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QUASICONFORMAL MAPS AND TEICMÜLLER THEORY - EXTREMAL MAPPINGS, OVERVIEW

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A b s t r a c t. In the first part we give a short review of Teicmüller theory. In the second part, we study the conditions under which unique extremality of quasiconformal mappings occurs, and we provide a broader point of view of the phenomenon of unique extremality. Furthermore, we make some contributions to what we refer to as the Teichmüller research question. In particular we report a positive answer to this question if μ *is uniquely extremal on a domain* G *and the lower oscillation of* μ *is less than* L^∞*- norm of* μ *expect on a discrete set in* G*. Additional information is obtained by means of specialized constructions. In particular we review some results from [Ma8], in which we generalize the construction theorem in [BLMM], thus providing a more basic understanding of it. We also announce some results related extremal mappings in 3 dimensions.*

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1. *Introduction*

We plan to write series of papers related to the most recent developments in theory of planar quasiconformal mappings the Teichmüller theory. Lars Ahlfors's Lectures on Quasiconformal Mappings [Ah] is the basic literature in the subject. These lectures develop the theory of quasiconformal mappings from the beginning on only 146 pages, and give a self-contained treatment of the Beltrami equation, and cover the basic properties of Teichmüller spaces, including the Bers embedding and introduction of complex structure on the Teichmüller spaces.

Besides these lovely lecture notes for foundations of the theories of quasiconformal mappings and Teichmüller spaces we also highly recommend books [LV, Leh].

In four decades since the publication of Lars Ahlfors's book [Ah] and the classical text of Olli Lehto and Kalle Virtanen [LV], there have been important developments.

The new edition [Ah1] of Ahlfors's book includes three new chapters. The first, by the title A supplement to Ahlfors's Lectures, written by Earle and Kra, describes further developments in the theory of Teichmüller spaces and provides many references to the numerous literature on Teichmüller spaces and quasiconformal mappings.

We refer to this chapter as EK-supplement to Ahlfors's Lectures, which is particularly relevant to our article. The second, by Shishikura, describes the role of quasiconformal mappings in the subject of complex dynamics. The third, by Hubbard, illustrates the role of these mappings in Thurston's theory of hyperbolic structures on 3-manifolds. Together, these three new chapters show that the theory of quasiconformal mappings is presence in current important trends of research. This article also summarizes further developments in some areas related to Ahlfors's book and there is some overlap with EK-supplement to Ahlfors's Lectures.

Note that Teichmüller's theory is related to complex analysis, hyperbolic geometry, the theory of discrete groups, algebraic geometry, low-dimensional topology, differential geometry, symplectic geometry, dynamical systems,topological quantum field theory, string theory, and many others subject. For further developments we also highly recommend the books [GaLa, FM1, AIM, IT, Leh, Mu].

In order to define the Teichmüller space of a Riemann surface, we introduce some notation.

A closed manifold is a type of topological space, namely a compact manifold without boundary. In contexts where no boundary is possible, any compact manifold is a closed manifold.

A Riemann surface S is a complex manifold of complex dimension one. This means that S is a connected Hausdorff topological space endowed with an atlas: for every point $p \in S$ there is a neighbourhood U containing p homeomorphic to an open subset of the complex plane $\mathbb{C}$ (the unit disk of the complex plane) by a map z defined on U. The map z (we use also notation (U, z)) carrying the structure of the complex plane to the Riemann surface is called a chart. Additionally, the transition maps between two overlapping charts are required to be holomorphic. If $\{U_i\}$, $i \in I$ form an open covering of R, the system $c := \{(U_i, z_i) : i \in I\}$ is called a complex structure for S (It is also said to define a *conformal (analytic) structure* on S). If S has a fixed complex structure, we write (S, c) (or S_c) to denote S with given complex structure. If $f : S \to S$ is a diffeomorphism, define the pullback of c by f: $f^*c := \{(f^{-1}(U_i), z_i \circ f) : i \in I\}$. The group of $Diff^+(S)$ orientation-preserving diffeomorphisms of S acts on the space of all complex structures $\mathcal{C}$ on S (from the right) by pullback: $\mathcal{C} \times Diff^+(S) \to C$, $(c, f) \mapsto f^*c$.

First we consider surfaces of finite type.

A Riemann surface S_0 is of finite type if there is a closed Riemann surface S such that S_0 is contained in S, $S \setminus S_0$ consists of finitely many points and of finitely many disjoint closed parametric discs of S.

Consider a compact (more precisely closed) oriented surface $S = S_{g,p}$ of genus $g \geqslant 0$ from which $p \geqslant 0$ points, so-called punctures, have been deleted. Such a surface is called of finite type (g, p). Homeomorphisms of the surface act in a natural manner on atlases, and two complex structures on $S_{g,p}$ are said to be equivalent if there exists a homeomorphism of the surface which is homotopic to the identity and which sends one structure to the other. The surface $S_{g,p}$ admits infinitely many non-equivalent complex structures, except if this surface is a sphere with at most three punctures.

Let $C_{g,p}$ be the space of all complex structures on $S_{g,p}$ and let $Diff^+(S_{g,p})$ be the group of orientation-preserving diffeomorphisms of $S_{g,p}$. We consider the action of $Diff^+(S_{g,p})$ by pullback on $C_{g,p}$. The quotient space $M_{g,p} = C_{g,p}/Diff^+(S_{g,p})$ is called Riemanns moduli space of deformations of complex structures on $S_{g,p}$.

The Teichmüller space $T_{g,p}$ (we also use notations $T(S)$ and $Teich(S)$) of $S = S_{g,p}$ was introduced in the 1930s by Oswald Teichmüller. It is defined as the quotient of the space $C_{g,p}$ of complex structures by the group $Diff_0^+(S_{g,p})$ of orientation-preserving diffeomorphisms of $S_{g,p}$ that are isotopic to the identity. The group $Diff_0^+(S_{g,p})$ is a normal subgroup of $Diff_+(S_{g,p})$, and the quotient group $\Gamma_{g,p} = Diff_+(S_{g,p})/Diff_0^+(S_{g,p})$ is called the mapping class group of $S_{g,p}$ (sometimes also called the modular group, or the Teichmüller modular group) of $S_{g,p}$ and denote by $mcgS$.

We can use the set $M(R)$ of Riemann metrics on R to reconstruct $Teich(R)$.

Two elements ds^2 and ds_1^2 in $M(R)$ are equivalent if there exists $\omega \in Diff_+(R)$ such that $\omega : (R, ds^2) \to (R, ds_1^2)$ is conformal; and strongly equivalent if this ω belongs to $Diff_0(R)$.

The mapping which send an element $[S, f]$ in $T(R)$ to the equivalence and strong equivalence class, respectively, of a metric corresponding to f give the following identifications:

$T(R) \cong M(R)/Diff_o(R), M_g \cong M(R)/Diff(R)$.

During a remarkably brief period of time (1935- 1941), Teichmüller wrote about thirty papers. These papers laid in the foundations of the theory which now bears his name. After Teichmüller's death in 1943 (at the age of 30), L. V. Ahlfors, L. Bers and several of their students and collaborators developed further Teichmüller's ideas. Roughly speaking, in more than two decades, the whole complex-analytic theory of Teichmüller space was built. In the 1970s, W. P. Thurston opened a new and wide area of research by introducing beautiful techniques of hyperbolic geometry in the study of Teichmüller space and of its asymptotic geometry.

Non-periodic, irreducible class on a closed surface is canonically represented by a pseudo-Anosov automorphism. The precise statement of Thurstons theorem (the classification theorem) is: Every homeomorphism of a compact surface is homotopic to a homeomorphism that either (a) has finite order, (b) is reducible (that is, fixes an essential 1-submanifold), or (c) is pseudo- Anosov. for further developments of this subject and literature we refer to cf [Marg]. Here we quote the following from [Marg] "even given the classification theorem, one thing is not at all obvious: do pseudo-Anosov homeomorphisms exist? If so, how do we construct them? Nielsen knew a few examples of infinite order, irreducible mapping classes, but few enough that he conjectured there were none acting trivially on the homology of the surface".

Thurston's introduction of invariant measure on the stable and an stable laminations and singular foliation associated with the laminations greatly clarified the structure of non-periodic, irreducible automorphisms.

Thurston gave a necessary and sufficient criterion for a surface bundle over the circle to be hyperbolic: the monodromy of the bundle should be pseudo-Anosov . This is part of his celebrated hyperbolization theorem for Haken manifolds.

The bases of the complex analytic theory of Teichmüller space were developed by Ahlfors and Bers. There are several ways of defining the complex structure of Teichmüller space. All of them use deep results from analysis. In Bers' embedding of $T_{g,p}$, the holomorphic cotangent space at a point is identified with the vector space of holomorphic quadratic differentials with simple poles at the punctures on a Riemann surface representing the point. By the theory of quasiconformal mappings, any complex structure on a surface is specified by a Beltrami differential of norm less than one, and this leads to a description of the holomorphic tangent space at a point of Teichmüller space as a vector space of Beltrami differentials of norm less than one divided by a subspace of differentials which induce trivial deformations.

Teichmüller's metric. This metric is obtained by first defining the distance between two conformal structures g and h on the surface $S_{g,p}$ to be $\frac{1}{2}\inf_f logK(f)$, where the infimum is taken over all quasiconformal homeomorphisms $f : (S_{g,p}, g) \to (S_{g,p}, h)$ that are isotopic to the identity and where K(f) is the quasiconformal dilatation of f . Teichmüller showed that the infimum is realized by a quasiconformal homeomorphism, and he gave a description of this homeomorphism in terms of a quadratic differential on the domain conformal surface $(S_{g,p}, g)$. This distance function on the set of conformal structures is invariant by the action of the group of diffeomorphisms isotopic to the identity on each factor, and it induces a distance function on Teichmüller space $S_{g,p}$, which is Teichmüller's metric. Teichmüller's metric is a complete Finsler metric which is not Riemannian unless the surface is a torus, in which case Teichmüller space, equipped with Teichmüller's metric, is isometric to the 2-dimensional hyperbolic plane. Teichmüller's metric is geodesically convex, that is, any two points are joined by a unique geodesic segment. The metric is also uniquely geodesic, that is, the geodesic segment joining two arbitrary points is unique.

There are some variations in the terminology of mapping class groups. Usually, the term extended mapping class group of a surface designates the group of isotopy classes of diffeomorphisms of that surface. The mapping class group is then the group of isotopy classes of orientation-preserving diffeomorphisms of an oriented surface.

The mapping class group of a surface is also the isometry group of the Teichmüller metric and of the Weil - Petersson metric on the Teichmüller space of that surface, and it is conceivable that similar results hold for other metrics on that space. The most enlightening study of the mapping class group is certainly the one that Thurston made through the analysis of the action of that group on the compactification of Teichmüller space by the space of projective classes of measured foliations on the surface.

In this connection, we recall that after Thurston's work was completed, Bers worked out a similar classification of mapping classes which is based on the action of the mapping class group equipped with the Teichmüller metric. Bers obtained this classification by analyzing the minimal set and the displacement function associated to a mapping class acting by isometries.

Teichmüller metric is defined using a solution of an extremal problem. In 1928 Grotzsch considered the following natural extremal problem,at least in the case of rectangles. Because Teichmüller later considered the case of general Riemann surfaces, this problem is sometimes referred to as Teichmüller's extremal problem.

Fix a homeomorphismf $f : X \to Y$ of Riemann surfaces and consider the set of dilatations of quasiconformal homeomorphisms $X \to Y$ in the homotopy class of f.
Question (a). Is the infimum of this set realized ?
Question (b). If so, is the minimizing map unique ?

Teichmüller's theorems (see below) give a positive solution to both questions (under the assumption of negative Euler characteristic). The minimizing map is called the Teichmüller map.

During the last several years, important progress has been made in characterizing the conditions under which unique extremality occurs (see [BMM], [BLMM],[Ma1], [Re9]). In particular, the Characterization Theorem which gives the characterization of unique extremality in functional-analytic fashion by special sequences of integrable holomorphic functions of what we call *Re*-sequences (*Re* being an abbreviation of *Reich*) has found interesting applications. In particular, we give a negative solution to Question (b) in general if Riemann surfaces are not of finite analytic type,[BLMM, Ma8].

In subsection 3.3, we report a positive answer to Teichmüller research question:

Theorem 1.1. *(I) Let G be a $C^{1,\alpha}$- domain, $0 < \alpha < 1$, and χ be uniquely extremal on G. Suppose that $\omega^-(\mu; a) < |\chi\|_\infty$ expect on a discrete set in G. Then χ is of Teichmüller type on G.*
(II) If μ is extremal on G and the lower oscillation of μ on the boundary is less than L^∞- norm of μ on G, then μ is of Teichmüller type on G.

In subsection 3.5, we also announce and outline some results related extremal mappings in 3- dimensions.

We plan to write a few paper related to this remarkable subject and to discuss connections between Teichmüller spaces, physics (especially string theory), complex dynamics, and Thurston's theory of hyperbolic structures on 3-manifolds. This is first one, which is mainly review of old and new results.

For a comperhensive surway of geometric function theory, see the - volume handbook [Ku], edited by R. Kühnau, and for a panorama of some of the most important aspects of Teichmüller theory see [Pa].

2. *Teicmüller theory*

In this section we give a short review of Teicmüller theory. For an expanded version, a review of Teicmüller theory and basic definitions see [Ma6].

2.1. Definitions. By $\mathbb{D}$, $\mathbb{H}$ and $\mathbb{H}^-$ we denote the unit disk, the upper half plane and the lower half plane respectively. Let us consider a Riemann surface $S = \mathbb{D}/G$, where G is of the second kind. We denote by Λ^c the complement of the limit set Λ of G with respect to the unit circle. Then $S \cup \Lambda^c/G$ is a bordered Riemann surface.

Let Riemann surface S has a half-plane as its universal covering surface (we take here $\mathbb{H}^-$).

Beltrami differential μ is a function μ is defined on $\mathbb{H}^-$ which is a Beltrami differential for the covering group Γ of $\mathbb{H}^-$ over S.

Let $Belt_1(\Gamma)$ (we use also notation $B_1(\Gamma)$, $B_1(S)$, $Belt_1(S)$) be the open unit ball in the Banach space $Belt(\Gamma)$ of Beltrami differentials.

The mapping $B = f_\mu \circ A \circ f_\mu^{-1}$ is Möbius transformation for every $A \in \Gamma$. It follows that

$$[f_\mu] = [B \circ f_\mu] = [f_\mu \circ A] = [f_\mu] \circ A\, A'^2$$

on $\mathbb{H}$.

If M, N are Riemann surfaces, and $f : M \to N$ is conformal, then we say M is conformally equivalent to N.

The automorphism group of a Riemann surface is $Aut(M) = \{f : M \to M$ such that f is conformal$\}$.

For example, $Aut(\mathbb{D}) = \{$ Möbius transformations $\mathbb{D} \to \mathbb{D}\}$. In some sense, $Aut(M)$ is usually the identity.

Example 1. *Suppose $f : M \to M$ conformal, where M is a closed Riemann surface of finite genus $\geqslant 2$. Prove that there is an integer $n > 0$ such that f^n =identity.*

If $f : M \to N$ is quasiconformal, then M, N are homeomorphic, but the converse is not true, eg. $M = \mathbb{D} \setminus \{0\}$ and $N = \mathbb{D} \setminus \{|z| \leqslant \frac{1}{2}\}$. However if M is a closed surface and M is homeomorphic to N, then there is a quasiconformal map $f : M \to N$.

Let $f : M \to N$ be quasiconformal, then we say $(N, f) \in Def(M)$, the *deformation space* of M, and a Riemann surface marked by M is a pair $(N, f) \in Def(M)$. More precisely, the *deformation space* $Def(M)$ of a Riemann surface M is the set of pairs (N, f) where N is a Riemann surface, and $f : M \to N$ is quasiconformal.

For every simply connected plane domain Ω different than $\mathbb{C}$, $(\Omega, f) \in Def(\mathbb{D})$ by taking the Riemann map f of $\mathbb{D}$ onto Ω.

The Teichmüller metric is defined by

$$d((f_1, N_1), (f_2, N_2)) = \frac{1}{2} \inf\{\log K(f) : \ f : N_1 \to N_2 \,\text{is homotopic to}\, f_2 \circ f_1^{-1}\}.$$

Remark. We follow Royden and use here the scaling factor 1/2. Ahlfors uses Teichmüller 's original scaling without the scaling factor 1/2.

The Teichmüller space is obtained from the space of all (N, f) by identifying those at distance zero.

Equivalently, following Ahlfors (chapter VI, [Ah1, Ah] we can define an equivalence relation $\sim$ on $Def(M)$ by $(N_1, f_1) \sim (N_2, f_2)$ if and only if $f_2 \circ f_1^{-1} : N_1 \to N_2$ is homotopic to a conformal map $g : N_1 \to N_2$. We call it reduced Teichmüller (RT) or Ahlfors's equivalence relation.

Let S_0 be any hyperbolic Riemann surface, and let $\pi_0 : \mathbb{H} \to S_0$ be a holomorphic universal covering of S_0.

We say that a qc f of M onto itself is *Teichmüller trivial* if it has a lift $\tilde{f} : \mathbb{H} \to \mathbb{H}$ that fixes the extended real axis pointwise.

Remark. We say $(N_1, f_1) \sim (N_2, f_2)$ (in Bers sense or strongly equivalent or modulo the boundary) if there is a conformal mapping g of N_1 onto N_2 such that
(I.1) $f_2 \circ f_1^{-1} : N_1 \to N_2$ is homotopic modulo the boundary to g, or equivalently
(I.2) $f_2^{-1} \circ g \circ f_1$ of M onto itself is Teichmüller trivial.

If $\Gamma = \Gamma(M)$ is of first kind the definitions are the same. When Γ is not of first kind, i.e M is bordered, Ahlfors's equvalence relation produces so-called reduced Teichmüller space $T^\sharp(M)$.

Example 2. *(a) Check that $T^{\#}(\mathbb{D})$ is trivial.*
(b) What is $Teich(S^2)$?

If M has finite conformal (analytic) type, Γ is of first kind.

In what follows, we shall not consider reduced Teichmüller spaces and we use the term "homotopy" to mean "homotopy modulo the boundary".

Recall that if $\Gamma = \Gamma_M = \Gamma(M)$ is of first kind and $f_1, f_2 : M \to N$ are homotopic, then the lifts $\tilde{f}_1, \tilde{f}_2$ on $\mathbb{D}$ once normalized to fix three points of $\partial\mathbb{D}$ will agree on $\partial\mathbb{D}$. This makes sense because the lifts are quasiconformal maps of $\mathbb{D}$ and so extend to homeomorphisms of the boundary of $\mathbb{D}$.

The space of equivalence classes of these pairs (that is, of Riemann surfaces (N, f) marked by M) is the *Teichmüller space* $Teich(M)$; we also use a short notation $T(M)$. Thus the *Teichmüller space* is $T(M) = Def(M)/\sim$, and we use notation $[(N, f)] = \tau \in T(M)$ for a point of Teichmüller space.

We also denote the equivalence class of a qc mapping $f : M \to N$ shortly by $[f]$ or $[\mu]$, where μ is the Beltrami coefficient of f. If M is hyperbolic it is often convenient to identify f by the lift $\tilde{f} : \mathbb{H} \to \mathbb{H}$. Two Beltrami differentials in $Belt(M)$ are equivalent if they induce qc mappings on M whose lifts to $\mathbb{H}$ have the same boundary values.

(M, I) is called the initial point in $T(M)$; term the base point is also used. In the other words, the equivalence class of zero differential is the base point in $T(M)$ and is denoted by $[0]$.

Every $\mu \in Belt(M)$ defines a complex structure on M, and the Teichmüller space $T(M)$ is the space of deformations of a given (basepoint) complex structure on M. The Teichmüller metric measures the distance between different conformal structures.

Example 3. $M = \mathbb{D}$.

For each $(N, f) \in Def(\mathbb{D})$, there is a $(\mathbb{D}, g) \sim (N, f)$ such that $g : \mathbb{D} \to \mathbb{D}$, g fixes 1, -1 and i. For each $[(N, f)]$ there is a 1-1 correspondance with a quasi-symmetric map which fixes 3 points on $\partial\mathbb{D}$.

Thus $T(\mathbb{D}) \cong \{$ normalised quasisymmetric maps of $\partial\mathbb{D}\}$.

Let $QS(\hat{\mathbb{R}})$ be the group of quasi-symmetric maps $g : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$. These are exactly the homeomorphisms of $\hat{\mathbb{R}}$ that arise as boundary values of quasiconformal maps. The case where Γ_M is the trivial group, and $M = \mathbb{H}$ is called universal Teichmüller space. In this case we have $T(\mathbb{H}) \simeq QS(\hat{\mathbb{R}})/PSL_2(\mathbb{H})$.

2.2. Teichmüller metric. For $\tau_1, \tau_2 \in T(M)$ and $\tau_i = [(N_i, f_i)]$ for $i = 1, 2$, the Teichmüller metric is given by

$$d_T(\tau_1, \tau_2) = \frac{1}{2} \inf_{f \sim f_1 \circ f_2^{-1}} (\log K_f).$$

If $f : N_2 \to N_1$ satisfies $\log K_f = d(\tau_1, \tau_2)$, f is an extremal quasiconformal map.

f is uniquely extremal if there is only one extremal map.

To show that the infimum can be attained, and so d_T really is a metric, go to the universal cover, $\mathbb{D}$.

$\widetilde{f}$ is the lift of f, and $\widetilde{f} = \widetilde{f_1 \circ f_2^{-1}}$ on $\partial\mathbb{D}$. $K_f = K_{\widetilde{f}}$, so taking any sequence converging to the infimum, $\inf \log K_{\widetilde{f}} = \lim_{n\to\infty} \log K_{\widetilde{f}_n}$.
Now $\widetilde{f}_n \to \widetilde{f}_0$ uniformly on compact sets (eg. on $\overline{\mathbb{D}}$) by the compactness of quasiconformal maps, so $\lim_{n\to\infty} \log K_{\widetilde{f}_n} = \log K_{\widetilde{f}_0}$.

The study of extremal quasiconformal maps is crucial for the study of the geometry and analytic properties of Teichmüller spaces, and we will see more of this later.

Example 4. *[Annulus ,Torus] (a) The reduced Teichmüller space of an annulus is isomorphic to $\mathbb{R}$ with euclidean metric.*
(b) Teichmüller space of a torus is isomorphic to R with hyperbolic metric. We outline a proof of (b). Let $T_1(z) = z + 1$, $T_\tau(z) = z + \tau$ for $\tau \in \mathbb{H}$, $\Gamma_\tau =< T_1, T_\tau >$ and $R_\tau = \mathbb{C}/\Gamma_\tau$. If R is a torus, then $R = R_\tau = \mathbb{C}/\Gamma_\tau$, $\tau \in \mathbb{H}$. More precisely there is $\tau \in \mathbb{H}$ such that R is conformally equivalent with R_τ. Here τ is not unique, but the lattice group $G_\tau = \{m+n\tau : m, n \in \mathbb{Z}\}$ is well defined. Let $i = \tau_0$, R_{τ_0} is a square torus. For $(N, f) \in Def(R_{\tau_0})$, there is $a \in \mathbb{H}$ and a conformal mapping $A : N \to R_a$. Set $g = A \circ f$. We can choose A such that that the lift $\tilde{g}$ of g is normalized by $\tilde{g}(0) = 0$ and $\tilde{g}(1) = 1$. Set $\tau = \tilde{g}(i)$. It is readable there is a unique $\tau \in \mathbb{H}$ such that $(N, f) \sim (R_\tau, g)$, the lift $\tilde{g}$ of g is normalized and $\tilde{g}(i) = \tau$. Set $T = T(R_{\tau_0})$.

So we have one-to-one mapping of T onto $\mathbb{H}$ ($[(N, f)] \leftrightarrow \tau \in \mathbb{H}$) and we will see that $(T, d_T) \cong (\mathbb{H}, d_{hyp})$. We outline a proof that $d_T = d_{\mathbb{H}}$ after this identification. To see that the Teichmüller metric d_T is the hyperbolic metric $d_{\mathbb{H}}$, take $\tau_1, \tau_2 \in \mathbb{H}$, and note there is an affine map $A : M_{\tau_1} \to M_{\tau_2}$ for $\tau_1 \neq \tau_2$ with the normalization $A_0(0) = 0$, which has dilatation $\ln K(A) = d_{\mathbb{H}}(\tau_1, \tau_2)$. A_0 conjugates the action of the groups, and it is not homotopic to a conformal map.

To see that A_0 is extremal, we consider the distortion of modulus of the corresponding family of curves.

Let $p, q \in \mathbb{Z}$ and $l(p, q)$ be the projection of segment $[0, p + q\tau]$ on R_τ and $\Gamma(p, q)$ the family of paths on the tori homotopic with $l(p, q)$. Verify that $M(\Gamma(p, q)) = \frac{s\tau_2}{\tau_2^2+(\tau_1+s)^2}$, where $\tau = \tau_1 + i\tau_2$, $s = p/q$ and in particular $Mod(\Gamma(1, 0)) = \mathrm{Im}\tau$.

Note here that (T, d_T) is not compact but it is a complete metric space.

If $g : M \to N$ is quasiconformal, take $(S, f) \in Def(N)$ and map it to $(S, f \circ g) \in Def(M)$. Thus we have an induced map I_g from $T(N)$ to $T(M)$. I_g and $I_{g^{-1}}$ are isometries of the corresponding Teichmüller metrics. (For M, N both tori, then I_g must be a conformal Möbius transformation of $\mathbb{H}$.)

If M is a Riemann surface, define $L^\infty(M) = \{\mu \frac{d\bar{z}}{dz} : ||\mu||_\infty < \infty\}$, and let $B_1(M)$ be the unit ball in $L^\infty(M)$. If $M = \mathbb{D}/\Gamma$, we identify μ with $\tilde{\mu} \in L^\infty(\mathbb{D})$, such that $(\tilde{\mu} \circ A)\frac{\overline{A}'}{A'} = \tilde{\mu}$ for all $A \in \Gamma$.

We leave to the reader to explain the notation $L^\infty(\Gamma)$ and $B(\Gamma)$.

Example 5. *[4-point interpolant, [LKF]]*

In section 2, they present the key ingredient of this paper: the 4-point interpolant (FPI) formula. Our goal is to answer the following question: given an ordered set of four source points $Z = \{z_1, z_2, z_3, z_4)\} \subset \mathbb{C}$, where $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}^2\}$ denotes the complex plane, and four target points $W = \{w_1, w_2, w_3, w_4)\} \subset \mathbb{C}$, what is the "most conformal" way to interpolate these points with a bijective map of the plane? Although finding optimal quasiconformal map is in general a very hard task, it turns out, surprisingly, that a closed-form solution to this problem can be devised for the particular case of 4 interpolation points.

The solution is given in terms of a very simple formula, defined as composition of two Mäbius transformations m_1, m_2 and an affine mapping A: $f(z) = m_2 \circ A \circ m_1(z)$. We will refer to this formula as the 4-Point Interpolant (FPI).

Recall, if M is a hyperbolic surface, we can extend lift of μ by symmetry to the lower half plane to define f^μ.

If $M = \mathbb{H}/\Gamma$, we consider a Beltrami differential μ on M, or what is the same, a function μ defined on $\mathbb{H}$ which is a Beltrami differential for the covering group Γ of $\mathbb{H}$ over M; we denote by f^μ quasiconformal self -mapping of $\mathbb{H}$ which has the complex dilatation μ and fixes the points 0, 1 and ∞, and by f_μ quasiconformal self -mapping of the plane which has the complex dilatation μ in the upper half-plane, is conformal in the lower half-plane and fixes the points 0, 1 and ∞.

For $\mu \in B_1(M)$ there is a quasiconformal mapping of M whose complex dilatation is μ, and by the uniqness part all such mapping define the same point in $T(M)$. $f^\mu : M \to N$ has $Belt(f^\mu) = \mu$.

For $\nu \in B_1(M)$, we write $\mu \sim \nu$ and $\mu \sim_{rt} \nu$ if $(N_\mu, f^\mu) \sim (N_\nu, f^\nu)$ are strongly equivalent and reduced equivalent, respectively.

Using the mapping $\mu \mapsto (N, f^\mu) \in Def(M)$, we define

$(B(M)/\sim) \to (Def(M)/\sim) = T(M)$. So $T(M) = \{[\mu] : \mu \in B_1(M)\}$.

If $M = \mathbb{H}/\Gamma$, $\mu, \nu \in B_1(\Gamma)$, then $\mu \sim \nu \Leftrightarrow f^\mu = f^\nu$ on $\overline{\mathbb{R}}$.

We call $T(\mathbb{H})$ is the *universal Teichmüller space*. Note also, if $\mu_n \to \mu$ in $L^\infty(\Gamma)$ and $\mu_n \in [\mu]$, then $f^{\mu_n} \to f^\mu$ since $\mu_n \to \mu$ pointwise.

Suppose that $M = \mathbb{H}/\Gamma$, and $\mu, \nu \in B_1(\Gamma)$.

Lemma 2.1. *$\mu \sim \nu \Leftrightarrow f^\mu = f^\nu$ on $\mathbb{R}$.*

Lemma 2.2. *$\mu \sim \nu \Leftrightarrow f_\mu = f_\nu$ on $\mathbb{H}^- \cup \mathbb{R}$.*

In the literature, authors often make assumption that $\Gamma = \Gamma(M)$ is of first kind. Under this condition we can prove that previous lemmas are valid for reduced equivalent dilatations, $\mu \sim_{rt} \nu$.

P r o o f. $f^\mu = f^\nu$ on $\mathbb{R}$ if and only if $f_\mu = f_\nu$ on $\mathbb{H}^- \cup \mathbb{R}$. $f^\mu = f^\nu \Rightarrow f_\mu = f_\nu$ on $\mathbb{H}^-$. Define $h = (f^\nu)^{-1} \circ f^\mu$ on $\mathbb{H}$ and $h = z$ on $\mathbb{H}^-$.
$A = f_\nu \circ h \circ (f_\mu)^{-1}$ implies A is conformal on $\mathbb{C}$, hence is a Möbius transformation and is thus the identity by the normalisation condition. Thus $f_\nu = f_\mu$ on $\mathbb{H}^-$.

□

Recall that the integral of 2-form over a parametric disc in a surface (a 2-dimensional manifold) is invarinatly defined and $(1,1)$-differentials on Riemann surfaces are in fact 2-forms. Set $\varpi = \rho^{-2}$, where $\rho = \rho_{hyp}$. It is readable that ϖ is $(-1,-1)$ -differential, and if φ is quadratic differential, that is $(2,0)$ -differential, then $|\varphi|$ $(1,1)$ -differential. Since $(-1,-1)+(1,1) = (0,0)$, $\varpi|\varphi|$ is a function ($(0,0)$ -differential).

The *Bers space*, $Q(\Gamma) = L^\infty Q(\rho^{-2};\Gamma) = Q^\infty(\varpi;\Gamma)$, is the space of $(2,0)$ holomorphic differentials on $M = \mathbb{H}/\Gamma$ with the norm $|\varphi|_Q = ||\varphi||_{\varpi,\infty} = ||\varphi||_\infty = \sup_{z\in\mathbb{H}} |\rho^{-2}(z)\varphi(z)|$, where $\rho = y^{-1}$ is the hyperbolic density (sometimes we also say metric) on $\mathbb{H}$ (we write $\varphi \in Q(\Gamma)$, $\varphi = \varphi\, dz^2$); thus $\varphi \in Q(\Gamma)$ if φ holomorphic, $(\varphi \circ A)(A')^2 = \varphi$ for $A \in \Gamma$ and $||\varphi||_\infty = \sup_{z\in\mathbb{H}} |y^2\varphi(z)|$.

In Lehto [Leh] the scaling is used such the norm is $4\sup_{z\in\mathbb{H}} |\rho^{-2}(z)\varphi(z)|$.

If φ quadratic differential ($(2,-0)$ -differential), then $|\varphi|^2$ is $(2,2)$ -differential and since $(-1,-1) + (2,2) = (1,1)$, $\varpi|\varphi|^2$ is a form ($(1,1)$ -differential).

We summarize $\varpi|\varphi|$ is a function, $\varpi|\varphi|^2$ and $|\varphi|$ are $(1,1)$ -differential is (2- forms).

$A^2 = A^2(\varpi) = A^2(\varpi;\Gamma)$ and $A^1 = A^1(\Gamma)$ are the spaces of $(2,0)$ holomorphic differentials on $M = \mathbb{H}/\Gamma$ with norm $||\varphi||_{\varpi,2} = \int_M \varpi|\varphi|^2$ and $||\varphi|| = \int_M |\varphi|$, respectively.

Example 6. *a) If M is closed, φ a $(2,0)$ holomorphic form, then $||\varphi||_\infty < \infty$.*
b) Let $p_n(z)z^{-n}$, $n \in \mathbb{Z}$ and $\varphi = p_2$. Check that $p_n \in Q(\mathbb{H}^-$ if and only if $n = 2$.
Since $|y| \leqslant |z|$, $\varphi(z) = z^{-2} \in Q(\mathbb{H}^-)$ and $|\varphi|_Q = 1$.

Let S be a closed (compact without boundary) Riemann surface of genus $g > 1$. By $Q(S)$ we denote the space of holomorphic quadratic differentials on S.

For the following facts see [Leh].

If f_1 and f_2 are the Teicmüller mappings with complex dilations $z_1\varphi/|\varphi|$ and $z_2\varphi/|\varphi|$, then the composition $f_2 \circ f_1^{-1}$ is also a Teicmüller mapping.

Since $Q(S)$ is $(3g-3)$-dimensional linear space over the complex numbers, we can fix a base $\varphi_1, \varphi_2, \cdots \varphi_{3g-3}$ in $Q(S)$. Then every $\varphi \in Q(S)$ has a representation $\varphi = \sum_{i=1}^{3g-3} z_i\varphi_i$, $z_i \in \mathbb{C}$, and φ can be identified with the point $(z_1, z_2, \cdots, z_{3g-3}) \in \mathbb{C}^{3g-3}$ or with the point $(x_1, \cdots, x_{3p-3}, y_1, \cdots, y_{3p-3})$ of the euclidean space $\mathbb{R}^{6g-6}$.

Every $p \in Teich(S)$ has the unique representation $p = [t\overline{\varphi}/|\varphi|]$, $||\varphi|| = 1$.

The mapping $[t\overline{\varphi}/|\varphi|] \mapsto (tx_1, \cdots, tx_{3p-3}, ty_1, \cdots, ty_{3p-3})$, where $\varphi \in Q(S)$, $|\varphi|_e = 1$, and $0 \leqslant t < 1$, is a homeomorphisam of T_S onto the open unit ball of the euclidean space $\mathbb{R}^{6p-6}$. We call it "Teicmüller imbedding".

However, the complex structure that the Teicmüller space inherit from $\mathbb{C}^{3g-3}$ through this imbedding is not a natural one. For two Riemann surface S and S' of genus g, the bijective isometry between $Teich(S)$ and $Teich(S')$ induced by a quasiconformal mapping of S onto S' is not usally biholomorphic with respect to these structures. The Bers imbedding introduced complex structure into an arbitrary Teicmüller space.

The Teichmüller space of S is a complex manifold. Its complex dimension depends on topological properties of S. If S is obtained from a compact surface of genus g by removing n points, then the dimension of TS is $3g-3+n$ whenever this number is positive. These are the cases of "finite type". In these cases, it is homeomorphic to a complex vector space of this dimension, and in particular is contractible.

Dauddy and Earle showed that every Teichmüller space is contractible, cf. [Leh]. For a fixed $\varphi \in Q(S)$, the mapping $t \mapsto [t\overline{\varphi}/|\varphi|]$ of $\mathbb{D}$ into the

Teichmüller space $Teich(S)$ is holomorphic.
The set $\Delta = \Delta_\varphi = \{[t\overline{\varphi}/|\varphi|]; t \in \mathbb{D}\}$ is called the Teicmüller disk induced by φ.
The mapping $t \mapsto [t\overline{\varphi}/|\varphi|]$ is an isometry of the hyperbolic unit disk $\mathbb{D}$ onto the Teicmüller disk Δ_φ.

As before, S is a compact Riemann surface of genius > 1. Let $f : S \to S'$ be a Teichmüller mapping determined by a pair (φ, k). Then there is a unique quadratic differential ψ on S' such that: 1^0 If φ has a zero of order $n \geqslant 0$ at p, then ψ has a zero of the same order at $f(p)$. 2^0 If ζ and ζ' are natural parameters of φ and ψ respectively at regular points p and $f(p)$, then $\zeta' \circ f = \frac{\zeta + k\bar{\zeta}}{1-k}$ in a neighborhood of p.

We call φ its initial and ψ its terminal differential. Let $f : S \to S'$ be a Teichmüller mapping determined by a pair (φ, k). Then the $f^{-1} : S' \to S$ is a Teichmüller mapping determined by a pair $(-\psi, k)$, where ψ is the terminal differential of f. Then $\zeta \circ f^{-1} = \frac{\zeta' - k\bar{\zeta}'}{1-k}$ in a neighborhood of a regular point of ψ and the terminal differential of f^{-1} is $-K^2\varphi$.

Let $f_1 : S \to S_1$ and $f_2 : S \to S_2$ be a Teichmüller mappings determined by pairs (φ_1, k_1) and (φ_2, k_2) such that φ_2/φ_1 is a constant. Then $f_2 \circ f^{-1}$ is a Teichmüller mapping. Up to constants, the initial differential of $f_2 \circ f^{-1}$ agrees with that of f^{-1}, and the terminal differential with that of f_2.

2.3. Uniquely extremal and geodesic. By definition, a geodesic segment in T_S is the image of an injective continuous map f from an interval $[a, b]$ into T_S such that

$$d_T(f(x), f(y)) = d_T(f(x), f(z)) + d_T(f(z), f(y))$$

whenever $a \leqslant x \leqslant z \leqslant y \leqslant b$.

There always at least one geodesic segment joining two points of T_S. We can apply a geometric automorphism and assume τ_2 is the base point. If μ is an extremal Beltrami coefficient representing τ_1, then the image of the map $f : [0, |\mu|_\infty)] \to T_S$ given by

$$f(t) = [t\frac{\mu}{|\mu|_\infty)}] \tag{1}$$

is geodesic segment if S of finite analytic type and μ has the special form $\mu = k|\varphi|/\varphi$, for some φ in $A(S)$. When S is of infinite analytic type, the situation is more complicated because there are extremal Beltrami coefficients that are not uniquely extremal. Therefore, we focus on infinite dimensional Teicmüller spaces.

Suppose μ is extremal in $Belt_1(S)$. In [EKK], it is proved that the following are equivalent are equivalent.

(a) μ is uniquely extremal and $|\mu| = |\mu|_\infty$ a.e

(b) there is only one geodesic segment joining the base point and $[\mu]$ in T,

(c) there is only one holomorphic isometry $\phi : \mathbb{D} \to T_S$ such that $\phi(0) = [0]$ and $\phi(|\mu|_\infty) = [\mu]$, and

(c) there is only one holomorphic map $\omega : \mathbb{D} \to B_1(S)$ such that $\omega(0) = [0]$ and $\omega(|\mu|_\infty) \in [\mu]$ in T.

In [EKK], it is proved that the following are equivalent are equivalent.

(a) μ is uniquely extremal and $|\mu| = |\mu|_\infty$ a.e

(b) there is only one geodesic segment joining the base point and $[\mu]$ in T

Theorem 2.1. *[Earle, Kra, Krushkal [EKK]]*
Let $f : \mathbb{D} \to T(M)$ be holomorphic, then there exists a lift $\widetilde{f} : \mathbb{D} \to B(M)$ such that $f = \pi \circ \widetilde{f}$.

Let Ω be a plane domain and denote by Λ its complement in the extended complex plane.

By a result of Slodkowski, the extended λ-lemma, any holomorphic motion can be extended to holomorphic motion of of the extended complex plane. This extension is not nenecessarily unique.

For any μ in $Belt_1(\Omega)$, we can define holomorphic motion of Λ: let h_t be the restriction to Λ of a qc homeomorphism of the extended complex plane which fixes $0, 1$, and ∞, and has Beltrami coefficient $t(\mu/|\mu|_\infty)\,\chi_\Omega$. We say that h_t so defined is a canonical holomorphic motion of Λ generated by μ.

Example 7. *[[BLMM]] Let Ω be a plane domain and denote by Λ its complement in the extended complex plane.*

1. *Any holomorphic motion of Λ has an extension to a holomorphic motion of the extended complex plane. This extension is not necessarily unique.*

   For instance, the identity mappings and the vertical stretchings are two different extensions of the holomorphic motion $h_t(z) = z$ of the complement of the upper half plane.

2. *Let Ω be a plane domain and denote by Λ its complement in the extended complex plane. Suppose that μ is uniquely extremal in $Teich(\Omega)$ and that h_t is a canonical holomorphic motion of Λ induced by μ. Then h_t has a unique extension to a holomorphic motion of the extended complex plane.*

3. *If $\Lambda = \{z : x \leqslant |y|^\alpha\}$, with $\alpha \geqslant 3$, then the affine motion*

$$h_t(z) = \frac{1+t}{1-t}x + iy \tag{2}$$

of Λ has a unique extension to a holomorphic motion of the extended complex plane.

2.4. Kobayashi metric. For any complex manifold X and Y, let $\mathcal{O}(X,Y)$ be the set of biholomorphic maps of X into Y. A pseudometric d on X is called Schwarz-Pick metric if

$$d(f(z), f(z')) \leqslant d_{\mathbb{D}}(z, z')$$

for all f in $\mathcal{O}(\mathbb{D}, X)$ and all z and z' in $\mathbb{D}$.

By definition, the Kobayashi metric k_X is the largest Schwarz-Pick metric on X.

For any complex manifold X and Y, it is obvious that

$$k_Y(f(z), f(z')) \leqslant k_X(z, z')$$

for all f in $\mathcal{O}(X,Y)$ and all z and z' in X.

In particular, biholomorphic maps preserve Kobayashi distances.

Let X be a complex Banach manifold. We denote by $d_K^X = d_K = \sigma_X = \sigma$ Kobayashi distance on X.

If X is a complex manifold, the Kobayashi pseudometric d_K may be characterized as the largest pseudometric on X such that

$$d(f(x), f(y)) \leqslant \rho(x, y),$$

for all holomorphic maps f from the unit disk $\mathbb{D}$ to X (where $\rho(x,y) = d_{hyp}$ denotes distance in the Poincaré metric on $\mathbb{D}$).

In order to give another definition of Kobayashi distance we recall. The inverse hyperbolic tangent is a multivalued function and hence requires a branch cut in the complex plane, which Mathematica's convention places at the line segments $(-\infty, -1]$ and $[1, \infty)$. This follows from the definition of

$tanh^{-1}z$ as $tanh^{-1}z = 1/2[ln(1+z) - ln(1-z)]$. For real $x < 1$, this simplifies to $tanh^{-1}(x) = \frac{1}{2}ln((1+x)/(1-x))$. The original Kobayashi metric is a pseudometric (or pseudodistance) on complex manifolds introduced by Kobayashi (1967). It can be viewed as the dual of the Carathéodory metric, and has been extended to complex analytic spaces and almost complex manifolds. On Teichmüller space the Kobayashi metric coincides with the Teichmüller metric; on the unit ball, it coincides with the Bergman metric.

Let p and q be points in X and $d^1_{kob,X}(p,q) = \tanh^{-1}(r)$, where r is the infinum of nonnegative number s for which there exists $f \in O(\mathbb{D}, X)$ with $f(0) = p$ and $f(s) = p$. If it is clear from the context we write shortly d_1 instead of $d^1_{kob,X}$. Set $d_n(p,q) = \inf \sum_{k=1}^{n} d_1(p_{k-1}, p_k)$, where the infinum is taken over all points $p_0, \cdots, p_n$ in X for which $p_0 = p$ and $p_n = q$.

it is clear $d_{n+1} \leqslant d_n$ and define $d_K = \lim d_n$. If d_1 satisfies the triangle inequality, then $d_1 = d_n$ for all n and so $d_1 = d_K$.

Note that for all Teichmüller spaces with complex structure which are modeled on a Fuchsian group, $d_1 = d_K = d_T$, where d_T is Teichmüller metric.

Let B be the unit ball in a complex Banach manifold and $v \in B$. Then $d^B_K(0,v) = tanh^{-1}(|v|)$.

The linear functional $l(t) = l_v(t) = tv/|v|$ maps the unit disk $\mathbb{D}$ into the unit ball M, $|v|$ into v, and 0 into 0. Therefore $d^M_K(v,0) \leqslant d_{hyp,\mathbb{D}}(0,|v|)$. By the Hahn-Banach, there exists a continuous linear functional L on B such that $L(v) = |v|$ and $|L| = 1$ so $d_{hyp,\mathbb{D}}(0,|v|) \leqslant d^M_K(v,0)$.

Now consider $M = B_1(R)$.

Proposition 2.1. *Let R be a Riemann surface and $M = Belt(R) = B_1(R)$. Then $d^1_{Kob,M}(\mu,\nu) = tanh^{-1}(|\frac{\mu-\nu}{1-\overline{\nu}\mu}|_\infty)$, $\mu,\nu \in M$.*

Set $d_{B_1(R)}(\mu,\nu) = \tanh^{-1}(|\frac{\mu-\nu}{1-\overline{\nu}\mu}|_\infty)$. Using the Schwarz lemma, one can show that $d_1(0,\nu) = d_M(o,\nu)$. Since $\mu \mapsto \hat{\alpha}_\nu(\mu) = \frac{\mu-\nu}{1-\overline{\nu}\mu}$ is a biholomorphic self-map of M, we have

$d^1_{Kob,M}(\mu,\nu) = d_M(\mu,\nu)$.

Since Teichmüller metric is the quotient metric,we will show that formula yields $d_K \leqslant d_T$.

Let $p, q \in T(R)$ and let $\mu_0 \in p$, $\nu_0 \in q$ be extremal representatives. Then $d_T(p,q) \leqslant \inf\{d_M(\mu,\nu) : \mu \in p, \nu \in q\} = d_M(\mu_0,\nu_0)$.

Since $\pi : M \to T(R)$ is holomorphic and $p = \pi(\mu_0)$, $q = \pi(\nu_0)$, then

$d_1(p,q) = d_1(\pi(\mu_0), \pi(\nu_0)) \leqslant d_1(\mu_0, \nu_0)$. By Proposition 2.1, $d_1(\mu_0, \nu_0) = d_M(\mu_0, \nu_0) = d_T(p,q)$. Hence $d_K \leqslant d_1 \leqslant d_T$.

Theorem 2.2. *[Royden-Gardiner] For any hyperbolic Riemann surface S, the Kobayashi and the Teichmüller metrics coincide on $T(S)$.*

We show that $d_T = d_1$.

If R and S are hyperbolic Riemann surfaces, then every biholomorphic map of $T(R)$ onto $T(S)$ preserves Teichmüller distances.

In [EKK], the equivariant version of Slodkowski extension theorem is used to prove: Let $f : \mathbb{D} \to T(M)$ be holomorphic, then there exists a lift $\widetilde{f} : \mathbb{D} \to B(M)$ such that $f = \pi \circ \widetilde{f}$.

For the opposite inequality , take $f \in O(\mathbb{D}, T(R))$ such $f(0) = p$ and $f(t) = q$ for some $t \in \mathbb{D}$. Then there exists a lift $g : \mathbb{D} \to B(M)$ such that $f = \pi \circ g$. Since $\pi : M \to T(R)$ is holomorphic,

$d_T(p,q) = d_T(\pi(g(0), \pi(g(t)) \leqslant d_M(g(0, g(t)) \leqslant d_{hyp,\mathbb{D}}(0,t)$. Taking the infinum over all such f, we obtain $d_1 \geqslant d_T$.

Teichmüller metric is the quotient metric on $T(M)$ induced by the Kobayashi metric on $B_1(\Gamma)$ and the Bers map.

2.5. Linear isometries of Teichmüller space. A Riemann surface M is said to be of exceptional type if it has finite conformal type (g,n) $2g + n \leqslant 4$, where g is the genus of M, and n is the number of punctures. All nonhyprbolic Riemann surfaces have exceptional type. It is of *non-exceptional type* if $2g + n > 4$.

It has been proved that for a Riemann surface of non-exceptional type, every biholomorphic automorphism of the Teichmüller space is induced by a quasiconformal automorphism of the Riemann surface.

Theorem A. *Let M be a hyperbolic surface of non exceptional type. Then the space of biholomorphic automorphisms $Aut(T(M))$ coincides with the mapping class group $mcg(M)$ of M.*

The proof is a combination of two theorems. One theorem states that the above statement is true if a Riemann surface has the isometry property, which was proved by Earle and Gardiner [EGa] and is called the automorphism theorem. Another theorem states that every Riemann surface of non-exceptional type has the isometry property, which was finally proved by Markovic.

By Riemann-Roch, it is possible to separate points of M if it is of non-exceptional type.

Let M, N be Riemann surfaces with the corresponding Bergman spaces $A^1(M)$ and $A^1(N)$. If $\alpha : N \to M$ conformal map of N onto M, each $\varphi \in A^1(N)$ can be pulled back to $\alpha^*(\varphi) = (\varphi \circ \alpha)(\alpha')^2$ on $A^1(M)$ and it is an isometry.

A surjective $\mathbb{C}$-linear isometry, where M and N are Riemann surfaces, is called geometric if there exists $c \in \mathbb{C}$ such that $|c| = 1$ and $\alpha : N \to M$ conformal such that $L^{-1}(\varphi) = c\,\alpha^*(\varphi)$.

Theorem 2.3. *[Royden-Lakic-Markovic] Let M and N be Riemann surfaces and let $L : A^1(M) \to A^1(N)$ be a surjective $\mathbb{C}$-linear isometry, where M and N are of non-exceptional type. Then L is geometric.*

Thus there exists $c \in \mathbb{C}$ such that $|c| = 1$ and $\alpha : M \to N$ conformal such that $L^{-1}(\varphi) = c\,\alpha^*(\varphi)$, for every $\varphi \in A^1(N)$. Thus if we set $\psi = L^{-1}(\varphi)$, then $\varphi = L(\psi)$, $\psi \in A^1(M)$ and $\beta^*(\psi) = (\psi \circ \beta)(\beta')^2 = c\,\varphi$, where $\beta : N \to M$ and $\beta =: \alpha^{-1}$. Hence L^{-1} is geometric.

It convenient to use notation $[XX] - [ab]$, the reference $[ab]$ in $[XX]$. Royden proved for M and N compact and hyperbolic. His method extends to Riemann surfaces of non-exceptional finite conformal type. Lakic refined it to prove for surfaces of infinite conformal type with finite genus. Markovic proved for all Riemann surfaces. Royden proved Theorem 2.3 in [FM1]-[16] in the case where M and N are compact and hyperbolic, and his method was extended to Riemann surfaces of non-exceptional finite type, even though M and N are not assumed to be homeomorphic, by Earle and Kra in [FM1]-[3] and Lakic in [FM1]-[12]. Some further special cases of Theorem 3.6 were proved by Matsuzaki in [FM1]-[13]. Markovic proved Theorem 2.3 in full generality, that is, for the infinite analytic type case, in [Mar3]. In [FM1]-[5] (cf also [MaSa]), C. Earle and V. Markovic, use the methods of [Mar3] to prove Theorem 2.3 in the finite analytic case, which gives a good indication of the methods used, without going into the technical detail required for the general case.

In [Fu], Fujikawa reviews the proof by Earle and Gardiner and gives another approach to a proof of the automorphism theorem.

Every Riemann surface of non-exceptional type has the isometry property. Let $E : T(R) \to T(R)$ be a biholomorphic automorphism of $T(R)$. For a point $p = [f_1] \in T(R)$, set $E(p) = [f_2] \in T(R)$. For each $i = 1, 2$, the quasiconformal homeomorphism $f_i : R \to S_i$ induces a geometric automorphism $I_i = \rho_{f_i} : T(R) \to T(S_i)$ that maps $[f] \mapsto [f \circ f_i^{-1}]$. Then $F := I_2 \circ E \circ I_1 : T(S_1) \to T(S_2)$ is a holomorphic isomorphism that preserves the base points.

We fix i and denote S_i by S. The dual space $A^*(S)$ of $A(S)$ is identified with the tangent space $T_0(T(S))$ of $T(S)$ at the base point. Let $\pi_S : Belt(S)_1 \to T(S)$ be the Bers projection. The derivative $\pi'_S(0) : Belt(S) \to T_0(T(S))$ of π_S at the base point is a surjective homomorphism. Thus $T_0(T(S))$ is isomorphic to the quotient space $Belt(S)/Ker(D\pi_S(0))$. On the other hand, let $P : Belt(S) \to A^*(S)$ be the surjective linear map defined by

$$P\mu(\varphi) = \iint \mu\varphi$$

for $\mu \in Belt(S)$ and for $\varphi \in A(S)$. Since $KerP = Ker(D\pi_S(0))$ (see [6]), we see that $A^*(S)$ is isomorphic to $Belt(S)/Ker(\pi'_S(0))$. Then the map $D\pi_S(0) \mapsto P\mu$ gives an isomorphism between $T_0(T(S))$ and $A^*(S)$. The tangent space $T_0(T(S))$ of $T(S)$ at the base point has the Teichmüller norm

$$|\pi'_S(0)\mu| = \lim_{t\to 0} \frac{d_T([t\mu],[0])}{t} .$$

We see that $F'([id])$ is a $\mathbb{C}$-linear isometry of $A^*(S_1)$ onto $A^*(S_2)$. Indeed, since F is biholomorphic, $F'([id])$ is an invertible $\mathbb{C}$-linear map. Furthermore it preserves the Teichmüller norm. This follows from the fact that the biholomorphic isomorphism F preserves the Kobayashi distance, and that the Kobayashi distance coincides with the Teichmüller distance (see [Fu]-[9]). The adjointness proposition states that $DF([id])$ induces a $\mathbb{C}$ -linear isometry of $A(S_2)$ onto $A(S_1)$. Note that $A(S_i)$ is not reflexive if it is infinite dimensional.

By applying the adjointness theorem to our case, there exists a $\mathbb{C}$-linear isometry $G : A(S_2) \to A(S_1)$ such that $F'([id]) = G^*$. Since R has the isometry property, there exist $\theta \in \mathbb{R}$ ($a = e^{i\theta}$ with $|a| = 1$) and a conformal homeomorphism $g : S_1 \to S_2$ such that $G(\varphi) = a \cdot (\varphi \circ g)(g')^2$ for all $\varphi A(S_2)$. Then g induces a geometric isomorphism $\rho_g : T(S_1) \to T(S_2)$, which preserves the base points. Set $\omega_p := (\rho_{f_2})^{-1} \circ \rho_g \circ \rho_{f_1} = \rho_{f_2^{-1}\circ g\circ f_1} : T(R) \to T(R)$, which is an element of $Mod(R)$. Then $\omega_p(p) = E(p)$. To complete a proof, we show that ω_p is independent of p, namely $\omega_p = E$ on $T(R)$.

Proposition 2.2 (the uniqueness theorem). Let R be a Riemann surface having the isometry property, and $H : T(R) \to T(R)$ a holomorphic automorphism satisfying $H(q) = q$ and $H'(q) = a \cdot id \, (a \in \mathbb{C})$ for some $q \in T(R)$. Then H is the identity.

Theorem 2.4. *[Fletcher [Fl, FM]] Suppose that M is a hyperbolic Riemann surfaces of infinite analytic type, then*
$A^1(M)$ is isomorphic to the sequence space l^1 and
$Q(M)$ is isomorphic to the sequence space l^∞.

2.6. Pants decomposition. Let R be a closed R of genus g ($\geqslant 2$). A maximal set $\mathcal{L} = \{L_j\}_{j=1}^n$ of mutually disjoint not freely homotopic simple closed geodesics on R is called s system of decomposing curves, and the family $\mathcal{P} = \{P_k\}_{k=1}^m$ consisting of all connected components of $R \setminus \cup_{j=1}^n L_j$ the pants decomposition corresponding to $\mathcal{L}$. Then $n = 3g-3$, $m = 2g-2$. The usual definition is that a Riemann surface is of finite type if it is conformally equivalent to a compact Riemann surface minus a finite set of points. For instance, under this usual definition, an annulus of finite modulus is not of finite type. Consider a compact oriented surface S of genus $g \geqslant 0$ from which $g \geqslant 0$ points, so-called punctures, have been deleted. Such a surface is called of finite type. We assume that S is non-exceptional, i.e. that $3g-3+m \geqslant 2$; this rules out a sphere with at most four punctures and a torus with at most one puncture.

Here we mean by an essential simple closed curve a simple closed curve which is not contractible nor homotopic into a puncture. Since $3g-3+m$ is the number of curves in a pants decomposition of S, i.e. a maximal collection of disjoint mutually not freely homotopic essential simple closed curves which decompose S into $2g-2+m$ open subsurfaces homeomorphic to a thrice punctured sphere.

Let R be a hyperbolic Riemann surface with hyperbolic metric ds_R^2. Consider a Fuchian model G of R acting on $\mathbb{D}$, and let $\pi : \mathbb{D} \to R$ be the projection of $\mathbb{D}$ onto $R = \mathbb{D}/G$. Since the Poincaré metric ds^2 is invariant under action by G, we obtain a Riemann metric ds_R^2 on R which satisfies $\pi^*(ds_R^2) = ds^2$. We call ds_R^2 the Poincaré metric, or the hyperbolic metric.

In mathematics, a pair of pants is a simple two-dimensional surface resembling a pair of pants: topologically, it is a sphere with three holes in it. Pairs of pants admit hyperbolic metrics, and their isometry class is determined by the lengths of the boundary curves (the cuff lengths), or dually the distances between the boundaries (the seam lengths).

In hyperbolic geometry all three holes are considered equivalent no distinction is made between "legs" and "waist".

Two pairs of pants can be sewn together to form an open surface with four boundary components which we call sewn two pairs of pants.

Six pairs of pants can be sewn together to form an open surface of genus two with four boundary components.

We call a relatively compact subsurfaces P of R a pair of pants of R if P is triple connected and if every connected component of the relative boundary of P in is a simple closed geodesic on R. Let $\tilde{P}$ be a connected component of $\pi^{-1}(P)$ and let $G_{\tilde{P}}$ be the subgroup of G consisting of all elements A of G for which $A(\tilde{P}) = \tilde{P}$. Then $G_{\tilde{P}}$ is a free group generated by two hyperbolic transformations, and $P = \tilde{P}/G_{\tilde{P}}$.

$\hat{P} = \mathbb{D}/G_{\tilde{P}}$ is called Nielsen extension of P and P the Nielsen kernel of $\hat{P}$.

Let L_1, L_2 and L_3 be the boundary components a pair of pants P and ℓ_j the length of L_j.

For any triple (a_1, a_2, a_3) of positive numbers, there exists a pair of pants admitting a reflection J_P such that the hyperbolic lengths of the ordered boundary components are the given triple.

The complex structure of a pair of pants is uniquely determined by the hyperbolic lengths of the ordered boundary components of P.

The set F_P of all fixed points of J_P consists of three geodesic o_1, o_2, o_3. Every o_j has the end points on and it is orthogonal to both L_j and L_{j+1}, where $L_4 = L_1$.

J_P is called the reflection of P. The set F_P has two points on L_j. for every j, let $P_{j,1}$ and $P_{j,2}$ be two pairs of pants having L_j as a boundary component with the reflection J_1 and J_2 respectively. Take a fixed point of J_k on L_j for each $P_{j,k}$ (k=1,2), and denote it by $c_{j,1}$ and $c_{j,2}$. Fix also an orientation on L_j, and let T_j be oriented arc on L_j from $c_{j,1}$ to $c_{j,2}$. We can define the signed hyperbolic length τ_j of T_j and $\theta_j = 2\pi\tau_j/\ell_j$ (so τ_j is positive if the orientation of T_j is compatible with that of L_j and negative if it is not the case).

For every $t \in T_g$ take a marking -preseving homeomorphisam $f_t : R \to R_t$. Let $L_j(t)$ be the unique geodesic in the free homotopy class of the closed curve $f_t(L_j)$ on R_t.We denote the hyperbolic length $\ell(L_j(t))$ of $L_j(t)$ simple by $\ell_j(t)$. Let $L_{j,k}(t)$ be the boundary component corresponding to $L_{j,k}$ and denote by $o_{j,k}(t)$ the geodesic joining $L_j(t)$ and $L_{j,k}(t)$ and by $c_{j,k}(t)$ the point of $o_{j,k}(t)$ on $L_j(t)$.

$L_j(t)$ has the natural orientation determined from that of L_j.

Let $T_j(t)$ be oriented arc on $L_j(t)$ from $c_{j,1}(t)$ to $c_{j,2}(t)$. We can define the signed hyperbolic length $\tau_j(t)$ of $T_j(t)$ and $\theta_j(t) = 2\pi\tau_j(t)/\ell_j(t)$ (so τ_j is positive if the orientation of $T_j(t)$ is compatible with that of $L_j(t)$ and negative if it is not the case).

Define $\Psi(t) = (\ell_1(t), \cdots, \ell_{3g-3}(t), \theta_j(t), \cdots, \theta_{3g-3}(t))$. This mapping is homeomorphisam of T_g onto $R_+^{3g-3} \times R^{3g-3}$.

Gluing pants P_1 and P_2 along curves $L_1 \in P_1$ and $L_1' \in P_2$ of the same length. Let G_j be a Fuchsian model of Nielsen extension $\hat{P}_j$. We can assume that the transformation $\gamma(z) = \lambda z$, $\lambda = \exp a$, belongs both G_1 and G_2, γ covers boundary components L_1 and L_2 and that the point $i \in H$ lies over both c_1 and c_2. Consider the element $\chi(z) = dz$, $d = \exp(a\alpha/2\pi)$. Fuchsian group generated by G_1 and $\chi \circ G_2 \circ \chi^{-1}$ is model of the gluing pants; $\theta = \alpha$.

2.7. Thurston Nielsen Bers classification theorem. Thurston's classification theorem characterizes homeomorphisms of a compact orientable surface. Thurston's theorem completes the work initiated by Jakob Nielsen (1944). A pseudo-Anosov map is a type of a diffeomorphism or homeomorphism of a surface. It is a generalization of a linear Anosov diffeomorphism of the torus. Its definition relies on the notion of a measured foliation invented by William Thurston, who also coined the term "pseudo-Anosov diffeomorphism" when he proved his classification of diffeomorphisms of a surface. Thurston constructed a compactification of the Teichmller space $T(S)$ of a surface S such that the action induced on $T(S)$ by any diffeomorphism f of S extends to a homeomorphism of the Thurston compactification. The dynamics of this homeomorphism is the simplest when f is a pseudo-Anosov map: in this case, there are two fixed points on the Thurston boundary, one attracting and one repelling, and the homeomorphism behaves similarly to a hyperbolic automorphism of the Poincaré half-plane. A "generic" diffeomorphism of a surface of genus at least two is isotopic to a pseudo-Anosov diffeomorphism.

A homeomorphism $f : S \to S$ of a closed surface S is called pseudo-Anosov if there exists a transverse pair of measured foliations on S, F^s (stable) and F^u (unstable), and a real number $\lambda > 1$ such that the foliations are preserved by f and their transverse measures are multiplied by $1/\lambda$ and λ. The number λ is called the stretch factor or dilatation of f.

Given a homeomorphism $f : S \to S$, there is a map f_0 isotopic to f such that at least one of the following holds:

f_0 is periodic; f_0 preserves some finite union of disjoint simple closed curves on S (in this case, f_0 is called reducible); or f_0 is pseudo-Anosov.

The case where S is a torus (i.e., a surface whose genus is one) is handled separately (see torus bundle) and was known before Thurston's work. If the genus of S is two or greater, then S is naturally hyperbolic, and the tools of

Teichmüller theory become useful. In what follows, we assume S has genus at least two, as this is the case Thurston considered. (Note, however, that the cases where S has boundary or is not orientable are definitely still of interest.)

The three types in this classification are not mutually exclusive, though a pseudo-Anosov homeomorphism is never periodic or reducible. A reducible homeomorphism g can be further analyzed by cutting the surface along the preserved union of simple closed curves Γ. Each of the resulting compact surfaces with boundary is acted upon by some power (i.e. iterated composition) of g, and the classification can again be applied to this homeomorphism.

Berss extremal problem. Bers gave an alternative proof of Thurstons classification in by adopting extremal approach in quasiconformal mappings. Since Bers classification is more suitable for us, we briefly review it.

Let S_0 be a hyperbolic Riemann surface of finite conformal type and let f be a homeomorphisam of S_0 onto itself. Bers's problem is to minimize $K(\sigma \circ \tilde{f} \circ \sigma^{-1})$ as $\tilde{f}$ varies over the homotopy class of f and σ varies over all homeomorphisams of S_0 onto variable Riemann surfaces S.

The equivalent problem: minimize the function χ_f defined by $t \to d_T(t, \rho_f(t))$, $t \in T(S_0)$, where ρ_f maps the class of $t = (S, g)$ to the class of $(S, g \circ f^{-1})$.

Denote by $\alpha(f)$ the infimum of $d_T(t, \rho_f(t))$ over all $t \in T(S_0)$. $\alpha(f)$ is zero or positive.

We consider three cases.

In Case 1 and Case 2, we suppose that the infimum is attained, that is χ_f has an absolute minimum at a point $t_0 \in T(S_0)$.

Case 1. $t_0 \in T(S_0)$ and $d_T(t_0, \rho_f(t_0) = \alpha(f) = 0$. Then $\rho_f(t_0) = t_0$. We may assume that t_0 is the base point of $T(S_0)$. Then f is homotopic to a conformal map f_0 of S_0 onto itself and $\rho_f = \rho_{f_0}$. Since f_0 has finite order, so ρ_f does.

Case 2. $t_0 \in T(S_0)$ and $d_T(t_0, \rho_f(t_0) = \alpha(f) > 0$.

ρ_f maps Teichmüller geodesic determined by t_0 and $\rho_f(t_0)$ onto itself.

If φ is the initial quadratic differential of the Teichmüller mapping $f_0 : S_0 \to S_0$ homotopic to f, then φ is the terminal quadratic differential.

That means f_0 maps horizontal trajectory onto itself and it expands in the horizontal directions of φ and contracts in the vertical directions by the same factor; so f_0 is a pseudo-Anosov mapping.

It is left to consider the case when the infimum of χ_f on $T(S_0)$ is not attained.

Case 3. Suppose that $\alpha(f)$ is zero or positive, and $d_T(t, \rho_f(t)) > \alpha(f)$, for all $t \in T(S_0)$. In this case ρ_f has infinite order and f_0 is reducible.

Bers used that the action of $mcg(S)$ also extends on the augmented Teichmüller space $\hat{T}(S)$.

Here we briefly discuss this subject. For the following results, terminology and more details see [IT] and the literature cited there.

Let S be a closed Riemann surface of genus $g \geqslant 2$. Then a mapping $f : S \to S$ is absolutely extremal if and only if
it is either a conformal mapping or
(i) f and f^2 Teichmüller mappings with $K(f^2) = K(f)^2$.
(i) is equivalent to
(ii) the initial and the terminal quadratic differentials of f coincide with each other up to a positive constant factor.

If $f$is a reducible self-mapping of S, we can deform f continuously to a complete reduced self mapping, or more precisely, to a self-mapping f_0 of S for which there is admissible set $\{C_1, C_2, \cdots C_m\}$ such that for every component S' of $S \setminus (C_1 \cup C_2, \cdots \cup C_m)$ and for the smallest positive integer n with $f_0^n(S') = S'$, the mapping $f_0^n|S'$ is irreducible.

Teichmüller mapping whose the initial and the terminal quadratic differentials coincide with each other up to a positive constant factor gives a so-called pseudo-Anosov homeomorphism.

Conversely, for any given pseudo-Anosov homeomorphism of S, there is a complex structure on S such that f is a Teichmüller mapping whose the initial and the terminal quadratic differentials coincide with each other up to a positive constant factor.

Teichmüller space $T(S)$ is actually globally homeomorphic to an open set in the complex Banach space $Q(G)$, which consists of the quadratic differentials with finite norm for the covering group G. Using this result we can define a natural complex structure for $T(S)$.

"Teichmüller imbedding" does not give a natural complex structure.

$\pi : B(G) \to T(S)$ is holomorphic, and it has local holomorphic sections everywhere in $T(S)$.

Every biholomorphic map between two Teichmüller spaces $T(S_0)$ and $T(S_1)$ is induced by a qc map between S_0 and S_1 except one of them has exceptional type.

3. *Extremality of quasiconformal mappings*

3.1. Introduction. We follow exposition in [Ma8].Let f be a quasiconformal map of a region G of the complex plane, and $\mu = f_{\overline{z}}/f_z$ its complex dilatation. The basic problem in the background is to characterize those dilatations μ that are uniquely extremal in their boundary class in the sense that the corresponding mappings are uniquely determined by the requirement that the essential sup of $|\mu|$ is minimal. Research on the problem started with Grötzsch and Teichmüller in the nineteen-thirties.

It was suggested by Teichmüller that boundary values on the boundary of the unit circle $\partial\Delta$ that allow a quasiconformal (qc) extension to the unit circle Δ always allow an extremal extension to what we now call a Teichmüller mapping. We refer to this statement, for short, as the Teichmüller question. One now knows that the answer to the Teichmüller question is "no". We get further results in this direction.

During the last several years, important progress has been made in characterizing the conditions under which unique extremality occurs (see [BMM], [BLMM],[Ma1], [Re9]). In particular, the Characterization Theorem which gives the characterization of unique extremality in functional-analytic fashion by special sequences of integrable holomorphic functions of what we call *Re*-sequences (*Re* being an abbreviation of *Reich*) has found interesting applications.

Note that there are many examples of extremal dilatations with nonconstant modulus, but all examples of uniquely extremal dilatations known up to our papers [BLMM] and [BMM] were of Teichmüller type. Moreover, many results obtained by studying the extremal problems spoke in favour of the conjecture that all uniquely extremal dilatations μ satisfy $|\mu(z)| = ||\mu||_\infty$, for almost all z. In [BMM] and [BLMM], we disproved this conjecture and showed that there are uniquely extremal dilatations with nonconstant modulus. We refer to the proof of this result of the construction of uniquely extremal dilatations with nonconstant modulus as "the construction", for short). Thus the form of a uniquely extremal complex dilatation can be very complicated.

Reich [Re3],[Re9] modified our original construction in [BLMM],[BMM], by using Runge's theorem instead of the approach through Mergelyan's theorem. Due to its highly technical nature, one can miss an intuitive understanding of the construction in [BLMM]. Among other things, this has motivated the author to continue further study of this subject and related properties of extremal and uniquely extremal dilatations.

We now give some comments related to our paper [Ma8]. In this paper, we look at uniquely extremal dilatations from a new point of view. Roughly speaking, we study how uniquely extremal dilatations on a domain are determined by their values on special sub-domains. In particular, we present a new construction (in Theorem 3.2 , below). It is more visual and a very special case leads to construction of a uniquely extremal complex dilatation which is of *Teichmüller* type outside a set K of positive measure with empty interior, and which has arbitrary values on K.

The significance of our new constructions is discussed in Subsection 3.4.

In [Ma1], we introduced the notion of a uniquely extremal complex dilatation on an extremal set which can be considered as a generalization of a uniquely extremal complex complex dilatation if the extremal set is of positive measure. Using this notion, we generalize the results related to uniquely extremal dilatations; see the Equivalence Theorem II for Pairs and Characterization Theorem II. In particular, a corollary of these results (Characterization Theorem II for Pairs) has applications. We also provide some simplifications with respect to the corresponding proof in [BLMM]. All together, this leads to a better understanding of unique extremality.

In Section 3.2, we discuss some of the background of the subject.

We have chosen to confine our discussion to subregions of the plane rather than general Riemann surfaces. This enables us to focus on the basics, but still allows for a rich variety of examples.

Part of this paper was published as a Warwick preprint [Ma5].

3.2. Definitions, Background; Extremal and uniquely extremal mappingsA. Extremal mappings. In this section, we give basic definitions and state the main result about extremal quasiconformal mappings. Note that some results presented in this paper, concerning unique extremality, have roots and analogues in the theory of extremal qc mappings.
The interested reader can learn more about extremal mappings from the excellent survey articles of Strebel's and Reich's [S5], [Re9] and Earle-Li Zhong's paper [ELi] (see also [Ma3]).

The study of extremal mappings has been one of the main topics in the theory of quasiconformal mappings, since its earliest days, when Grötzsch solved the extremal problem for two rectangles. In order to discuss them we need to review some familiar definitions.

For a function h Lebesgue integrable on a set $M \subset \mathbb{C}$, we define

$$||h||_M = \iint_M |h| \, dxdy.$$

A homeomorphism f from a domain G onto another is called quasiconformal (shortly qc) if f is ACL (absolutely continuous on lines) in G and $|f_{\bar z}| \leqslant k|f_z|$ a.e. in G, for some real number k, with $0 \leqslant k < 1$. In this setting, it is well known that partial derivatives f_z, $f_{\bar z}$ are locally square integrable and that the directional derivatives satisfy

$$\max |D_\alpha f(z)| \leqslant K \min |D_\alpha f(z)| \tag{3}$$

for a.e. $z \in G$, where $K = (1+k)/(1-k)$. Roughly speaking, (3) means that at almost all points z of G infinitesimal circles are mapped onto infinitesimal ellipses with axis ratio $D_f(z) \leqslant K$. It is also well known that if f is a quasiconformal mapping defined on the region G then the function f_z is nonzero a.e. in G. The function

$$\mu_f = \frac{f_{\bar z}}{f_z}$$

is therefore a well defined bounded measurable function on G, called the complex dilatation (shortly dilatation) or Beltrami coefficient of f. Note that, in the context of Riemann surfaces, it is usually called differential instead of complex dilatation.
The L^∞ norm of every Beltrami coefficient is less than one. Conversely, every μ in $L^\infty(G,\mathbb{C})$ with norm less than one is the Beltrami coefficient of some qc mapping whose domain is G. A computation shows that

$$D_f(z) = \frac{1+|\mu_f(z)|}{1-|\mu_f(z)|}.$$

The positive number

$$K(f) = \frac{1+||\mu_f||_\infty}{1-||\mu_f||_\infty}$$

is called the maximal dilatation of f. We say that f is K-qc if f is a qc mapping and $K(f) \leqslant K$.

Let $QC(G)$ (shortly QC) denote the family of all quasiconformal mappings from G into $\mathbb{C}$ and let $QC_0(G)$ denote the group of all quasiconformal mappings from $\mathbb{C}$ onto itself that fix every point of $\mathbb{C} \setminus G$ and are homotopic to the identity by a homotopy g_t in which each g_t is a homemorphism of $\mathbb{C}$ onto itself that fix every point of $\mathbb{C} \setminus G$. Two elements $f, g \in QC(G)$ are equivalent (in $Teichmüller's$ sense) if $f^{-1} \circ g \in QC_0(G)$.

This means that the equivalence class of f is the set

$$Q_f = [f] = \{f \circ (\phi_0|G) : \phi_0 \in QC_0(G)\}.$$

Let $\mathcal{S}$ be a set of qc mappings whose domain is G. The mapping f_0 in $\mathcal{S}$ is said to be extremal in $\mathcal{S}$ if $K(f_0) \leqslant K(g)$ for all g in $\mathcal{S}$.
In particular, a qc map f_0 is extremal in its Teichmüller class $[f]$ (abbreviated an EQC map) if $K(f_0) \leqslant K(g)$ for every mapping g in the same class. A qc map f_0 is uniquely extremal in its Teichmüller class $[f]$ if every other mapping g in the same class satisfies $K(f_0) < K(g)$.

We closely follow approach of Earle-Li Zhong [ELi] concerning the definition of QC_0 and extremal quasiconformal mappings on plane regions (see also below).

However, we find more convenient in most of this paper to express the results in terms of extremal and uniquely extremal dilatations.

We shall refer to a complex dilatation as extremal or uniquely extremal meaning that a mapping with that complex dilatation is extremal or uniquely extremal.

If two elements $f, g \in QC(G)$ are equivalent (in *Teichmüller*'s sense) we also say that their dilatations $\mu = \mu_f$ and $\nu = \mu_g$ are equivalent We denote the equivalence class of μ by $[\mu]$.

We also use the notation,

$$k_0([f]) = k_0([\mu_f]) = \inf\{||\mu_g||_\infty : g \in Q_f\}$$

and

$$K_0([f]) = K_0([\mu_f]) = \inf\{K(g) : g \in Q_f\}.$$

In studying extremal qc mappings of a region, the L^1 norms of functions analytic(holomorphic) in that region play a special role.

We denote by $L^1_a = L^1_a(G)$ the Banach space consisting of all holomorphic functions φ, belonging to $L^1 = L^1(G)$, with norm

$$||\varphi|| = ||\varphi||_G = \iint_G |\varphi(z)|\, dxdy < \infty .$$

Instead of L^1_a the short notation $\mathcal{A} = \mathcal{A}(G)$ is also used.
From now on, we shall assume that the complement $\mathbb{C} \setminus G$ of G contains at least three points. This assumption provides that the space $L^1_a(G)$ has positive dimension.
By Ω we denote a domain in $\mathbb{C}$, and by $L^\infty(\Omega)$ the space of all measurable and essentially bounded functions on Ω; and if $\mu \in L^\infty(\Omega)$, we say that μ is a complex dilatation on Ω. Let $\|\mu\|_\infty = \|\mu\|_{\infty,\Omega}$ denote the L^∞- norm of μ

on Ω.
By $\mathcal{M} = \mathcal{M}(\Omega)$ we denote the open unit ball in L^∞. Thus, if $\mu \in L^\infty(\Omega)$ and $k = ||\mu||_{\infty,\Omega} < 1$, we write $\mu \in \mathcal{M}$.
For $k \geqslant 0$, by $\overline{\mathcal{M}}_k = \overline{\mathcal{M}}_k(\Omega) = \{\mu \in L^\infty(\Omega) : ||\mu||_{\infty,\Omega} \leqslant k\}$ we denote the closed ball of radius k in L^∞.
We say that a sequence φ_n c-uniformly converges on Ω if it converges uniformly on every compact subset of Ω.

It is convenient to use short notation

$$\Lambda_\mu(\varphi) = (\mu, \varphi) = \iint\limits_G \mu\,\varphi\,dxdy, \quad \lambda_\mu(\varphi) = Re\int_G \mu\varphi,$$

where $\mu \in L^\infty(G)$, $\varphi \in \mathcal{A}$; and we say that the linear functional $\Lambda_\mu \in \mathcal{A}^*$ is induced by μ.
$\eta \in L^\infty$ is an annihilator of $\mathcal{A}$ in L^∞ if $(\eta, \varphi) = 0$ for every $\varphi \in \mathcal{A}$.
We denote by $\mathcal{N} = \mathcal{N}(G)$ the set of all annihilators of $\mathcal{A}$ in L^∞.
We say that $\mu \in L^\infty(G)$ and $\nu \in L^\infty(G)$ are infinitesimally equivalent (belong to the same equivalence class in the tangent space $\mathcal{B} = \mathcal{B}(G)$) if $\mu - \nu \in \mathcal{N}(G)$.
By the Hahn-Banach theorem and the Riesz's representation theorem, $(L^1_a)^*$ is isometrically isomorphic to the Banach space $\mathcal{B}$ of equivalence classes of elements in L^∞.

We say that $\chi \in L^\infty(G)$ is extremal in its infinitesimal class (abbreviated by $\chi \in ED_a$) if the norm of the linear function $\Lambda_\chi \in \mathcal{A}^*$ induced by χ, is the same as sup norm $||\chi||_\infty$ of χ. It means that $||\chi||_\infty \leqslant ||\mu||_\infty$ for every complex dilatation infinitesimally equivalent by χ.

For $\mu \in L^\infty$, we denote by $||\mu||_*$ the norm of the functional Λ_μ on $\mathcal{A} = L^1_a(G)$.

We say that a $\mu \in L^\infty$ satisfies the Hamilton-Krushkal condition if

$$||\mu||_* = ||\mu||_\infty.$$

We are now ready to state the main result about extremal complex dilatations.

Theorem HKRS. (Hamilton-Krushkal and Reich-Strebel) *Let G be a plane region whose complement $\mathbb{C} \setminus G$ contains at least three points. Let f be a qc mapping whose domain is G, and let $\mu = \mu_f$ be its Beltrami coefficient.*

A necessary and sufficient condition that f is an EQC (extremal) mapping in [f] is that

$$||\mu||_* = ||\mu||_\infty.$$

The proof that the Hamilton-Krushkal condition is sufficient is based on the Reich-Strebel inequality (so-called the Main Inequality). Various forms of this inequality play a major role in the theory of quasiconformal mappings and have many applications. In particular, using the Main Inequality in [BLMM] the generalized Delta inequality (Theorem 3 in [BLMM]) is proved, which is a very convenient tool in the theory of uniquely extremal qc mappings.

A Hamilton sequence for μ_f, is a sequence in $\mathcal{A}$, such that $||\varphi_n|| = 1$, for all n, and

$$\lim_{n\to\infty}(\mu, \varphi_n) = ||\mu||_\infty.$$

Now we can state the theorem of Hamilton-Krushkal and Reich-Strebel in the form: *f is extremal in its class $[f]$ if and only if μ_f has a Hamilton sequence.*

Theorem HKRS gives, via Hamilton sequences, what may be called an "analytic" method to test for extremality in distinction to the earlier methods that were more "geometric" in character.

If $\chi \in L^\infty(G)$ is extremal in its infinitesimal class we say shortly χ is extremal or χ is ED_a (we also write $\chi \in ED_a$). By the Hamilton-Krushkal and Reich-Strebel theorem, if $\chi \in \mathcal{M}$, it means that χ is extremal in its Teichmüller class.

The Equivalence Theorem I, which follows, is the parallel statement to Theorem HKRS for unique extremality.

B. Unique extremality

We say that $\chi \in L^\infty(G)$ is uniquely extremal in its infinitesimal class (abbreviated by $\chi \in HBU_a$) if it is extremal and the linear functional $\Lambda_\chi \in \mathcal{A}^*$ induced by χ,

$$\Lambda_\chi(\varphi) = (\chi, \varphi) = \iint_G \chi\varphi\, dxdy,$$

has a unique norm-preserving extension from $\mathcal{A}$ to a bounded linear functional on $L^1(G)$. It means that every other $\mu \in L^\infty(G)$, which is in the same infinitesimal class, satisfies $||\chi||_\infty < ||\mu||_\infty$.

It is convenient to use notation

$$\delta_n = \delta_\mu(\varphi_n; G) = \|\mu\|_\infty \int_G |\varphi_n|\, dxdy - \, Re \int_G \varphi_n \mu \, dxdy$$

and short notation $\delta_\mu[\varphi_n]$, $\delta_G[\varphi_n]$, $\delta[\varphi_n]$ instead of $\delta_\mu(\varphi_n; G)$ if the meaning of this is clear from the context.

We say that a sequence $\varphi_n \in L^1_a$ is a weak Hamilton sequence for μ if $\delta_\mu[\varphi_n]$ converges to 0.

The next two theorems have been proved by Božin, Lakić, Marković and Mateljević in [BLMM], [BMM] and [MM1].

Theorem B. *[*The Equivalence Theorem I*] Let $\chi \in \mathcal{M}$. Then χ is uniquely extremal in its Teichmüller class if and only if χ is uniquely extremal in its infinitesimal class.*

If $\chi \in L^\infty(G)$ is uniquely extremal in its infinitesimal class we say shortly χ is uniquely extremal or χ is HBU_a (we also write $\chi \in HBU_a$). By the Equivalence Theorem I, if $\chi \in \mathcal{M}$, then χ is uniquely extremal in its Teichmüller class.

The proof of Equivalence Theorem I has been based on estimates which allow us to compare two Beltrami coefficients μ and ν in the same global(Teichmüller) equivalence class and two complex dilatations in the same infinitesimal equivalence class. Note that the Equivalence Theorem I was a very important step in understanding the notion of uniquely extremal complex dilatation.

The next important step has been to analyze the proof of Hahn-Banach theorem and its applications to our setting. In particular, using the Equivalence Theorem I, we have obtained the following necessary and sufficient criterion for the unique extremality of a given Beltrami coefficient χ.

Theorem C. *[*The Characterization Theorem I*, [BLMM], [BMM]] The Beltrami coefficient χ is uniquely extremal if and only if for every admissible variation $\hat{\chi}$ of χ there exists a sequence φ_n in $L^1_a(G)$ such that*

(a) $\delta[\varphi_n] = \|\varphi_n\| \|\hat{\chi}\|_\infty \, - \, Re \int_G \varphi_n \hat{\chi} \to 0$

(b) $\liminf_{n\to\infty} \mid \varphi_n(z) \mid > 0,$ for almost all z *in* $E(\hat{\chi})$.

Here, an admissible variation $\hat{\chi}$ of χ is any complex dilatation that does not increase the $L^\infty-$norm of χ, and which is allowed to differ from χ only on the set $E_s = \mid \chi(z) \mid \leqslant s < k$, where $k = \|\chi\|_\infty$ and s is a constant, and

the extremal set $E(\hat{\chi})$ is the set on which $\hat{\chi}(z) = \|\hat{\chi}\|_\infty$; in this setting, if $\hat{\chi}$ is different from χ only on a set $F \subset E_s$, we say that $\hat{\chi}$ is an admissible variation of χ on F.
Note that we do not require that $\hat{\chi}$ and χ are equivalent.

We say that $\mu \in L^\infty(G)$ satisfies the Reich condition (Re-condition) on a set $S \subset G$ or, we say that φ_n is a Reich sequence (Re-sequence) for μ on S (relative to G if it is not clear from the context) if

(1) there exists a sequence $\varphi_n \in L^1_a(G)$ such that $\delta_\mu(\varphi_n; G) \to 0$ (i.e. there is a weak Hamilton sequence φ_n for λ_μ)

(2) $\liminf |\varphi_n(z)| > 0$ a.e. in S.

Thus by the Characterization Theorem I, the Beltrami coefficient χ is uniquely extremal if and only if for every admissible variation $\hat{\chi}$ of χ there exists a Re- sequence φ_n in $\mathcal{A}(G)$ on the extremal set $E(\hat{\chi})$.

Note that we mainly use the Characterization Theorem I and II in some special situations.
In particular, if $|\chi|$ is constant on G, χ is uniquely extremal on G if and only if there exists a Re-sequence for χ on G.

The Characterization Theorem gives, via Re- sequences, what may be called an "analytic" method to test for unique extremality. Roughly speaking, we may say that there is analogue between Theorem HKRS (via Hamilton sequences) and The Characterization Theorem (via what we call Re-sequences).

Let $\chi \in L^\infty(G)$. Since χ is uniquely extremal in its infinitesimal Teichmüller class (that is $\chi \in HBU_a$) if the normalization $\chi_k = k\,\chi/||\chi||_\infty$ is uniquely extremal in its Teichmüller class for some $0 < k < 1$ (and hence for every $0 < k < 1$), it is convenient in some settings to express some results by means of the HBU_a property; or shortly to say $\chi \in L^\infty(G)$ is uniquely extremal if it is uniquely extremal in its infinitesimal Teichmüller class.

Definition 3.1. [complex dilatation of Teichmüller type] *Let G be a domain in $\mathbb{C}$. If $\mu = s(z)|\psi|/\psi$, where s is a nonnegative measurable function from G into $[0, 1)$ and ψ is an analytic function, not identically zero, on G we shortly say that μ is of general Teichmüller type (s, ψ) on G. If, in addition, s is a constant k a.e. on G, we say that μ is of Teichmüller type (k, ψ) on G and if ψ is an analytic integrable function on G we shortly say that μ is a Teichmüller complex dilatation.*

Thanks to the Characterization Theorem we can study uniquely extremal dilatations using an infinitesimal cotangent space $\mathcal{A} = L^1_a$, which is the space of holomorphic integrable functions. Using new tools, some properties of uniquely extremal dilatations of general Teichmüller type have been described. In particular, we use compactness of certain families of holomorphic functions and the mean value theorem to prove the following results [Ma5].

Theorem D. (The second removable singularity Theorem) *Let Ω be a bounded domain (multiply connected in general), Ω_∞ the unbounded component of Ω^c and $\Omega_0 = (\Omega_\infty)^c$. Let χ be a uniquely extremal complex dilatation of general Teichmüller type (s, φ) on Ω. Then*

(a) *$\chi = k|\varphi|/\varphi$ a.e. in Ω, where k is a constant.*
If, in addition, χ has uniquely extremal extension to Ω_0, then

(b) *φ has an analytic extension $\tilde{\varphi}$ from Ω to Ω_0*

(c) *$\chi = k|\tilde{\varphi}|/\tilde{\varphi}$ a.e. in Ω_0.*

If D is a simply-connected domain and K a compact set such that $K \subset D$, we say that (K, D) is a pair.

Theorem E. *Let (K, D) be a pair and $V = D \setminus K$. Then*
(A) The following condition
(a) $|\chi|$ is a constant a.e. on D and χ is HBU_a on D
implies
(b) there is a Re -sequence consisting of polynomials for χ on D (and, in particular, on V).
(B) Suppose

(b_1) there is a Re -sequence consisting of polynomials for χ on V.
(c) χ is a complex dilatation of general Teichmüller type (s, φ) on V .
Then (b_1) and (c) imply that there is a unique complex dilatation χ_0, which is uniquely extremal extension of χ to D (and consequently which is of Teichmüller type).

Note that there is the difference between the hypothesis in part (B) of this theorem and Theorem D (The second removable singularity Theorem). Namely, the assumption in Theorem D that χ is uniquely extremal in $D = \Omega_0$, is replaced by the assumption in Theorem E that holds: (b) there is a *Re* -sequence consisting of polynomials for χ on V .

3.3. Unique extremality and oscillation. Let a complex valued function μ is defined on a domain G. The oscillation of μ on a set D is :

$\omega(\mu; D) = \operatorname{esssup}\{|\mu(z_1) - \mu(z_2)| : z_1, z_2 \in D \cap G\}$, $\omega_r(\mu; a) = \omega(\mu; B)$, where $B = B(a; r)$,
and the oscillation of a complex valued function μ at a point $a \in \overline{G}$ is defined as the limit as $r \to 0$:
$\omega(\mu; a) = \lim_{r\to 0} \omega_r(\mu; a)$.
For $b \in \mathbb{C}$, set $|\mu - b|_{\infty,B} = \operatorname{esssup}\{|\mu(z) - b| : z \in B\}$. It is clear that there is $b \in \mathbb{C}$ such that $|\mu - b|_{\infty,B} \leqslant \omega(\mu; B)$. In a similar way we define $\omega_r^-(\mu; a) = \inf\{|\mu(z) - b|_{\infty,B} : b \in \mathbb{C}\}$ and the lower oscillation $\omega^-(\mu; a)$ of μ at a. We call the function ω_μ^- defined on $\overline{G}$ by $a \mapsto \omega^-(\mu; a)$ the lower oscillation of μ.

In the statements that follow G denotes a $C^{1,\alpha}$- domains, $0 < \alpha < 1$.

Suppose that μ is a dilatation on G and $V \subset \overline{G}$. If $\omega^-(\mu; a) < ||\mu||_{\infty;V}$ for every $a \in V$, we say that the lower oscillation of μ is less than L^∞- norm of μ on V. If $\omega^-(\mu; a) < ||\mu||_{\infty;G}$ for every $a \in \partial G$, we say that the lower oscillation of μ on the boundary is less than L^∞- norm of μ on G.

Roughly we prove

(a1) If μ is extremal on G and the lower oscillation of μ on the boundary is less than L^∞- norm of μ on G, then μ is of Teichmüller type on G.

(a2) Let χ be uniquely extremal on G, $\|\chi\|_\infty = 1$. Suppose that $\omega^-(\mu; a) < 1$ (the lower oscillation of μ) is strictly lees 1) expect on a discrete set in G. Then χ is of Teichmüller type on G.

If $\mu(z) = k$ for $z \in \mathbb{D}^+$ and $\mu(z) = -k_1$ for $z \in \mathbb{D}^-$, where $0 < k_1 < k < 1$, then by (a1) μ is not extremal on $\mathbb{D}$.

Theorem 3.1. *[[Ma9]] Let χ be uniquely extremal on G, $\|\chi\|_\infty = 1$, $a \in G$. Suppose that there exists $B = B(a; r) \subset G$ such that*
(ii) $s = |\mu - b|_{\infty,B} < 1$ for some $b \in \mathbb{C}$ (more generally, respectively $\omega^-(\mu; a) < 1$).
Then χ is of Teichmüller type on B (respectively in some neighborhood of a).

Let χ be uniquely extremal on G, $\|\chi\|_\infty = 1$. Suppose that $\omega^-(\mu; a) < 1$ expect on a discrete set in G. Then χ is of Teichmüller type on G.

In particular if χ is continuous at a, then (ii) holds.

3.4. Construction.Outline of new construction.

Let (K, D) be a doubly-connected pair. We will show that there exists a sequence of Jordan-domains J_n such that

$$J_n \subset Int J_{n+1},\ \cup_1^\infty \overline{J_k} = D \setminus K. \tag{4}$$

In this setting, we say that sequence of Jordan-domains J_n exhausts $D \setminus K$. Namely, since $D \setminus K$ is doubly connected and K contains at least two point, then there is the closed disk $\overline{B}$ of radius r and a conformal mapping Φ of $\Delta \setminus \overline{B}$ onto $D \setminus K$, where Δ denotes the unit disk. Let $r_n = r + 1/n$, $J'_n = \{\rho\, e^{i\theta} : r_n < \rho < 1,\, 0 < \theta < 2\pi - 1/n\}$ and $J_n = \Phi(J'_n)$. It is clear that the sequence of Jordan-domains J_n satisfies condition (4).

Definition 3.2 [simply-connected triple] *Let $I_r = (r, 1)$ be an interval, $\Lambda = \Phi(I_r)$, $V = D \setminus K$, and $V' = V \setminus \Lambda$. We call (K, D, V') a simply-connected triple (a doubly-connected pair (K, D) with cut Λ). If, in addition, K has empty interior we say that (K, D, V') is a special simply-connected triple and that (K, D) is a special doubly-connected pair.*

Suppose that (K, D) is a pair and χ is of a general Teichmüller type (s, φ) on $V = D \setminus K$. In this setting, we conclude:
a) The proof of Theorem E (see also Theorem D) tells us that if $\chi \neq 0$ a.e. on V, and has a *Re*- sequence consisting of polynomials on V, then the corresponding normalized sequence of polynomials P_n *c*-uniformly converges on D and gives the analytic continuation $\tilde{\varphi}$ of φ to D. Since $\tilde{\varphi}$ is not identically zero on D, we can define $\tilde{\chi} = k\dfrac{|\tilde{\varphi}|}{\tilde{\varphi}}$. It is not difficult to verify that P_n is a *Re*- sequence for $\tilde{\chi}$ on D and therefore that $\tilde{\chi} \in HBU_a(D)$ (see Theorem E).
In particular, if χ is uniquely extremal on D and $\chi = 0$ on K, then $\chi = 0$ on D a.e. or K is a finite set.

b) If, in addition, the normalized sequence of polynomials P_n is a special for χ and pair (K, D), then first we conclude that $\tilde{\varphi}$ is zero on K. Therefore χ cannot be of Teichmüller type on V unless if K is a finite set or $s = 0$ a.e. on V.

Theorem 3.2. *[Construction Theorem] Let (K, D, V') be a simply-connected triple and ψ_0 a holomorphic function on K.*
a) *There is a sequence of polynomials φ_n, which uniformly converge to ψ_0 on K and which c-uniformly converges to a holomorphic nonzero function φ_o on V', and which is a Re-sequence on V.*
b) *If, in addition, ψ_0 is different than 0 a.e. in K, and if χ is defined on D by $\chi = \frac{|\psi_0|}{\psi_0}$ on K and $\chi = \frac{|\varphi_0|}{\varphi_o}$ on V', then the sequence φ_n is a Re-sequence for χ on D, and therefore χ is HBU_a on D.*

The situation described in the theorem is one that we will often encounter. Hence it is convenient to introduce new terminology.

Definition 3.3 [(φ_o, ψ_0)**- special sequence of polynomials**] *We call the sequence* φ_n *described in theorem,* (φ_o, ψ_0)*- special sequence of polynomials for a simply-connected triple* (K, D, V')*.*

Before we prove this theorem, we state some remarks and corollaries which partially explain the significance of the result.

Remarks and corollaries of Theorem 3.2.

1. The theorem holds if we only suppose that ψ_0 is a continuous function on K and a holomorphic function in the interior of K.

2. the assumption that φ_n is a Re-sequence of polynomials on V', which c-uniformly converges to a holomorphic not identically zero function φ_o on V', implies φ_o is different than 0 on V'.

3. (*holomorphic function as germ of uniquely extremal complex dilatation*) If the holomorphic function ψ_0 is different than 0 a.e. in K, and if χ is defined on K by $\chi = \frac{|\psi_0|}{\psi_0}$, then it follows from part b) of the theorem that χ has an extension which is uniquely extremal on D.
Namely, if we extend χ to D by $\chi = \frac{|\varphi_0|}{\varphi_o}$ on V', where the function φ_0 is defined by the sequence φ_n on V', then φ_n is a Re-sequence for χ on D. Therefore, by a very special case of The Characterization Theorem (Proposion B, see Section 3.4), χ is HBU_a on D; that is χ is uniquely extremal.
In this way, we can roughly state that every ψ_0 is a germ of a uniquely extremal complex dilatation on D.
Continuous function as germ of uniquely extremal complex dilatation.
In particular, if K is a special set, f is a continuous function on K, which is different than 0 a.e. in K, and if $\chi_0 = \frac{|f|}{f}$ on K, then, using the method of the proof of Theorem 3.2 and the Mergelyan theorem instead of Runge, one can show that χ_0 has a uniquely extremal extension to D (see Corollary 2).

4. (*construction of uniquely extremal complex dilatation with nonconstant modulus*) If $\psi_o \equiv 0$, and if χ is defined on D by $\chi = \frac{|\varphi_0|}{\varphi_o}$ on V' and $\chi = 0$ on K, using the Delta inequality in the tangent space $\mathcal{B}$ (as in [BLMM]), one can show that χ is uniquely extremal on V relative to D.
If, in addition, the set K has empty interior and positive two-dimensional measure, then using Lemma R, which roughly states that annihilators with support on special sets vanish, one can show that χ is uniquely extremal (HBU_a) on D.

This gives an example of uniquely extremal complex dilatation with non-constant modulus.

Note that one can verify that χ defined in this item is uniquely extremal by means of Theorem V. 3.1 [Re9], which is a special case of the Characterization theorem. That has sense because the proof of the theorem is much simpler than the proof of the Characterization theorem.

5. If ψ_0 has no analytic continuation to D, then, by Theorem D, the complex dilatation χ is not of Teichmüller type on V (i.e. the function φ_o has no analytic continuation to V). Thus we can construct a uniquely extremal complex dilatation μ on D, which is of Teichmüller type on both V' and K, but not on $K \cup V'$. □

The next statement is an immediate corollary of Theorem 3.2 .

Corollary 1. (the HBU_a continuation of holomorphic function) *Let (K, D) be a doubly-connected pair, ψ_0 a holomorphic function on K, which is different than 0 a.e. in K, and $\chi_0 = \frac{|\psi_0|}{\psi_0}$ on K. Then there exists a $\chi \in HBU_a$ on D such that $\chi = \chi_0$ on K.*

Corollary 2. (the HBU_a continuation of continuous function) *Let (K, D) be a special doubly-connected pair, and let f be a continuous function on K, which is different than 0 a.e. in K ; If $\chi_0 = \frac{|f|}{f}$ on K, then there is $\chi \in HBU_a$ on D such that $\chi = \chi_0$ on K.*

Using Mergelyan's theorem as in [BLMM], one can show that this corollary holds if (K, D) is a pair and K a special M-set.

Outline of proof of Corollary 2. Proceeding in a similar way as in the proof of Theorem 3.2 below one can prove Corollary 2. The difference is only in the application of the Mergelyan theorem (instead of the Runge theorem) which gives
(2") $\varphi_n - f = 0(1)$ on K.

The proof of the part b) of Theorem 3.2 is based on an important corollary of the Characterization Theorem :

Proposition B. *If $|\chi|$ is a constant on G, then χ is uniquely extremal if and only if χ satisfies the Re- condition on G.*

3.5. Extremal mappings in 3 dimensions. Suppose that $f : \Omega \to \Omega^*$ is a homeomorphism. Consider a path family Γ in Ω and its image family $\Gamma^* = \{f \circ \gamma : \gamma \in \Gamma\}$. We introduce the quantities

$$K_I(f) = \sup \frac{M(\Gamma^*)}{M(\Gamma)}, \ \ K_O(f) = \sup \frac{M(\Gamma)}{M(\Gamma^*)},$$

where the supremum are taken over all path families Γ in Ω such that $M(\Gamma)$ and $M(\Gamma^*)$ are not simultaneously 0 or ∞.

Definition 3.4 *Suppose that $f : \Omega \to \Omega^*$ is a homeomorphism; we call $K_I(f)$ the inner dilatation and $K_O(f)$ the outer dilatation of f. The maximal dilatation of f is $K(f) = max(K_O(f), K_I(f)$. If $K(f) \leqslant K < \infty$, we say f is K-quasiconformal (abbreviated qc).*

The inner coefficient of D with respect to D' is the number $K_I(D, D') = \inf K_I(f)$ over all homeomorphisms $f : D \to D'$. Similarly, the outer coefficient is $K_O(D, D') = \inf K_O(f)$. If $D' = B^n$, we abbreviate $K_I(D, B^n) = K_I(D)$ and $K_O(D, B^n) = K_O(D)$.

A mapping $f : D \to D'$ is extremal for K_I or K_O if $K_I(f) = K_I(D, D')$ or $K_O(f) = K_O(D, D')$, respectively.

Quasiconformal mappings in Euclidean n-space, $n > 2$, have been studied rather intensively in recent years by several authors. It turns out that these mappings have many properties similar to those of plane quasiconformal mappings. On the other hand, there are also striking differences. Probably the most important of these is that there exists no analogue of the Riemann mapping theorem when $n > 2$. This fact gives rise to the following two problems. Given a domain D in Euclidean n-space, does there exist a quasiconformal homeomorphis f of D onto the n-dimensional unit ball $\mathbb{B}^n$. Next, if such a homeomorphism f exists, how small can the dilatation of f be; more precisely determine $K_I(D)$ and $K_O(D)$? Complete answers to these questions are known when $n = 2$. For a plane domain D can be mapped quasiconformally onto the unit disk $\mathbb{D} = \mathbb{B}^2$ if and only if D is simply connected and has at least two boundary points. The Riemann mapping theorem then shows that if D satisfies these conditions, there exists a conformal homeomorphism f of D onto $\mathbb{D}$. The situation is very much more complicated in higher dimensions cf [GeVa].

Define $K_*(f) = K_I(f) + K_O(f)$. Suppose that $Q = Q^3$ is a unit cube (a cube whose sides are 1), and $R = R(a, b, c) = [0, a] \times [0, b] \times [0, c]$, $a, b, c > 0$, is a rectangular parallelepiped.

In the literature we did not find even the answer to the following question (which looks the simple):

Question 1. What is the vale for $K_I(Q, R)$ and $K_O(Q, R)$.

Let F be the family of qc maps $f : Q \to R$, which maps vertices to vertices, then $K_O(f) \geqslant c^2/ab$. If $c \geqslant a, b$, then the affine map $A_0(x) = (ax_1, bx_2, cx_3)$ is K_O extremal. Is it uniquely extremal?

Occasionally, it is convenient to use (a, b, c) instead of (a_1, a_2, a_3). Suppose that $a \geqslant b \geqslant c$.

Concerning Question 1, we announce the following result:

Theorem 3.3. *If $f \in F$, then $K_*(f) \geqslant \frac{a_1^2}{a_2 a_3} + \frac{a_1 a_2}{a_3^2}$. The affine map A_0 is K_* uniquely extremal in F.*

In further research, we plan to generalize the method of extremal length and derive analogies with the two dimensional Teichmüller theory. In particular, we relate extremal problems to the existence of special Lagrangian fibrations, proposing a method for constructing them in the large complex structure limit of Calabi-Yau manifolds and consider extremal problems for rectangular parallelepiped and tori. This subject is initiated in [Bo]. An extremal quasiconformal homeomorphisms in a class of homeomorphisms between two CR 3-manifolds is one which has the least conformal distortion among this class. Lempert proposed to develop Teichmüller theory in the setting of Cauchy-Riemann (CR) manifolds. We plan to continue studies of extremal quasiconformal homeomorphisms between CR 3-manifolds.

Note only here that harmonic quasiregular (briefly, *hqr*) mappings in the plane are also active area of research. In particular, there is a group of mathematicians related to Belgrade Analysis Seminar who are interested in this subject. The subject has grown to include study of *hqr* maps in higher dimensions, cf. for example [Ma10, KaMM, KaM, AAMS] and references cited there.

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