \sum_{i=0}^{m-1} \int_\tau^t \frac{(t-\tau)^{p-1}}{\Gamma(p)} \frac{(s-\tau)^{\beta_i - 1}}{\Gamma(\beta_i)} A_i(s)\, \mathrm{d}s, \quad \text{if } \alpha_m - \alpha_{m-1} > p,$$

or

$$K(t,\tau) = \frac{(t-\tau)^{p-1}}{\Gamma(p)} A_{m-1}(\tau) + \sum_{i=0}^{m-2} \int_\tau^t \frac{(t-s)^{p-1}}{\Gamma(p)} \frac{(s-\tau)^{\beta_i - 1}}{\Gamma(\beta_i)} A_i(s)\, \mathrm{d}s,$$

if $\alpha_m - \alpha_{m-1} = p$, $\beta_i = \alpha_m - \alpha_i - p$ *and*

$$F(t,\mathbf{c}) = {}_0I_t^p f(t) + \sum_{\nu=0}^{p-1} c_\nu \frac{t^\nu}{\Gamma(\nu+1)}.$$

Solution $u(t,\mathbf{c})$ *is given by*

$$u(t,\mathbf{c}) = F(t,\mathbf{c}) + \lambda \int_0^t R(t,\tau,\lambda) F(\tau,\mathbf{c_0})\,\mathrm{d}\tau,$$

where R *is given by* (3.13).

B) *The functions* $y(t,\mathbf{c})$ *are related to* $u(t,\mathbf{c})$ *by:*

$${}_0I_t^{1-\gamma_m} y(t,\mathbf{c}) = {}_0I_t^{[\alpha_m]+1-p} u(t,\mathbf{c}), \quad t \in [0,b]$$

and are solution to (2.1) *for any* $\mathbf{c} \in \mathbb{R}^p$.

C) *The solutions* $y(t,\mathbf{c})$ *satisfy the following initial conditions* C1) *and* C2), *given in terms of derivatives* $({}_0I_t^{1-\gamma_m} y(t,\mathbf{c}))^{(i)}(0^+,\mathbf{c})$ *and* $y^{(i)}(0^+,\mathbf{c})$, *respectively,* $i = 0,1,\dots,[\alpha_m]$. *Namely,*

C1) $({}_0I_t^{1-\gamma_m} y(t,\mathbf{c}))^{(i)}(0^+,\mathbf{c})$ *is given by*

$$= \begin{cases} \big({}_0I_t^{[\alpha_m]+1-p-i} u(t,c)\big)(0^+,\mathbf{c}) = 0, & \text{if } i = 0,1,\dots,[\alpha_m]-p, \\ c_{k-1}, & \text{if } i = [\alpha_m]-p+k, \quad k = 1,\dots,p-1, \\ \lim\limits_{t\to 0^+} \int_0^t f(\tau)\,\mathrm{d}\tau + c_{p-1}, & \text{if } i = [\alpha_m]; \end{cases}$$

C2) $y^{(i)}(0^+,\mathbf{c}) = 0$, $i = 0,1,\dots,[\alpha_m]-p$, *and* $y^{([\alpha_m]-p+k)}(0^+,\mathbf{c}) = \infty$, $k = 1,\dots,p$.

Proof. As in Part I the proof consists in construction of the corresponding Volterra integral equation and construction of the solutions y to (2.1) using solutions u to such integral equation.

Let y be such that y and u are related as

$${}_0I_t^{1-\gamma_m} y(t) = {}_0I_t^{[\alpha_m]+1-p} u(t), \quad t \in [0,b]. \tag{3.2}$$

Then,

$$\begin{aligned} {}_0D_t^{\alpha_m} y &= {}_0D_t^{[\alpha_m]+1} {}_0I_t^{1-\gamma_m} y = D^{[\alpha_m]+1} {}_0I_t^{[\alpha_m]+1-p} u \\ &= D^p D^{[\alpha_m]+1-p} {}_0I_t^{[\alpha_m]+1-p} u = D^p u, \end{aligned} \tag{3.3}$$

and for $i = 0, 1, \ldots, m-1$,

$$\begin{aligned} {}_0D_t^{\alpha_i} y(t) &= D^{[\alpha_i]+1} {}_0I_t^{1-\gamma_i} y(t) \\ &= D^{[\alpha_i]+1} {}_0I_t^{\gamma_m-\gamma_i+1-\gamma_m} y(t) \\ &= D^{[\alpha_i]+1} {}_0I_t^{\gamma_m-\gamma_i+[\alpha_m]+1-p} u(t) \\ &= {}_0I_t^{\alpha_m-\alpha_i-p} u(t) \\ &= {}_0I_t^{\alpha_m-\alpha_{m-1}+\alpha_{m-1}-\alpha_i-p} u(t) \\ &= {}_0I_t^{\beta_i} u(t), \quad \beta_i = \alpha_m - \alpha_{m-1} + \alpha_{m-1} - \alpha_i - p. \end{aligned} \tag{3.4}$$

If a) $\alpha_m - \alpha_{m-1} > p$, we have $\beta_i > 0$, $i = 0, 1, \ldots, m-1$; b) $\alpha_m - \alpha_{m-1} = p$, $\beta_i > 0$, $i = 0, 1, \ldots, m-2$, but $\beta_{m-1} = 0$.

Since $f \in \mathcal{J}_0([0, b])$,

$$D^p \left({}_0I_t^p f + \sum_{j=0}^{p-1} c_j \frac{t^j}{\Gamma(j+1)} \right) = f.$$

As in Part 3 of [3], if we take ${}_0I_t^0 y = y$, in case $[\alpha_m - \alpha_{m-1}] = p$, $p \in \mathbb{N}$ (case $p = 0$ is treated in [3]), equation (2.1) can be written as

$$D^p \left(u(t) - \lambda\, {}_0I_t^p \sum_{i=0}^{m-1} A_i(t) {}_0I_t^{\beta_i} u(t) \right) = D^p \left({}_0I_t^p f(t) + \sum_{\nu=0}^{p-1} c_\nu \frac{t^\nu}{\Gamma(\nu+1)} \right)$$

or

$$u(t) - \lambda \sum_{i=0}^{m-1} \left({}_0I_t^p A_i(\cdot) {}_0I_t^{\beta_i} u(\cdot) \right)(t) = F(t, \mathbf{c}), \quad t \in [0, b]. \tag{3.5}$$

where

$$F(t, \mathbf{c}) = {}_0I_t^p f(t) + \sum_{\nu=0}^{p-1} c_\nu \frac{t^\nu}{\Gamma(\nu+1)}, \qquad \mathbf{c} = (c_0, \ldots, c_{p-1}). \tag{3.6}$$

In case a) we have

$$\begin{aligned} u(t) - \lambda \sum_{i=0}^{m-1} \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} A_i(\tau) \frac{1}{\Gamma(\beta_i)} \times \\ \times \int_0^\tau (\tau - s)^{\beta_i - 1} u(s)\, \mathrm{d}s\, \mathrm{d}\tau = F(t, \mathbf{c}), \qquad t \in [0, b]; \end{aligned} \tag{3.7}$$

while in case b)

$$\begin{aligned} u(t) &- \lambda\, {}_0I_t^p \left(A_{m-1}(\cdot)u(\cdot)\right)(t) \\ &-\lambda \sum_{i=0}^{m-2} \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} A_i(\tau) \frac{1}{\Gamma(\beta_i)} \int_0^\tau (\tau-s)^{\beta_i-1} u(s)\,\mathrm{d}s\,\mathrm{d}\tau \\ &= F(t,\mathbf{c}), \quad t\in[0,b]. \end{aligned} \tag{3.8}$$

Case a) Interchanging the order of integrations, by Dirichlet's formula we arrive to

$$\begin{aligned} u(t) - \lambda \sum_{\nu=0}^{m-1} \int_0^t u(s) \int_s^t \frac{(\tau-s)^{\beta_i-1}}{\Gamma(\beta_i)} \frac{(t-\tau)^{p-1}}{\Gamma(p)} A_i(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\ = F(t,\mathbf{c}), \quad t\in[0,b]. \end{aligned} \tag{3.9}$$

Transposing variables τ and s, we have finally

$$\begin{aligned} u(t) - \lambda \int_0^t u(\tau) \sum_{i=0}^{m-1} \int_\tau^t \frac{(t-s)^{p-1}}{\Gamma(p)} \frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)} A_i(s)\,\mathrm{d}s\,\mathrm{d}\tau \\ = F(t,\mathbf{c}), \quad t\in[0,b], \end{aligned} \tag{3.10}$$

or

$$u(t) - \lambda \int_0^t K(t,\tau)u(\tau)\,\mathrm{d}\tau = F(t,\mathbf{c}), \tag{3.11}$$

where

$$K(t,\tau) = \sum_{i=0}^{m-1} \int_\tau^t \frac{(t-s)^{p-1}}{\Gamma(p)} \frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)} A_i(s)\,\mathrm{d}s. \tag{3.12}$$

Equation (3.11) is a Volterra integral equation with the kernel $K(t,\tau)$ having the property

$$|K(t,\tau)| \le M \sum_{i=0}^{m-1} \int_\tau^t \frac{(t-s)^{p-1}}{\Gamma(p)} \frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}\,\mathrm{d}s,$$

where

$$M = \max_{s\in[0,b],\, i=0,1,\dots,m-1} |A_i(s)| .$$

Then

$$|K(t,\tau)| \le M \sum_{i=0}^{m-1} \frac{(t-\tau)^{p+\beta_i-1}}{\Gamma(p+\beta_i)}.$$

<u>*Case* b)</u> Equation (3.8) can be transform in

$$u(t) - \lambda \int_0^t \frac{(t-\tau)^{p-1}}{\Gamma(p)} A_{m-1}(\tau) u(\tau)\,\mathrm{d}\tau$$

$$-\lambda \int_0^t u(\tau) \sum_{i=0}^{m-2} \int_\tau^t \frac{(t-s)^{p-1}}{\Gamma(p)} \frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)} A_i(s)\,\mathrm{d}s\,\mathrm{d}\tau.$$

The kernel $K(t,\tau)$ of integral equation (3.11) is now

$$K(t,\tau) = \frac{(t-\tau)^{p-1}}{\Gamma(p)} A_{m-1}(\tau) + \sum_{i=0}^{m-2} \int_\tau^t \frac{(t-s)^{p-1}}{\Gamma(p)} \frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)} A_i(s)\,\mathrm{d}s\,\mathrm{d}\tau.$$

Hence, $K(t,\tau)$ is a bounded function on $[0,b]$ in both cases a) and b).

With assumptions of Theorem 3.1, Volterra integral equation (3.11) is with bounded kernel $K(t,\tau)$, $(t,\tau) \in [0,b] \times [0,b]$ and bounded function $F(t,\mathbf{c})$, $t \in [0,b]$. Let us remark that these properties of Volterra integral equation (3.11) is valued for every $\mathbf{c} \in \mathbb{R}^p$.

Now we can apply the already cited Theorem A in [17], p. 13. With the suppositions in Theorem 3.1, equation (3.11) has the unique bounded solution $u(t,\mathbf{c})$ for every $\mathbf{c} \in \mathbb{R}^p$:

$$u(t,\mathbf{c}) = F(t,\mathbf{c}) + \lambda \int_0^t R(t,\tau,\lambda) F(\tau,\mathbf{c})\,\mathrm{d}\tau,$$

$$R(t,\tau;\lambda) = K(t,\tau) + \sum_{n=1}^{\infty} \lambda^n K_n(t,\tau), \tag{3.13}$$

where $F(t,\mathbf{c})$ is given by (3.6) and $K(t,\tau)$ is given by (3.12).

This proves A).

With the bounded solution $u(t,\mathbf{c})$ we can construct a solution $y(t,\mathbf{c})$ to (1.1) by using the function $u(t,\mathbf{c})$ and relation (3.2),

$$y(t,\mathbf{c}) = {}_0D_t^{1-\gamma_m}\, {}_0I_t^{[\alpha_m]+1-p}u(t,\mathbf{c}) = {}_0I_t^{\alpha_m-p}u(t,\mathbf{c}). \qquad (3.14)$$

To prove that $y(t,\mathbf{c})$ is a family of solutions to (1.1) we begin with the existence of ${}_0D_t^{\alpha_i}y(t,\mathbf{c})$, $i=0,1,\dots,m$ (cf. Lemma 2.1). We will start with ${}_0D_t^{\alpha_m}y(t,\mathbf{c})$.

Applying the operator ${}_0D_t^{[\alpha_m]+1}$ to (3.2) gives

$${}_0D_t^{\alpha_m}y(t,\mathbf{c}) = D^p u(t,\mathbf{c}) \quad (\text{cf. } (3.3)). \qquad (3.15)$$

Now, to prove that ${}_0D_t^{\alpha_m}y(t,\mathbf{c}) \in \mathcal{J}_0([0,b])$ it is enough to prove that $D^p u(t,\mathbf{c}) \in \mathcal{J}_0([0,b])$. We will use this fact. Since this proof is the same for cases a) and b), we present it only for case a).

The function $u(t,\mathbf{c})$ is given by two addends (cf. (3.13)). Let us start with the first one,

$$D^i F(t,\mathbf{c}) = {}_0I_t^{p-i}f(t) + \sum_{\nu=i}^{p-1} c_\nu \frac{t^{\nu-i}}{\Gamma(\nu-i+1)}, \quad i=0,1,\dots,p-1. \qquad (3.16)$$

This gives

$$D^p F(t,\mathbf{c}) = f(t).$$

The derivatives of the second addend $D^i\lambda\int_0^t R(t,\tau,\lambda)F(\tau)\,\mathrm{d}\tau$, we also consider in two parts, because

$$R(t,\tau,\lambda) = K(t,\tau) + \sum_{n=1}^{\infty}\lambda^n K_n(t,\tau).$$

Then

$$\begin{aligned} D\lambda\int_0^t K(t,\tau)F(\tau)d\tau &= \lambda D\int_0^t \sum_{i=0}^{m-1}\int_\tau^t \frac{(t-x)^{p-1}}{\Gamma(p)}\frac{(x-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}A_i(x)\,\mathrm{d}xF(\tau)\,\mathrm{d}\tau \\ &= \lambda D\sum_{i=0}^{m-1}\int_0^t A_i(x)\int_0^x \frac{(t-x)^{p-1}}{\Gamma(p)}\frac{(x-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}F(\tau)\,\mathrm{d}\tau\,\mathrm{d}x \end{aligned}$$

We can use the formula for the derivative with respect to t of an integral $\int_0^t \varphi(t,\tau)\,\mathrm{d}\tau$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \varphi(t,\tau)\,\mathrm{d}\tau = \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t,\tau)\,\mathrm{d}\tau + \varphi(t,t).$$

Thus,

$$\begin{aligned}
D^1\lambda\int_0^t K(t,\tau)F(\tau)\,\mathrm{d}\tau &= \lambda\sum_{i=0}^{m-1} A_i(s)\int_0^s \frac{(t-s)^{p-1}}{\Gamma(p)}\frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}F(\tau)\,\mathrm{d}\tau\Big|_0^t \\
&+ \lambda\sum_{i=0}^{m-1}\int_0^t A_i(s)\int_0^s \frac{(t-s)^{p-2}}{\Gamma(p-1)}\frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}F(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&= -\lambda\sum_{i=0}^{m-1} A_i(0)\lim_{s\to 0}\frac{(t-s)^{p-1}}{\Gamma(p)}\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta_i)}F(\tau)\,\mathrm{d}\tau \\
&+ \lambda\sum_{i=0}^{m-1}\int_0^t A_i(s)\int_0^s \frac{(t-s)^{p-2}}{\Gamma(p-1)}\frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}F(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&= \lambda\sum_{i=0}^{m-1}\int_0^t A_i(s)\int_0^s \frac{(t-s)^{p-2}}{\Gamma(p-1)}\frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}F(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}$$

In the same way we can continue with the derivatives $i = 2,\dots,p-2$. For the $(p-1)$-th derivative we have

$$D^{p-1}\int_0^t K(t,\tau)F(\tau)\,\mathrm{d}\tau = \lambda\sum_{i=0}^{m-1}\int_0^t A_i(s)\int_0^s \frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}F(\tau)\,\mathrm{d}\tau \qquad (3.17)$$

and for the p-th derivative it is

$$D^{p}\int_0^t K(t,\tau)F(\tau)\,\mathrm{d}\tau = \lambda\sum_{i=0}^{m-1} A_i(t)\int_0^t \frac{(t-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}F(\tau)\,\mathrm{d}\tau. \qquad (3.18)$$

We proved that the function $D^p\int_0^t K(t,\tau)F(\tau)\,\mathrm{d}\tau$ is bounded on $[0,b]$.

It is left to prove that the function

$$D^p\lambda\int_0^t\sum_{n=1}^{\infty}\lambda^n K_n(t,\tau)F(\tau)\,\mathrm{d}\tau \tag{3.19}$$

is bounded on $[0,b]$, for every $\lambda\in\mathbb{R}$. This we do in the same way as in Theorem 4.1 in [3]: First we have to prove that we can interchange the integral and the series in (3.19), i.e.,

$$\int_0^t\sum_{n=1}^{\infty}\lambda^n K_n(t,\tau)F(\tau)\,\mathrm{d}\tau=\sum_{n=1}^{\infty}\lambda^n\int_0^t K_n(t,\tau)F(\tau)\,\mathrm{d}\tau.$$

Second, starting with the relation

$$K_n(t,\tau)=\int_\tau^t K(t,s)K_{n-1}(s,\tau)\,\mathrm{d}s,\ K_0=K,$$

we have to prove that we can interchange the integrals, i.e.,

$$\int_0^t\sum_{n=1}^{\infty}\lambda^n K_n(t,\tau)F(\tau)\,\mathrm{d}\tau=\sum_{i=0}^{\infty}\lambda_n\int_0^t\int_\tau^t K(t,s)K_{n-1}(s,\tau)\,\mathrm{d}s\,F(\tau)\,\mathrm{d}\tau. \tag{3.20}$$

All proofs proceed just in the same manner as for the proof of Theorem 4.1 in [3]. The result is that ${}_0D_t^{\alpha_m}y(t,\mathbf{c})=D^pu(t,\mathbf{c})\in\mathcal{J}_0([0,b])$, for every $\mathbf{c}\in\mathbb{R}^p$.

Now it remains to prove that ${}_0D_t^{\alpha_i}y(t,\mathbf{c}),\ i=0,1,\dots,m-1$, also exist and belong to $\mathcal{J}_0([0,b])$. This we realize by using Lemma 2.1.

We introduced in Part 3.1 of this paper numbers $p=[\alpha_m-\alpha_{m-1}]$ and $q=[\alpha_m]-[\alpha_{m-1}]$. Since $\alpha_m-\alpha_{m-1}=[\alpha_m]-[\alpha_{m-1}]+\gamma_m-\gamma_{m-1},\ p=q$ if $\gamma_m\geq\gamma_{m-1}$ and $p=q-1$ if $\gamma_{m-1}>\gamma_m$. This relation between p and q we will use in the following.

By Lemma 2.1 if ${}_0D_t^{\alpha_m}y(t,\mathbf{c})\in\mathcal{J}_0([0,b])$ and

$$({}_0I_t^{1-\gamma_m}y(t,\mathbf{c}))^{(j)}(0^+,c)=0,\quad j=0,1,\dots,j_0, \tag{3.21}$$

then ${}_0D_t^{\alpha_i}y(t,\mathbf{c})\in\mathcal{J}_0([0,b]),\ i=0,1,\dots,m-1$, as well.

Consequently, we have to prove yet (3.21) only.

From (3.2) we have

$$({}_0I_t^{1-\gamma_m} y(t,\mathbf{c}))^{(j)} = {}_0I_t^{[\alpha_m]+1-p-j} u(t,\mathbf{c}), \quad j = 0,1,\dots,[\alpha_m]-p.$$

Since $1 \le [\alpha_m]+1-p-j$, for $j = 0,1,\dots,[\alpha_m]-p$, it follows that

$$({}_0I_t^{1-\gamma_m} y(t,\mathbf{c}))^{(j)}(0^+,\mathbf{c}) = 0, \quad j = 0,1,\dots,[\alpha_m]-p, \tag{3.22}$$

because $u(t,\mathbf{c})$ is a bounded function.

Now we can prove that the suppositions in Lemma 2.1 are satisfied and consequently that there exist ${}_0D_t^{\alpha_i} y(t,\mathbf{c})$, $i = 0,1,\dots,m-1$.

In Lemma 2.1 it is supposed that

$$({}_0I_t^{1-\gamma_m} y(t,\mathbf{c}))^{(j)}(0^+,\mathbf{c}) = 0,\ j = 0,1,\dots,[\alpha_m]-q-1,\ \gamma_m > \gamma_{m-1};$$

$$({}_0I_t^{1-\gamma_m} y(t,\mathbf{c}))^{(j)}(0^+,\mathbf{c}) = 0,\ j = 0,1,\dots,[\alpha_m]-q,\ \gamma_{m-1} \ge \gamma_m;$$

By (3.22) and the relations between numbers p and q: If $\gamma_2 \ge \gamma_1$, then $p = q$ and if $\gamma_1 > \gamma_2$, then $p = q-1$, we have

$$({}_0I_t^{1-\gamma_m} y(t,\mathbf{c}))^{(j)}(0^+,\mathbf{c}) = 0,\ j = 0,1,\dots,[\alpha_m]-q,\ \gamma_m \ge \gamma_{m-1},$$

$$({}_0I_t^{1-\gamma_m} y(t,\mathbf{c}))^{(j)}(0^+,\mathbf{c}) = 0,\ j = 0,1,\dots,[\alpha_m]-q+1,\ \gamma_m < \gamma_{m-1}.$$

Consequently, the supposed relations in Lemma 2.1 are satisfied.

We proved that there exist ${}_0D_t^{\alpha_i} y(t,\mathbf{c})$ and belongs to $\mathcal{J}_0([0,b])$, $i = 0,1,\dots,m$, $\mathbf{c} \in \mathbb{R}^p$.

We have seen that every solution $y(t,\mathbf{c})$ to equation (1.1) which satisfies condition of Theorem 3.1 defines a function $u(t,c)$ related to $y(t,\mathbf{c})$ by (3.2), which is solution to the integral equation (3.11), (3.12).

On the contrary it is easy to show that the function $y(t,\mathbf{c})$ such that ${}_0D_t^{\alpha_i} y(t,\mathbf{c})$ exist and belong to $\mathcal{J}_0([0,b])$, $i = 0,1,\dots,m$, is related with $u(t,\mathbf{c})$ by (3.2) and $u(t,\mathbf{c})$ is a bounded solution to integral equation (3.11), (3.12), this function $y(t,\mathbf{c})$ is a solution to (1.1). We already proved the properties of $y(t,\mathbf{c})$ constructed in described way, so it remains only to use the inverse way from $y(t,\mathbf{c})$ to $u(t,\mathbf{c})$, which is easy to do.

This proves part B).

In B) we proved (3.22), i.e.,

$$({}_0I_t^{1-\gamma_m} y(t,\mathbf{c}))^{(j)}(0^+,\mathbf{c}) = 0, \quad j = 0,1,\dots,[\alpha_m]-p.$$

We will see what happens when $j > [\alpha_m] - p$.

With (3.22) we have

$$({}_0I_t^{1-\gamma_m} y(t,\mathbf{c}))^{([\alpha_m]-(p-k))}(0^+,\mathbf{c}) = u^{(k-1)}(0^+,\mathbf{c}), \quad k = 1,\dots,p. \quad (3.23)$$

Since

$$\begin{aligned} u^{(k-1)}(0^+,\mathbf{c}) &= \lim_{t\to 0} {}_0I_t^{p-k+1} f(t) + \lim_{t\to 0^+} \sum_{\nu=k-1}^{p-1} c_\nu \frac{t^{\nu-k+1}}{\Gamma(\nu-k+2)} \\ &= \lim_{t\to 0} {}_0I_t^{p-k+1} f(t) + c_{k-1}, \quad k = 1,\dots,p, \end{aligned}$$

by Remark 2.1,

$$({}_0I_t^{1-\gamma_m} y(t,\mathbf{c}))^{([\alpha_m]-p+k)}(0^+,\mathbf{c}) = c_{k-1}, \quad k = 1,\dots,p-1,$$

$$({}_0I_t^{1-\gamma_m} y(0^+,\mathbf{c}))^{([\alpha_m])} = \lim_{t\to 0} \int_0^t f(\tau)\,\mathrm{d}\tau + c_{p-1}.$$

This proves C1).

For C2) we use (3.14) to obtain

$$y^{(i)}(0^+,\mathbf{c}) = \left({}_0I_t^{[\alpha_m]-p+\gamma_m-i} u(t,\mathbf{c})\right)(0^+,\mathbf{c}) = 0, \quad i = 0,1,\dots,[\alpha_m]-p,$$

because $u(t,\mathbf{c})$ is bounded on $[0,b]$. But

$$\begin{aligned} y^{([\alpha_m]-p+k)}(0^+,\mathbf{c}) &= \left({}_0I_t^{\gamma_m} u(t,\mathbf{c})\right)^{(k)}(0^+,\mathbf{c}) \\ &= \lim_{t\to 0^+} \left(\frac{t^{\gamma_m-1}}{\Gamma(\gamma_m)} * u^{(k)}(t,\mathbf{c}) \right. \\ &+ \left. \sum_{\nu=0}^{k-1} \left(\frac{t^{\gamma_m-1}}{\Gamma(\gamma_m)}\right)^{(\nu)} u^{(k-1-\nu)}(0^+,\mathbf{c}) \right) = \infty, \quad (3.24) \end{aligned}$$

$k = 1,\dots,p$, for every $\mathbf{c} \in \mathbb{R}^p$.

Thus, Theorem 3.1 is proved. □

Remark 3.1. Theorem 3.1 indicates the difference between two initial conditions and also the difference between the theory of linear differential equations and the theory of linear fractional differential equations.

The initial condition C1) better corresponds to equations of the form (1.1) even though it has to contain a required part coming from Lemma 2.1. The initial condition C2) shows that as a general rule the solution $y(t, \mathbf{c})$ can not be a function of the class $AC^n([0, b])$.

The number of solutions to (1.1) and the analytic form of solutions, depend, first of all, on $\alpha_m - \alpha_{m-1}$.

Remark 3.2. If we permit that $0 < \alpha_m < \alpha_{m-1} < 1$, i.e., $p = 0$, the corresponding Volterra integral equation becomes singular Voltera equation (cf. [3]) and we can find $u(t, \mathbf{c})$ as it is done in [3]. But the analyses of this case request additional technical works.

3.2. *Case* $\alpha_m = [\alpha_m],\ [\alpha_m - \alpha_m] = p,\ p \le \alpha_m - \alpha_{m-1} < p + 1,\ p \in \mathbb{N}$

First we show the changes in Part 3.1, because of new assumption $\alpha_m = [\alpha_m]$, i.e., $\gamma_m = 0$. This is listed by 1°, 2° and 3° below. Then we give a theorem which follows from Theorem 3.1, enumerated by 4°.

1° By definition

$$ {}_0D_t^{\alpha_m} y = D^{[\alpha_m]+1} {}_0I_t^{1-\gamma_m} y. $$

If $\gamma_m = 0$, ${}_0D_t^{[\alpha_m]} = D^{[\alpha_m]+1} {}_0I_t^1 y = D^{[\alpha_m]} D_0 I_t^1 y = D^{[\alpha_m]} y$. Consequently, ${}_0D_t^{[\alpha_m]} y = D^{[\alpha_m]} y$.

2° The supposition that ${}_0D_t^{\alpha_m} y(t) \in \mathcal{J}_0([0, b])$ in case $\alpha_m = [\alpha_m]$ gives ${}_0D_t^{\alpha_m} y = D^{[\alpha_m]} D\ {}_0I_t^1 y \equiv D^{[\alpha_m]} y \in \mathcal{J}_0([0, b])$, that is $y \in \mathcal{J}_{[\alpha_m]}([0, b])$.

3° If $y \in \mathcal{J}_{[\alpha_m]}([0, b])$, then there exists every ${}_0D_t^{\alpha_i} y$, and ${}_0D_t^{\alpha_i} y \in \mathcal{J}_0([0, b])$, $i = 0, 1, \ldots, m$. We have no reason for additional limitation to guarantee the existence of some $D^{\alpha_i} y$. (cf. Lemma 2.1).

4° Construction of the Volterra integral equation.

Because of 3°, instead of (3.2), we introduce

$$ y(t) = {}_0I_t^{[\alpha_m]-p} u(t) + P(t),\ P(t) = \sum_{\nu=0}^{[\alpha_m]-1} c_\nu \frac{t^\nu}{\Gamma(\nu+1)}. $$

Then

$$ D^{[\alpha_m]} y = D^p D^{[\alpha_m]-p} y = D^p u(t) + D^{[\alpha_m]} p, \quad p \ge 1, \tag{3.25} $$

and for $i = 0, 1, \ldots, m-1$:

$$\begin{aligned} D^{\alpha_i} y &= D^{[\alpha_i]+1} {}_0I_t^{1-\gamma_i} y = D^{[\alpha_i]+1} {}_0I_t^{1-\gamma_i+[\alpha_m]-p} u + D^{\alpha_i} P \\ &= {}_0I_t^{[\alpha_m]+1-\gamma_i-p-[\alpha_i]-1} u + D^{\alpha_i} P \\ &= {}_0I_t^{[\alpha_m]-\alpha_i-p} u + {}_0D_t^{\alpha_i} P \\ &= I^{\beta_i} u + {}_0D_t^{\alpha_i} P, \quad \beta_i = [\alpha_m] - \alpha_i - p \geq 0. \end{aligned} \tag{3.26}$$

As in Part 3.1 we have to differ two cases:

a) $p = [[\alpha_m] - \alpha_{m-1}] = [\alpha_m] - [\alpha_{m-1}] - 1 < [\alpha_m] - \alpha_{m-1}$, and

b) $p = [\alpha_m] - \alpha_{m-1}$ ($\gamma_{m-1} = 0$).

With (3.25) and (3.26) we transform equation (1.1) in a Volterra integral equation in the following way

$$D^p \left(u(t) - \lambda \, {}_0I_t^p \sum_{i=0}^{m-1} A_i(t) {}_0I_t^{\beta_i} u(t) \right) = D^p \left({}_0I_t^p f(t) + \sum_{\mu=0}^{p-1} c_\mu \frac{t^\mu}{\Gamma(\mu+1)} \right)$$
$$+ \; D^p {}_0I_t^p \left({}_0D_t^{\alpha_m} - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} \right) P$$

and

$$\begin{aligned} &u(t) - \lambda \, {}_0I_t^p \sum_{i=0}^{m-1} A_i(\cdot) {}_0I_t^{\beta_i} u(t) \\ &\quad = {}_0I_t^p \left(\left({}_0D_t^{\alpha_m} - \lambda \sum_{i=0}^{m-1} A_i(t) D^{\alpha_i} \right) P + f(t) \right) + \sum_{\mu=0}^{p-1} c'_\mu \frac{t^\mu}{\Gamma(\mu+1)}. \end{aligned} \tag{3.27}$$

We consider

$${}_0D_t^{\alpha_i} P = \sum_{\nu=0}^{[\alpha_m]-1} c_\nu \, {}_0D_t^{\alpha_i} \frac{t^\nu}{\Gamma(\nu+1)}.$$

Let i_0 be the biggest number i such that $\alpha_i \neq [\alpha_i]$, i.e., $\gamma_i \neq 0$. Then

$${}_0D_t^{\alpha_{i_0}} P(t) = \sum_{\nu=0}^{[\alpha_m]-1} c_\nu \frac{t^{\nu-\alpha_{i_0}}}{\Gamma(\nu+1-\alpha_{i_0})},$$

and

$$\left|A_{i_0}(t)_0D_t^{\alpha_{i_0}}P(t)\right| \le \max_{0\le t\le b}|A_{i_0}(t)| \sum_{\nu=0}^{[\alpha_m]-1} c_\nu \frac{t^{\nu-\alpha_{i_0}}}{\Gamma(\nu+1-\alpha_{i_0})}. \tag{3.28}$$

In order to have that $A_{i_0}(t)_0D_t^{\alpha_{i_0}}P(t)$ belongs to $\mathcal{J}_0([0,b])$ for any A_{i_0} bounded function we take $\nu \ge [\alpha_{i_0}]$. Hence $c_\nu = 0,\ \nu = 0,1,\dots,[\alpha_{i_0}]-1$, and

$$P = \sum_{\nu=[\alpha_{i_0}]}^{[\alpha_m]-1} c_\nu \frac{t^\nu}{\Gamma(\nu+1)}. \tag{3.29}$$

With P given by (3.29) we have:

$${}_0D_t^{\alpha_m}P = 0;$$

${}_0D_t^{\alpha_i}P(t)$ for $i_0 < i < m$ is a bounded function on $[0,b]$;

${}_0D_t^{\alpha_0}P(t) \in \mathcal{J}_0([0,b])$ (cf. (3.28));

${}_0D_t^{\alpha_i}P(t)$ for $i = 0,1,\dots,i_0-1$ belongs to $\mathcal{J}_0([0,b])$. (3.30)

We go back to (3.27). The function at the right of equation (3.27) will be denoted by $F_1(t)$. By (3.30) it as a bounded function,

$$F_1(t) = {}_0I_t^p\left(c_{[\alpha_m]} - \lambda\sum_{i=0}^{m-1} A_i(t)_0D_t^{\alpha_i}P + f(t)\right) + \sum_{\mu=0}^{p-1} c'_\mu \frac{t^\mu}{\Gamma(\mu+1)}.$$

Equation (3.27) is

$$u(t) - \lambda_0 I_t^p \sum_{i=0}^{m-1} A_i(\cdot)_0I_t^{\beta_i}u(t) = F_1(t). \tag{3.31}$$

The left side of (3.31) is the same as the left side of (3.5). Therefore, (3.31) may be written as

$$u(t) - \lambda\int_0^t K(t,\tau)u(\tau)\,\mathrm{d}\tau = F_1(t),$$

where $K(t,\tau)$ is given by (3.12). Since $F_1(t)$ is bounded on $[0,b]$, we can apply Theorem A in [17], p. 13.

Theorem 3.2. *We suppose that in equation* (1.1)*:*

1) $\alpha_m = [\alpha_m]$;
2) $[\alpha_m - \alpha_{m-1}] = p$, $1 \le p \le \alpha_m - \alpha_{m-1} < p+1$ *and* $[\alpha_m] - [\alpha_{m-1}] = q$;
3) $\mathbf{c}$ *is any element of* $\mathbb{R}^{p+1}$;
4) $f \in \mathcal{J}_0([0,b])$ *and* 5) $A_i(t)$, $i = 0, 1, \ldots, m-1$, *are bounded and integrable on* $[0,b]$.

Then

A) *There exist* $u(t,\mathbf{c})$ *bounded solutions to Volterra integral equation*

$$u(t) - \lambda \int_0^t K(t,\tau)u(\tau)\,\mathrm{d}\tau = F(t,\mathbf{c}),$$

where

$$K(t,\tau) = \sum_{i=0}^{m-1} \int_\tau^t \frac{(t-\tau)^{p-1}}{\Gamma(p)} \frac{(s-\tau)^{\beta_i - 1}}{\Gamma(\beta_i)} A_i(s)\,\mathrm{d}s, \quad \text{if } \alpha_m - \alpha_{m-1} > p,$$

or

$$K(t,\tau) = \frac{(t-\tau)^{p-1}}{\Gamma(p)} A_{m-1}(\tau) + \sum_{i=0}^{m-2} \int_\tau^t \frac{(t-s)^{p-1}}{\Gamma(p)} \frac{(s-\tau)^{\beta_i - 1}}{\Gamma(\beta_i)} A_i(s)\,\mathrm{d}s,$$

if $\alpha_m - \alpha_{m-1} = p$, $\beta_i = [\alpha_m] - \alpha_i - p$, *and*

$$F_1(t,\mathbf{c}) = {}_0I_t^p f(t) + \sum_{\nu=0}^{p-1} \mathbf{c}'_\nu \frac{t^\nu}{\Gamma(\nu+1)} + {}_0I_t^p \left(-\lambda \sum_{i=0}^{m-1} A_i(t)\, {}_0D_t^{\alpha_i} P \right).$$

Solution $u(t,\mathbf{c})$ *is given by*

$$u(t,\mathbf{c}) = F_1(t,\mathbf{c}) + \lambda \int_0^t R(t,\tau;\lambda) F_1(\tau,c)\,\mathrm{d}\tau,$$

where

$$R(t,\tau;\lambda) = K(t,\tau) + \sum_{n=1}^{\infty} \lambda^n K_n(t,\tau).$$

B) *The functions* $y(t,\mathbf{c})$ *which are related to* $u(t,\mathbf{c})$, *are given by*

$$y(t,\mathbf{c}) = {}_0I_t^{[\alpha_m]-p}u(t) + \sum_{\nu=[\alpha_{i_0}]}^{[\alpha_m]-1} c_\nu \frac{t^\nu}{\Gamma(\nu+1)}.$$

C) *The solutions* $y(t,\mathbf{c})$ *satisfy the initial conditions*

$$y^{(i)}(0^+,\mathbf{c}) = \lim_{t\to 0^+} D^i \sum_{\nu=[\alpha_{i_0}]}^{[\alpha_m]-1} c_\nu \frac{t^\nu}{\Gamma(\nu+1)}, \quad i = 0,1,\dots,[\alpha_m]-p-1,$$

$$y^{([\alpha_m]-p-1+k)}(0^+,\mathbf{c}) = u^{(k)}(0^+,\mathbf{c}) + \sum_{\nu=[\alpha_m]-p+k}^{[\alpha_m]-1} c_\nu t^{\nu-[\alpha_m]+p-k},$$

where $k = 1,\dots,p-1$.

3. *Applications to Mathieu's equation*

4.1. *Solutions of the damped Mathieu's equation*

Equation

$$D^2y(t) = -\omega^2[1+\varepsilon\cos t]y(t), \quad 0 < t \le b, \tag{4.1}$$

is called Mathieu's equation (cf. [2], p. 117 and [6]). Damped Mathieu's equation with damper νDy is

$$D^2y(t) + \nu Dy(t) + [\omega^2(1+\varepsilon\cos t)]y(t) = 0, \quad 0 < t \le b. \tag{4.2}$$

(cf. [14]). In [9] authors replaced in equation (4.2) the addend $\nu Dy(t)$ by the fractional derivative $\nu {}_0D_t^{1/2}y(t)$, i.e., they considered the equation

$${}_0D_t^2y(t) + \nu {}_0D_t^{1/2}y(t) + \omega^2[1+\varepsilon\cos t]y(t) = 0, \quad 0 < t \le b, \tag{4.3}$$

with initial condition

$$y(0) = 1, \quad Dy(t)|_{t=0} = 0. \tag{4.4}$$

Using the Adomian method (cf. [1]) they constructed the five term-approximate solution $\phi(t)$ to (4.3), (4.4), $\phi(t) = \sum_{n=0}^{4} y_n(t)$. They also presented approximate solution is assumed in the form of the series $\sum_{n=0}^{\infty} y_n(t)t^{n/2}$.

We will show how the results in this paper give the solutions which can be used as the solutions for equations (4.1), (4.2) and (4.3) with a general initial condition.

We apply Theorem 3.2 to equation, named Mathieu's damped equation,

$$D^2y(t) + \nu_0 D_t^\alpha y(t) + \omega^2(1+\varepsilon\cos t)y(t) = 0, \quad 0 < t \leq b. \tag{4.5}$$

Equation (4.5) is of the form as equation (1.1) in which $m = 2;\ \alpha_2 = 2;\ \alpha_1 = \alpha,\ 0 < \alpha < 1,\ \alpha_0 = 0,\ \lambda = -1,\ A_1 = \eta;\ A_0(t) = \omega^2(1+\varepsilon\cos t)$ and $f = 0$.

We study the case $[\alpha_1] = 0,\ \gamma_1 = \alpha$. Recall: $\alpha_2 - \alpha_1 = [\alpha_2] - 1 + 1 - \alpha$, hence $p = 1 < \alpha_2 - \alpha_1;\ q = [\alpha_2] - [\alpha_1] = 2;\ \beta_1 = 2 - \alpha - 1 = 1 - \alpha;\ \beta_0 = 2 - 0 - 1 = 1$.

The kernel $K(t,\tau)$, and the functions $P(t),\ F_1(t,\mathbf{c})$ and $u(t,\mathbf{c})$ are

$$\begin{aligned} K(t,\tau) &= \int_\tau^t \eta\frac{(s-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\,\mathrm{d}s + \int_\tau^t \omega^2(1+\varepsilon\cos s)\,\mathrm{d}s \\ &= \eta\frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)} + \omega^2\left(t-\tau+\varepsilon\sin t - \varepsilon\sin\tau\right) \\ &= \eta\frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)} + \omega^2\left(t-\tau\right)\left(1+\varepsilon\frac{\sin(t-\tau)/2}{(t-\tau)/2}\cos(t+\tau)/2\right) \\ &= \eta\frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)} + (t-\tau)M(t,\tau), \end{aligned} \tag{4.6}$$

where

$$M(t,\tau) = \omega^2\left(1+\varepsilon\frac{\sin(t-\tau)/2}{(t-\tau)/2}\right)\cos\frac{t+\tau}{2}$$

is a bounded function on $[0,b]$;

$$P(t) = c_0 + c_1 t; \tag{4.7}$$

$$F_1(t,\mathbf{c}) = c_0' + {}_0I_t^1\left(\eta c_1\frac{1}{\Gamma(2-\alpha)}t^{1-\alpha} + \eta c_0\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \omega^2(1+\eta\cos t)(c_0+c_1t)\right); \tag{4.8}$$

$$u(t,\mathbf{c}) = F_1(t,\mathbf{c}) + \int_0^t R(t,\tau,1)F_1(\tau,\mathbf{c})\,\mathrm{d}\tau;$$

$$R(t,\tau,1) = K(t,\tau) + \sum_{n=1}^{\infty}(-1)^n K_n(t,\tau),$$

$$K_n(t,\tau) = \int_{\tau}^{t} K(t,\tau)K_{n-1}(t,\tau)\,\mathrm{d}\tau.$$

Then by B)

$$y(t,\mathbf{c}) = \int_0^t u(x,\mathbf{c})\,\mathrm{d}x + c_0 + c_1 t$$

or

$$y(t,\mathbf{c}) = \int_0^t F_1(x,\mathbf{c})\,\mathrm{d}x + \int_0^t \int_0^x R(x,\tau,1)F_1(\tau,\mathbf{c})\,\mathrm{d}x\,\mathrm{d}\tau + c_0 + c_1 t.$$

We give another form of $y(t,\mathbf{c})$:

$$\int_0^t \int_0^x R(x,\tau,1)F_1(\tau,\mathbf{c})\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t \int_0^x \sum_{n=0}^{\infty}(-1)^n K_n(x,\tau)F_1(\tau,\mathbf{c})\,\mathrm{d}x\,\mathrm{d}\tau,$$

$$K_0(x,\tau) = K(x,\tau). \tag{4.9}$$

Let $L_n(x)$ denotes

$$L_n(x) = \int_0^x K_n(x,\tau)F_1(\tau,\mathbf{c})\,\mathrm{d}\tau,$$

then, by definition of $K_n(t,\tau)$, we have

$$\begin{aligned} L_n(x) &= \int_0^x \int_{\tau}^x K(x,s)K_{n-1}(s,\tau)\,\mathrm{d}s\,F_1(\tau,\mathbf{c})\,\mathrm{d}\tau \\ &= \int_0^x K(x,s)\int_0^s K_{n-1}(s,\tau)F_1(\tau,\mathbf{c})\,\mathrm{d}\tau\,\mathrm{d}s \\ &= \int_0^x K(x,s)L_{n-1}(s)\,\mathrm{d}s. \end{aligned} \tag{4.10}$$

Consequently,

$$\int_0^t \int_0^x R(x,\tau,1)F_1(\tau,\mathbf{c})\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t \sum_{n=0}^{\infty}(-1)^n L_n(x)\,\mathrm{d}x$$

and we have an analytical form for $y(t,\mathbf{c})$, which is preferable for calculating,

$$y(t,\mathbf{c}) = \int_0^t F_1(x,\mathbf{c})\,\mathrm{d}x + c_0 + c_1 t + \int_0^t \sum_{n=0}^{\infty}(-1)^n L_n(x)\,\mathrm{d}x. \tag{4.11}$$

The initial condition we get by (4.11)

$$y(0^+,\mathbf{c}) = c_0, \quad y^{(1)}(0^+,\mathbf{c}) = c_0' + c_1.$$

unfortunately, $y^{(2)}(0^+,\mathbf{c})$ does not exist and one of two constants in $c_0' + c_1$ remain undefined. Thus, c_0' can be arbitrary and, $y(t,\mathbf{c})$ is defined by c_0 and c_1.

4.2. *Some properties of the solutions to equation* (4.5)

1. Matieu's equation (4.1) and Damped Mathieu's equation (4.2) have been extensively elaborated especially in relation to the periodicity of solutions (cf. [2]) and [6]). Solutions (4.5) with a damper $\eta_0 D_t^{\alpha} y$ unfortunately loose periodicity.

2. We can give a new result on equation (4.5). Namely an asymptotic development of the function $y(t,\mathbf{c}) - \int_0^t F_1(x,\mathbf{c})\,\mathrm{d}x - c_0 - c_1 t$ at zero $(t \to 0^+,\ t \in (0,t_0),\ t_0 < 1)$, with respect to an asymptotic scale (for the asymptotic development see for example [7] and [5]).

We start with the asymptotic expansion. The sequence of functions

$$\psi_n(t) = \frac{t^{(n+1)(2-\alpha)+1}}{\Gamma((n+1)(2-\alpha)+1)}, \quad n \in \mathbb{N}, \tag{4.12}$$

is an asymptotic expansion for $t \to 0^+,\ 0 < t \le t_0$, for a fixed $t_0 < 1$. Namely, there exists a fixed $t_0 < 1$ such that $\psi_{n+1}(t) = o\left(\psi_n(t)\right),\ t \to 0^+,\ 0 < t < t_0$. This follows form

$$\lim_{t\to 0^+}\left(\frac{t^{(n+2)(2-\alpha)+1}}{\Gamma((n+2)(2-\alpha)+2)}\Big/\frac{t^{(n+1)(2-\alpha)+1}}{\Gamma((n+1)(2-\alpha)+2)}\right) \to 0,$$

for every $n \in \mathbb{N}$. Hence we have the asymptotic scale (4.12).

We will show that the series

$$\sum_{n=0}^{\infty}(-1)^n \int_0^t L_n(x)\,\mathrm{d}x,$$

is the the asymptotic expansion that we are looking for.

By (4.6) and (4.8) we can always find a t_0, $0 < t_0 < 1$, and also M_1 and N such that

$$|K(t,\tau)| \le M_1 \frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)} \quad \text{and} \quad |F_1(\tau)| \le N, \quad 0 \le \tau \le t, \quad 0 < t \le t_0.$$

Using (4.10) and (4.11) we have

$$|L_0(t)| \le NM_1 \frac{t^{2-\alpha}}{\Gamma(3-\alpha)}, \quad 0 < t \le t_0.$$

By the well-known formula

$$\int_0^t \frac{(t-\tau)^{\mu-1}}{\Gamma(\mu)} \frac{\tau^{\nu-1}}{\Gamma(\nu)}\,\mathrm{d}\tau = \frac{t^{\mu+\nu-1}}{\Gamma(\mu+\nu)}, \quad t > 0; \quad \mu, \nu > 0,$$

and by (4.11) we have successively:

$$|L_n(t)| \le NM_1^{n+1} \frac{t^{(n+1)(2-\alpha)}}{\Gamma((n+1)(2-\alpha)+1)}. \tag{4.13}$$

With (4.13) the series $\sum\limits_{n=0}^{\infty}(-1)^n L_n(x)$ converges uniformly on $[0, t_0]$ and we have:

$$\int_0^t \sum_{n=0}^{\infty}(-1)^n L_n(x)\,\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n \int_0^t L_n(x)\,\mathrm{d}x, \quad 0 \le t \le t_0.$$

By [5], p. 33, we have only to prove that

$$\begin{aligned} \sum_{\nu=0}^{\infty}(-1)^n \int_0^t L_\nu(x)\,\mathrm{d}x \quad &= \quad \sum_{\nu=0}^{n-1}(-1)^\nu \int_0^t L_\nu(x)\,\mathrm{d}x \\ &+ \quad O\left(\frac{t^{(n+2)(2-\alpha)+1}}{\Gamma((n+2)(2-\alpha)+2)}\right), \quad n \in \mathbb{N}. \end{aligned}$$

This follows by (4.13) and

$$\left|\sum_{\nu=n}^{\infty}(-1)^{\nu}\int\limits_{0}^{t}L_{\nu}(x)\,\mathrm{d}x\right| \le NM_1\frac{t^{(n+1)(2-\alpha)+1}}{\Gamma((n+1)(2-\alpha)+2)}\sum_{k=0}^{\infty}\left(\frac{t_0}{k_0}\right)^k, \quad k_0\in\mathbb{N},$$

where k_0 is fixed.

5. *Conclusion*

We presented solution to linear fractional differential equation with variable coefficients. As a special case solution to generalized Mathieu's equation arising in stability problems with periodic excitation, is presented.

For the case of variable coefficients, we presented solution in two cases:

1. $[\alpha_m-\alpha_{m-1}]=p$, where $p\le\alpha_m-\alpha_{m-1}<p+1$, $p\in\mathbb{N}$;
2. $\alpha_m=[\alpha_m]$.

The solution is obtained in two steps. In the first step we constructed the function ${}_0I_t^{1-\gamma_m}y$ by using the solution of the Voltera integral equation of the second kind and then, in the second step, by using the Abel integral equation we found the analytical form of y. In such a way we constructed solutions to (1.1) which are continous on $[0,b]$ having some derivatives which are not finite at $t=0$.

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