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LINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS I

T. M. ATANACKOVIĆ, B. STANKOVIĆ

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A b s t r a c t. We give the existence and the analytic form of the solutions to the linear fractional differential equations in which coefficients are bounded function on $[0, b]$. The method which is used is based on the theory of Volterra's integral equations.

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1. *Introduction*

Volterra's integral equations occur in the consideration of equations with fractional derivatives. It is a consequence of the fact that in the definition of fractional derivative is the integral operator ${}_0I_t^\alpha$ and for Volterra's integral equation we have a nice and complete theory. The book [4], Chapter 3 and 4 shows this fact. In all cases for construction corresponding Volterra's integral equation one uses the property (cf. [4], p.75):

"If $f(x) \in L_1(a,b)$ and $f_{n-\alpha} \equiv {}_0I_t^{n-\alpha} f \in AC^n[0,b]$,
then

$$ {}_0I_t^{\alpha}\, {}_0D_t^{\alpha} f(x) = f(x) - \sum_{j=1}^{n} \frac{f_{n-\alpha}^{(n-j)}}{\Gamma(\alpha - j + 1)}(x-a)^{\alpha-j}, \text{ a.e.,} $$

where $n = [\alpha] + 1$." Here $n \in \mathbb{N}$.

This method is proved useful for the generalized solutions as well. See for example [6]. In this paper we prove the existence and uniqueness of solution to the equation $D^q x = f(t,x),\ 0 < q < 1$ in a Banach Space.

We quote a general result for linear equations with fractional derivatives (cf. [3], [10] Part 4 2.2, [4] Part 3.1 and [7] 3.1).

Let differential equation of fractional order be of the form

$$ D^{\sigma_n} y(x) + \sum_{k=0}^{n-1} p_k(x) D^{\sigma_{n-k-1}} y(x) + p_n(x) y(x) = f(x),\ 0 < x \le b, \quad (1.1) $$

where $D^{\sigma_k} = {}_0D_t^{\alpha_k - 1}\, {}_0D_t^{\alpha_{k-1}} \ldots {}_0D_t^{\alpha_0},\ k = 1, \ldots, n,\ D^{\sigma_0} = {}_0D_t^{\alpha_0 - 1},\ \sigma_k = \sum_{j=0}^{k} \alpha_j - 1,\ k = 0, \ldots, n,\ 0 < \alpha_j \le 1,\ j = 0, 1, \ldots, n.$

Let the initial condition be as

$$ D^{\sigma_k} y(x)\Big|_{x=0^+} = b_k,\ k = 0, 1, \ldots, n-1. \quad (1.2) $$

First, we prove:

Theorem A. *Let the functions $p_k(x),\ k = 0, \ldots, n$ be Lipshitzien on $[0,b]$, i.e., they satisfy Hlder's condition of order $\lambda = 1$ and let $f(x)$ be continuous function in $[0,b]$; which can be represented as*

$$ f(x) = D^{\alpha_n - 1} \bar{f}(x),\ 0 < \alpha_n < 1,\ \bar{f} \in L(0,b). \quad (1.3) $$

If $\alpha_0 > 1 - \alpha_n$, then the Cauchy-type problem defined in (1.1), (1.2) has a unique solution continuous on $[0,b]$.

Theorem B. *Let $f \in L(0,b)$ and (1.3) is satisfied with $\alpha_n < 1$, then the conclusion of Theorem A is valid for equation*

$$ {}_0D_t^{\alpha_n} y(x) = f(x) $$

and the solution $y(x)$ is:

$$ y(x) = \sum_{k=0}^{n} b_k \frac{x^{\sigma_k}}{\Gamma(1+\sigma_k)} + \int_0^x \frac{(x-\tau)^{\sigma_n - 1}}{\Gamma(\sigma_n)} f(t)\,dt. $$

An other result is also interesting for our consideration. In [4] and [5] it is analysed equation

$$ {}_0D_t^{\alpha} y(x) - \lambda (x-a)^{\beta} y(x) = 0,\ a < x \le b,\ \alpha > 0 \tag{1.4}$$

with initial condition

$$\left({}_0D_t^{\alpha-k} y\right)(a^+) = b_k,\ k = 1, ..., n = -[-\alpha]. \tag{1.5}$$

Theorem C. *Let* $\alpha > 0, n = -[-\alpha], \lambda \in \mathbb{R}$ *and* $\beta \ge 0$. *Then* the Cauchy type problem *(1.4),(1.5) has a unique solution* $y(x)$ *in the space* $C^{\alpha}_{n-\alpha}$ *and this solution is given by:*

$$y(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} (x-a)^{\alpha - j} \left[1 + \sum_{k=1}^{\infty} C_k \left(\lambda (x-a)^{\alpha+\beta}\right)^k\right] \tag{1.6}$$

where Ck are specially determined constants ((cf. [4], p.227).)

In [5] it is also proved that for $a = 0$, the function $y(x)$ given by (1.6) is the solution of (1.4), (1.5) in the space of localy integrable function on $[0, d],\ d < \infty$.

As one can see for equationes (1.1), (1.2) one has the existence and the uniqueness of the solution, supposing strong conditions on functions $p_k(x),\ k = 0, ..., n-1$ and on α_n. For some interesting special cases of (1.1), (1.2) one can in addition give the analytical form or more properties of the solution (cf. for example [9]).

That was the reason to consider equation (1.1) and to assume less limitations on functions $p_k(x),\ k = 0, ..., n-1$ and on numbers $\alpha_k,\ k = 0, ..., n-1$, as well as to give the analytical form to the solutions.

So in this paper we suppose on functions A_i only to be bounded and integrable on $[0, b]$. But we do not suppose anything on numbers $\alpha_i,\ i = 0, ..., m$. Interesting special cases and questions related to the equation (2.1) are missing because of the length of the paper. This will be elaborated in the next paper.

2. *Preliminaries*

We consider equation

$$\left({}_0D_t^{\alpha_m} - \lambda \sum_{i=1}^{m-1} A_i(t)\, {}_0D_t^{\alpha_i} - \lambda A_0(t)\right) y(t) = f(t),\ 0 \le t \le b, \tag{2.1}$$

where $\alpha_i = [\alpha_i] + \gamma_i,\ [\alpha_i] \in \mathbb{N}_0,\ \gamma_i \in [0,1),\ 0 < \alpha_{i-1} < \alpha_i,\ n_i = [\alpha_i]+1,\ i = 1,...,m-1$. If $\alpha = [\alpha]$, then $\gamma = 0$ and ${}_0D_t^{\alpha} = D^{[\alpha]}$: $[\alpha]$ means the integral part of α.

The operators ${}_0I_t^{\alpha}$ and ${}_0D_t^{\alpha}$ are Riemann-Liouville fractional integral and fractional derivative (cf. [10] and [4]).

We will use the spaces $\mathcal{J}_0([0,b])$ and $\mathcal{J}_n([0,b])$ (cf. [1]).

$\mathcal{J}_0([0,b])$ is the space locally bounded functions f on $(0,b]$, and $f \in L^1([a,b])$.

$\mathcal{J}_n([0,b]) = \left\{f;\ f^{(n)} \in \mathcal{J}_0([0,b])\right\},\ n \in \mathbb{N}$. If $n \geq 2$, then $\mathcal{J}_n([0,b]) \subset AC^n\,[0,b]$, (cf. [2], p.100).

A necessary and sufficient condition that ${}_0D_t^{\alpha} f \in \mathcal{J}_0([0,b])$ is that ${}_0I_t^{1-\gamma} f \in \mathcal{J}_n([0,b])$, where $\alpha = n-1+\gamma,\ n \in \mathbb{N},\ \gamma \in (0,1)$.

For the properties of the space $\mathcal{J}_n([0,b])$ see [1].

We consider an auxiliary equation using the following properties of fractional operators:

1.) If $\alpha > 0$ and $\beta > 0$, then

$$ {}_0I_t^{\alpha}\ {}_0I_t^{\beta} f(x) = {}_0I_t^{\alpha+\beta} f(x), \text{ a.e.,} $$

for $f \in L^1(0,b)$.

2.) If $\alpha > \beta > 0$, then

$$ {}_0D_t^{\beta}\ {}_0I_t^{\alpha} f(x) = {}_0I_t^{\alpha-\beta} f(x), \text{ a.e.} $$

for $f \in L^1(0,b)$ (cf. [4], p.73-74). In particular when $\alpha = \beta$, we have

$$ {}_0D_t^{\beta}\ {}_0I_t^{\beta} f(x) = f(x), \text{ a.e.} $$

3.) Now it is easily seen that for $y \in L^1(0,b)$:

$$ \begin{aligned} {}_0D_t^{\alpha_i} y(t) &= D^{[\alpha_i]+1}{}_0I_t^{1-\gamma_i} y(t) \\ &= D^{[\alpha_i]+1} D^{[\alpha_m]-[\alpha_i]}{}_0I_t^{[\alpha_m]-[\alpha_i]}{}_0I_t^{1-\gamma_i} y(t) \\ &= D^{[\alpha_m]+1}{}_0I_t^{1-\gamma_m}{}_0I_t^{[\alpha_m]-[\alpha_i]+1-\gamma_i-1+\gamma_m} y(t) \\ &= {}_0D_t^{\alpha_m}{}_0I_t^{\alpha_m-\alpha_i} y\,(t)\,. \end{aligned} $$

4.) If ${}_0D_t^{\alpha} y$ and ${}_0D_t^{\alpha+m} y$ exist, then (cf. [4], p.74)

$$ D^m\ {}_0D_t^{\alpha} y = {}_0D_t^{\alpha+m} y,\ \ m \in \mathbb{N}. \tag{2.2} $$

3. *Reduction of equation (2.1) to an integral equation*

Equation (2.1) we write as

$$\Big({}_0D_t^{\alpha_m} - \lambda \sum_{i=0}^{m-1} A_i(t)\ {}_0D_t^{\alpha_i}\Big)y(t) = f(t),\ 0 \le t \le b, \tag{3.1}$$

where we suppose that for $\alpha_0 = 0$ we have ${}_0D_t^0 y(t) = y(t)$. We consider solutions which belong to the space $\mathcal{J}_{n_m}([0,b])$.

We have to distinguishof this two cases: $0 < \alpha_m - \alpha_{m-1} < 1$ and $1 \le \alpha_m - \alpha_{m-1}$. In both two cases we use the Riemann-Liouville fractional integral ${}_0I_t^{\alpha}$ and derivative ${}_0D_t^{\alpha}$ (cf. [4]).

3.1 *Case* $1 \le \alpha_m - \alpha_{m-1}$

To transform equation (3.1) with y bounded on $[0,b]$ we use the following properties of the fractional derivatives

$$ {}_0D_t^{\alpha_m} y(t) = D^{[\alpha_m]+1}{}_0I_t^{1-\gamma_m} y(t) : \tag{3.2}$$

There exists the function $u(t)$ such that

$$ {}_0I_t^{1-\gamma_m} y(t) = {}_0I_t^{[\alpha_m]} u(t), \quad u\,(t) = D^{\alpha_m - 1} y\,(t)\,, \tag{3.3}$$

(cf. [1], Lemma 3.5). Then by (3.2) and (3.3) we have

$$ {}_0D_t^{\alpha_m} y(t) = D\ D^{[\alpha_m]}\ {}_0I_t^{[\alpha_m]} u(t) = Du(t). \tag{3.4}$$

Also by (3.3) for ${}_0D_t^{\alpha_i} y(t),\ 0, ..., m-1$ it holds

$$\begin{aligned} {}_0D_t^{\alpha_i} y(t) &= D^{[\alpha_i]+1}\ {}_0I_t^{1-\gamma_i} y(t) \\ &= D^{[\alpha_i]+1}\ {}_0I_t^{\gamma_m - \gamma_i + 1 - \gamma_m} y(t) \\ &= D^{[\alpha_i]+1}\ {}_0I_t^{\gamma_m - \gamma_i + [\alpha_m]} u(t) \\ &= {}_0I_t^{\gamma_m - \gamma_i + [\alpha_m] - \alpha_{m-1} + \alpha_{m-1} - [\alpha_i] - 1} u(t) \\ &= {}_0I_t^{\beta_i} u(t), \end{aligned} \tag{3.5}$$

where $\beta_i = \alpha_m - \alpha_{m-1} + \alpha_{m-1} - \alpha_i - 1$. We have to show that $\beta_i > 0,\ i = 0, ..., m-1$.

Since $\alpha_{m-1} - \alpha_i > 0,\ i = 0, ..., m-2$ and if $\alpha_m - \alpha_{m-1} > 1$, all the numbers $\beta_i,\ 0, ..., m-2$, are positive. It remains to examine β_{m-1}. If $\alpha_m - \alpha_{m-1} > 1$, then $\beta_{m-1} = \alpha_m - \alpha_{m-1} - 1 > 0$, as well.

Also if $\alpha_m - \alpha_{m-1} = 1$, the numbers $\beta_i,\ i = 0, ..., m-2$ remain positive; but $\beta_{m-1} = 0$ and we have

$$ {}_0D_t^{\alpha_{m-1}} y = {}_0D_t^{\alpha_m - 1} y = D^{[\alpha_m]-1+\gamma_m} y = D^{[\alpha_m]}{}_0I_t^{1-\gamma_m} y. $$

With (3.3) this gives ${}_0D_t^{\alpha_{m-1}} y = u$.

Consequently for $\alpha_m - \alpha_{m-1} = 1$ we can use (3.5) also for $i = m-1$ but accepting that ${}_0I_t^0 u(t) = u(t)$.

With (3.3), (3.4), (3.5) for $1 < \alpha_m - \alpha_{m-1}$ equation (3.1) changes its analytical form in

$$ Du(t) - \lambda \sum_{i=0}^{m-2} A_i(t){}_0I_t^{\beta_i} u(t) - \lambda A_{m-1}(t){}_0I_t^{\beta_{m-1}} u(t) = f(t), \tag{3.6} $$

$$ \begin{aligned} D\Big(u(t) \quad - \quad & \lambda \sum_{i=0}^{m-2} \int_0^t A_i(\tau) \frac{1}{\Gamma(\beta_i)} \int_0^\tau (\tau - s)^{\beta_i - 1} u(s) ds d\tau - \\ - \quad & \lambda \int_0^t A_{m-1}(\tau){}_0I_t^{\beta_{m-1}} u(\tau) d\tau - \int_0^t f(\tau) d\tau - c\Big) = 0,\ 0 \le t \le b. \end{aligned} $$

The form of this equation is different for $1 < \alpha_m - \alpha_{m-1}$:

$$ u(t) - \lambda \sum_{i=0}^{m-1} \int_0^t A_i(\tau) \frac{1}{\Gamma(\beta_i)} \int_0^t (\tau - s)^{\beta_i - 1} u(s) ds d\tau = \int_0^t f(\tau) d\tau + c \tag{3.7} $$

and for $1 = \alpha_m - \alpha_{m-1}$:

$$ \begin{aligned} u(t) \quad - \quad & \lambda \sum_{i=0}^{m-2} \int_0^t A_i(\tau) \frac{1}{\Gamma(\beta_i)} \int_0^\tau (\tau - s)^{\beta_i - 1} u\,(s)\, ds d\tau - \\ - \quad & \lambda \int_0^t A_{m-1}(\tau) u(\tau) d\tau = \int_0^t f(\tau) d\tau + c. \end{aligned} \tag{3.8} $$

Interchanging the order of integration in (3.7) and (3.8) supposing that all $A_i(t),\ i = 0, ..., m-1$ are bounded on $[0, b]$, and by Dirichlet's formula we arrive from (3.7) at:

$$ u(t) - \lambda \sum_{i=0}^{m-1} \int_0^t u(s) \int_s^t A_i(\tau) \frac{(\tau - s)^{\beta_i - 1}}{\Gamma(\beta_i)} d\tau ds = \int_0^t f(\tau) d\tau + c. $$

Upon transposing the variables τ and s we finally have for (3.7)

$$u(t)-\lambda\sum_{i=0}^{m-1}\int_0^t u(\tau)\int_\tau^t A_i(s)\frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}dsd\tau \;=\; \int_0^t f(\tau)d\tau+c,\ 0\le t\le b. \tag{3.9}$$

In the same way for (3.8) we arrive at

$$\begin{aligned} u(t) \;-\; &\lambda\int_0^t u(\tau)\left(\sum_{i=0}^{m-2}\int_\tau^t A_i(s)\frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}ds-A_{m-1}(\tau)\right)d\tau = \\ =\; &\int_0^t f(\tau)d\tau+c,\ 0\le t\le b, \end{aligned} \tag{3.10}$$

or

$$u(t)-\lambda\int_0^t K(t,\tau)u(\tau)=F(t),\quad t\in[0,b].$$

Equations (3.9) and (3.10) are the Volterra integral equations of the second kind. The kernels have different form. Therefore we start first with the kernel $K(t,\tau)$ of equation (3.9),

$$K(t,\tau)=\sum_{i=0}^{m-1}\int_\tau^t A_i(s)(s-\tau)^{\beta_i-1}\frac{1}{\Gamma(\beta_i)}ds. \tag{3.11}$$

Since $\beta_i>0,\ i=0,...,m-1$, and since

$$|K(t,\tau)|\le M\sum_{i=0}^{m-1}\int_\tau^t\frac{(s-\tau)^{\beta_i-1}}{\Gamma(\beta_i)}=M\sum_{i=0}^{m-1}\frac{(t-\tau)^{\beta_i}}{\Gamma(\beta_i+1)}, \tag{3.12}$$

where $M=\max\limits_{0\le t\le b,\ i=0,...,m-1}|A_i(t)|$, the kernel. $K(t,\tau)$ is bounded and integrable on $T(0,b)=0\le\tau\le t,\ 0\le t\le b$, (cf. (3.11),(3.12)).

It is easily seen that the same properties has the kernel for the equation (3.10).

Theorem 3.1. *Suppose that in equation (3.1): 1)* $1\le\alpha_m-\alpha_{m-1}$*; 2) the functions* $A_i(t),\ i=0,...,m-1,$ *are bounded functions on* $[0,b]$ *and 3) the function* $f\in\mathcal{J}_0([0,b])$*. Then the Volterra integral equations (3.9) and*

(3.10) have one and only one family of bounded solutions $u(t,c_1)$ *depending on* $c_1 \in \mathbb{R}$ *and is given by:*

$$u(t,c_1) = F(t) + \lambda \int_0^t \mathcal{R}(t,\tau,\lambda)F(\tau)d\tau,\ 0 \le t \le b, \tag{3.13}$$

where the resolvent kernel $\mathcal{R}$ *is given by the series*

$$\mathcal{R}(t,\tau\lambda) = K(t,\tau) + \sum_{n=1}^{\infty} \lambda^n K_n(t,\tau) \tag{3.14}$$

convergent for all values of λ. *The function* $F(t) = \int_0^t f(\tau)d\tau + c$.

P r o o f. If we suppose that $f \in \mathcal{J}_0([0,b])$, then the function $F(t) \equiv \int_0^t f(\tau)d\tau + c$ is bounded on $[0,b]$ and integrable (cf. [2], p. 100). Then in equations (3.9), (3.10) the kernel $K(t,\tau)$ is bounded and integrable on the triangle $T(0,b)$ and the function $F(t)$ is bounded and integrable on $[0,b]$. We can apply the theorem for the Volterra integral equation (cf. for example [8], p.13). This proves Theorem 3.1. □

3.2 *Case* $0 < \alpha_m - \alpha_{m-1} < 1$

Let $0 < \alpha_m - \alpha_{m-1} < 1$, which gives

$$0 < [\alpha_m] - [\alpha_{m-1}] + h_m - h_{m-1} < 1.$$

This can be if:

$$\text{a)}\quad [\alpha_m] - [\alpha_{m-1}] = 0,\ h_m > h_{m-1}, \tag{3.15}$$

or

$$\text{b)}\quad [\alpha_m] - [\alpha_{m-1}] = 1,\ h_{m-1} > h_m. \tag{3.16}$$

We will transform equation (3.1) as in previous case.

With the supposition that there exists $v(t)$ such that

$${}_0D_t^{\gamma_m} y = {}_0I_t^{[\alpha_m]-\varepsilon} v(t),\ 0 \le t \le b, \tag{3.17}$$

where $\varepsilon > 0$ and it will be fixed later on.

We start with ${}_0D_t^{\alpha_m}y$ using (3.17)

$$\begin{aligned}{}_0D_t^{\alpha_m}y(t) &= D^{[\alpha_m]}{}_0D_t^{\gamma_m}y(t) = D^{[\alpha_m]}{}_0I_t^{[\alpha_m]-\varepsilon}v(t)\\ &= {}_0D_t^{\varepsilon}v(t) \quad \text{(cf. Part 2, property 3 and 4)).} \qquad (3.18)\end{aligned}$$

$$\begin{aligned}{}_0D_t^{\alpha_i}y(t) &= {}_0D_t^{\alpha_i}\left({}_0I_t^{\gamma_m}{}_0D_t^{\gamma_m}\right)y(t) + \frac{{}_0I_t^{1-\gamma_m}y(0^+)}{\Gamma(\gamma_m)}t^{\gamma_{m-1}} \quad \text{(cf. [4], p.75)}\\ &= {}_0D_t^{\alpha_i}{}_0I_t^{\gamma_m}{}_0D_t^{\gamma_m}y(t) \quad \text{(If } y(t) \text{ is supposed to be bounded)}\\ &= {}_0D_t^{\alpha_i}{}_0I_t^{\gamma_m}{}_0I_t^{[\alpha_m]-\varepsilon}v(t)\\ &= {}_0D_t^{[\alpha_m]}{}_0I_t^{[\alpha_m]-\alpha_i}{}_0I_t^{\gamma_m+[\alpha_m]-\varepsilon}v(t) \quad \text{(cf. (2.2))}\\ &= I^{[\alpha_m]-[\alpha_i]+\gamma_m-\gamma_i-\varepsilon}v(t) = I^{\delta_i}v(t), \qquad (3.19)\end{aligned}$$

where

$$\delta_i = [\alpha_m] - [\alpha_i] + \gamma_m - \gamma_i - \varepsilon. \qquad (3.20)$$

It is easily seen that ε can be fixed in such a way that $\delta_i > 0,\ i = 0, ..., m-1$ in both subcases a) and b).

Subcase a). The number ε has to be ε_1,

$$0 < \varepsilon_1 < h_m - h_{m-1}. \qquad (3.21)$$

Since $\delta_i = [\alpha_m] - [\alpha_i] + \gamma_m - \gamma_i - \varepsilon_1 = \alpha_m - \alpha_i - \varepsilon_1$, we have with (3.21) and (3.15) that $\delta_{m-1} > 0$ and that $\alpha_m - \alpha_{m-1} - \varepsilon_1 > 0$. Also for $i = 0, ..., m-2$ is

$$0 < \alpha_m - \alpha_{m-1} - \varepsilon_1 < \alpha_m - \alpha_i - \varepsilon_1 = \delta_i,\ i = 0, ..., m-2. \qquad (3.22)$$

Consequently $\delta_i > 0$ for $i = 0, ..., m-1$.

Subcase b). The number ε has to be ε_2,

$$\gamma_i - \gamma_m + \varepsilon_2 < 1,\ 0 < \varepsilon_2 < 1 + \gamma_m - \gamma_i,\ i = 0, ..., m-1.$$

Let

$$0 < \varepsilon_2 < 1 + \gamma_m - \gamma_{m-1}, \qquad (3.23)$$

then by (3.20) $\delta_{m-1} > 0$ and $\alpha_m - \alpha_{m-1} - \varepsilon_2 > 0$. Since $\alpha_m - \alpha_i - \varepsilon_2 > \alpha_m - \alpha_{m-1} - \varepsilon_2 > 0$, we have $\delta_i > 0$ for every $i = 0, ..., m-1$. Now, we can give an other form to (3.1).

With (3.18) and (3.19) equation (3.1) changes as

$$ {}_0D_t^{\varepsilon}\left(v(t)-\lambda\sum_{i=0}^{m-1}{}_0I_t^{\varepsilon}A_i(\cdot){}_0I_t^{\delta_i}v(t)-{}_0I_t^{\varepsilon}f(t)-c_2t^{\varepsilon-1}\right)=0, \tag{3.24}$$

or

$$\begin{aligned} v(t) \quad &- \quad \lambda\sum_{i=0}^{m-1}\int_0^t\frac{(t-\tau)^{\varepsilon-1}}{\Gamma(\varepsilon)}A_i(\tau)\int_0^{\tau}\frac{(\tau-s)^{\delta_i-1}}{\Gamma(\delta_i)}v(s)dsd\tau \\ &= \quad {}_0I_t^{\varepsilon}f(t)+c_2t^{\varepsilon-1}. \end{aligned}\tag{3.25}$$

We will use equation (3.25) with $c_2=0$.

Interchanging the order of integration in (3.25), supposing that all $A_i(t)$ and $v(t)$ are bounded functions on $[0,b]$ we arrive that

$$\begin{aligned} v(t) \quad &- \quad \lambda\sum_{i=0}^{m-1}\int_0^t v(s)\int_s^t A_i(\tau)\frac{(\tau-s)^{\delta_i-1}}{\Gamma(\delta_i)}\frac{(t-\tau)^{\varepsilon-1}}{\Gamma(\varepsilon)}d\tau ds= \\ &= \quad {}_0I_t^{\varepsilon}f(t),\ 0\le t\le b. \end{aligned}\tag{3.26}$$

(cf. [8], p.15).

Equation (3.26) is the Volterra integral equation of the second kind. Upon transposing the variables τ and s in it, the kernel of equation (3.26) becomes

$$K(t,\tau)=\sum_{i=0}^{m-1}\int_0^t A_i(s)\frac{(s-\tau)^{\delta_i-1}}{\Gamma(\delta_i)}\frac{(t-s)^{\varepsilon-1}}{\Gamma(\varepsilon)}ds. \tag{3.27}$$

Hence equation (3.26) can be written as

$$v(t)-\lambda\int_0^t K(t,\tau)v(\tau)d\tau=F(t). \tag{3.28}$$

The function $K(t,\tau)$ is absolutly integrable on $T(0,b)$ and the function $F(t)=\int_0^t f(\tau)d\tau$ bounded and integrable on $[0,b]$. In addition the function $K(t,\tau)$ satisfies

$$\begin{aligned} |K(t,\tau)| \quad &\le \quad M\sum_{i=0}^{m-1}\int_{\tau}^t\frac{(t-s)^{\varepsilon-1}}{\Gamma(\varepsilon)}\frac{(s-\tau)^{\delta_i-1}}{\Gamma(\delta_i)}d\tau \\ &\le \quad M\sum_{i=0}^{m-1}\frac{(t-\tau)^{\delta_i+\varepsilon-1}}{\Gamma(\delta_i+\varepsilon)}, \end{aligned}\tag{3.29}$$

where $M = \max_{0 \le t \le b,\, i=0,...,m-1} |A_i(t)|$ and $0 < \delta_i + \varepsilon + 1 < 1,\ i = 0, ..., m-1$.

In the both subcases a) and b) it is $0 < \alpha_m - \alpha_{m-1} < 1$. Also $\alpha_m - \alpha_{m-1} - \varepsilon < 1$. Then $\delta_{m-1} \le \alpha_m - \alpha_{m-1} < 1$. It follows that the kernel in the both subcases a) and b) is weakly singular, i.e.,

$$K(t,\tau) = \frac{N(t,\tau)}{(t-\tau)^{1-\delta_{m-1}}},$$

where $N(t,\tau)$ is bounded and integrable function on the triangle $T(0,b)$.

Theorem 3.2. *Suppose that in equation (3.1): $0 < \alpha_m - \alpha_{m-1} < 1$, the functions $A_i(t),\ i = 0, ..., m-1$ and the function $f(t)$ have the same properties as in Theorem 3.1. Then the singular Volterra's integral equation (3.28) has a solution $\bar{v}(t)$ which is obtained as the solution to (p-1) iterated equation (3.28) defined for a $p \in \mathbb{N}$ (cf. (3.31)). The solution $\bar{v}$ is given by*

$$\bar{v}(t) = F_p(t) + \lambda^p \int_0^t \mathcal{R}_p(t,\tau,\lambda) F_p(\tau) d\tau,\ 0 \le t \le b,$$

where

$$\begin{aligned} \mathcal{R}_p(t,\tau\,\lambda) &= (K)_p(t,\tau) + \sum_{n=1}^{\infty} \lambda^n ((K_p))_n(t,\tau); \\ F_p(t) &= F(t) + \sum_{\nu=1}^{p-1} \lambda^\nu \int_0^t K_\nu(t,\tau) F(\tau) d\tau, \quad F_1 = F; \end{aligned}$$

and

$$K_p\,(t,\tau) = \int_0^t K_{p-1}\,(t,s)\,K\,(s,\tau)\,ds, \quad K_1 = K.$$

P r o o f. Let $\bar{N}(t,\tau)$ denote the function defined on $P = [0,b] \times [0,b]$, given by

$$\bar{N}(t,\tau) = \begin{cases} N(t,\tau), & (t,\tau) \in T(0,b) \\ 0, & (t,\tau) \in P \setminus T(0,b), \end{cases}$$

and $\bar{K}(t,\tau) = \frac{\bar{N}(t,\tau)}{(t-\tau)^{1-\delta_{m-1}}}$. This is a bounded and integrable function on P. Equation (3.28) can be now written as

$$v(t) - \lambda \int_0^b v(\tau) \bar{K}(t,\tau) d\tau = F(t),\ t \in [0,b], \tag{3.30}$$

where $F(t) = \int_0^t f(\tau)d\tau$. This is weakly singular Fredholm equation (cf. [8], chapter III), with the singular Kernel $\bar{K}(t,\tau) = \bar{N}(t,\tau) \setminus (1-\tau)^{1-\delta_{m-1}}$, $0 < \delta_{m-1} < 1$. Equation (3.30) may be reduced to the iterated equation with a bounded kernel. (cf. [8], p.81)

$$v(t) - \lambda^p \int_0^b v(\tau)(\bar{K})_p(t,\tau)d\tau = F_p(t), \tag{3.31}$$

where $(\bar{K})_p(t,\tau) = \int_0^b (\bar{K})_{p-1}(t,s)\bar{K}(s,\tau)ds$, $\bar{K}_1 = \bar{K}$, and

$$F_p(t) = F(t) + \sum_{\nu=1}^{p-1} \lambda^\nu \int_0^b (\bar{K})_\nu(t,\tau)F(\tau)d\tau,\ F_1 = F.$$

Equation (3.31) is equivalent to equation (3.28); every solution to (3.28) is a solution to (3.31) and every solution to equation (3.31) is solution to (3.28). (cf. [8] c. III 3).

We can show that equation (3.31) is also an integral equation of the second kind but of Volterra's type.

By definition, for $p = 2, ...$ we have

$$(\bar{K})_p(t,\tau) = \int_0^b (\bar{K})_{p-1}(t,s)\bar{K}(s,\tau)ds,\ K_1 = K.$$

For $p = 2$ it is

$$(\bar{K})_2(t,\tau) = \int_0^b \bar{K}(t,s)\bar{K}(s,\tau)ds.$$

By the property of $\bar{K}(t,\tau)$

$$\begin{aligned}(\bar{K})_2(t,\tau) &= \begin{cases} \int_\tau^t K(t,s)K(s,\tau)d\tau, & (t,\tau) \in T(0,b) \\ 0, & (t,\tau) \in P \setminus T(0,b) \end{cases} \\ &= \left\langle \begin{array}{ll} K_2(t,\tau), & (t,\tau) \in T(0,b) \\ 0, & P \setminus T(0,b) \end{array} \right. \\ &= \overline{(K_2)}(t,\tau),\ (t,\tau) \in P.\end{aligned}$$

Now we proced by induction to obtain that $(\bar{K})_p = \overline{(K_p)}$ if we suppose that $(\bar{K})_{p-1} = \overline{(K_{p-1})}$.

Since $(\bar{K})_p = \overline{(K_p)}$ is bounded, (3.31) is a Volterra integral equation of the second kind with bounded kernel. But that is the case elaborated in 3.1. Thus we can apply Theorem 3.1 to equation (3.31) and the proof of Theorem 3.2 is completed. □

4. *Solutions to linear equation 2.1*

Theorem 4.3. *Suppose that* $1 \le \alpha_m - \alpha_{m-1}$ *and that* $f(t) \in \mathcal{J}_0([0,b])$. *If the functions* $A_i(t),\ i = 0, ..., m-1$, *are bounded and integrable on* $[0,b]$, *then equation (3.1) has a family of solutions* $y(t,c_1) = {}_0I_t^{\alpha_m-1}u(t,c_1) \in \mathcal{J}_{[\alpha_m]+1}([0,b])$, *where* $\alpha_m \neq [\alpha_m]$; $u(t,c_1)$ *is the unique solution to (3.9) and (3.10), for every* $c_1 \in \mathbb{R}$, *and which is given by (3.13).*

P r o o f. We recall the following facts:

A necessary and sufficient condition that ${}_0D_t^{\alpha_m}y(t,c_1)$ exists and that ${}_0I_t^{1-\gamma_m}y(t,c_1)$ belongs to $\mathcal{J}_{[\alpha_m]+1}([0,b])$ is that ${}_0D_t^{\alpha_m}y(t,c_1) \in \mathcal{J}_0([0,b])$. (cf. [1]), Lemma 2.3).

If $y(t,c_1) \in \mathcal{J}_{[\alpha_m]+1}([0,b])$, then $y^{(j)}(t,c_1),\ j = 0, ..., [\alpha_m]$, are bounded and integrable functions (cf. [1], Lemma 2.2).

If ${}_0D_t^{\alpha_m}y(t,c_1)$ exists, then ${}_0D_t^{\alpha_i}y(t,c_1) \in \mathcal{J}_{[\alpha_m]+1}([0,b]),\ i = 0, ..., m-1$, as well. This is a consequence of the following: Since $y^{(j)}(t,c_1),\ j = 0, ..., [\alpha_m]$ are bounded,

$$\left({}_0I_t^{1-\gamma_i}y(\cdot,c_1)\right)^{(j)}(0^+) = 0,\ j = 0, ..., [\alpha_m]-1.$$

By Lemma 3.5 in [1] it follows that ${}_0D_t^{\alpha_i}y(t,c_1)$, exist and ${}_0D_t^{\alpha_i}y(t,c_1) \in \mathcal{J}_0([0,b]),\ i = 0, ..., m-1$.

Thus, if we have a solution $u(t,c)$ to integral equation (3.9) or (3.10) such that ${}_0I_t^{1-\gamma_m}y(t,c) = {}_0I_t^{[\alpha_m]}u(t,c)$, then $y(t,c)$ is a solution to the equation of the form (3.1) if ${}_0D_t^{\alpha_m}y(t,c) \in \mathcal{J}_0([0,b])$. Therefore we have first to prove that ${}_0D_t^{\alpha_m}y(t,c) \in \mathcal{J}_0([0,b])$, where $y(t,c_1) = {}_0I_t^{\alpha_m-1}u(t,c_1)$. This relation between y and u follows from (3.3): ${}_0I_t^{1-\gamma_m}y = {}_0I_t^{[\alpha_m]}u$, which gives $y = {}_0D_t^{1-\gamma_m}\ {}_0I_t^{[\alpha_m]}u = {}_0I_t^{\alpha_m-1}u$.

Since

$$\begin{aligned} {}_0D_t^{\alpha_m}y(t) &= D\ D^{[\alpha_m]}\ {}_0I_I^{1-\gamma_m}y(t) = D\ D^{[\alpha_m]}\ {}_0I_t^{[\alpha_m]}u(t,c_1) \\ &= Du(t,c_1), \end{aligned}$$

we have to prove that $Du(t, c_1) \in \mathcal{J}_0([0, b])$.

It is easily seen that

$$D\,F(t) = f(t) \in \mathcal{J}_0([0, b]). \tag{4.1}$$

Consider

$$\begin{aligned}\int_0^t \mathcal{R}(t, \tau, \lambda) F(\tau) d\tau &= \int_0^t K(t, \tau) F(\tau) d\tau + \\ &+ \int_0^t \sum_{n=1}^{\infty} \lambda^n K_n(t, \tau) F(\tau) d\tau. \end{aligned} \tag{4.2}$$

Then by (3.11)

$$\begin{aligned}\int_0^t K(t, \tau) F(\tau) d\tau &= \int_0^t \sum_{i=0}^{m-1} \int_\tau^t A_i(s)(s-\tau)^{\beta_i - 1} \frac{1}{\Gamma(\beta_i)} F(\tau) ds d\tau \\ &= \int_0^t \int_0^s \sum_{i=0}^{m-1} A_i(s)(s-\tau)^{\beta_i - 1} \frac{1}{\Gamma(\beta_i)} F(\tau) d\tau ds.\end{aligned}$$

and

$$\frac{d}{dt} \int_0^t K(t, \tau) F(\tau) d\tau = \int_0^t \sum_{i=0}^{m-1} A_i(\tau)(t-\tau)^{\beta_i - 1} \frac{1}{\Gamma(\beta_i)} F(\tau) d\tau. \tag{4.3}$$

Thus

$$\frac{d}{dt} \int_0^t K(t, \tau) F(\tau) d\tau,$$

is a bounded and integrable function.

Using (3.13) we can prove that

$$\begin{aligned}\int_0^t \sum_{n=1}^{\infty} \lambda^n K_n(t, \tau) F(\tau) d\tau &= \sum_{n=1}^{\infty} \lambda^n \int_0^t K_n(t, \tau) F(\tau) d\tau = \\ &= \sum_{n=1}^{\infty} \lambda^n \int_0^t \int_\tau^t K(t, s) K_{n-1}(s, \tau) ds F(\tau) d\tau,\end{aligned} \tag{4.4}$$

i.e., that the interchanging of the integral and the series, and the order of two integrals are correct.

The procedure to find the derivative of the function (4.4) and to show that it is a bounded and integrable function is just the same as for (4.3). Only additionally we have to show that we can take the derivative of the series in (4.4) by taking the derivatives of every member of the series, i.e., that the following operations are valid:

$$\begin{aligned} & \frac{d}{dt}\int_0^t\int_\tau^t K(t,s)K_{n-1}(s,\tau)dsF(\tau)d\tau \\ = & \frac{d}{dt}\int_0^t\int_0^s K(t,s)K_{n-1}(s,\tau)F(\tau)d\tau ds \\ = & \int_0^t K(t,s)K_{n-1}(s,\tau)F(\tau)d\tau \\ + & \int_0^t\int_0^s K'_t(t,\tau)K_{n-1}(s,\tau)F(\tau)d\tau ds. \end{aligned}$$

This is true because of the properties of the functions $K(t,\tau)$, $K'_t(t,\tau)$ and $F(\tau)$. This completes the proof. □

Theorem 4.4. *Suppose that in equation (3.1) we have:* 1) $0 < \alpha_m - \alpha_{m-1} < 1$; 2) *the functions* $A_i(t)$, $i = 0, ..., m-1$ *are bounded on* $[0,b]$*;* 3) $f(t) \in \mathcal{J}_0([0,b])$*. Then equation (3.1) has a solution* $y(t)$ *such that* $y(t) = {}_0I_t^{\alpha_m-\varepsilon}v(t)$*, for a fixed* $\varepsilon > 0$*, where* $\alpha_m \neq [\alpha_m]$ *and* $v(t)$ *is the solution to the* $(p-1)$ *iterated weakly singular Fredholm integral equation (3.31).*

P r o o f. There always exists p_0 such that for $p > p_0$ the iterated weakly singular kernels $(\bar{K})_p$ are bounded (cf. [8], p.81). Also all the solutions of the iterated equation (3.31) are solutions to (3.30) (cf. [8], Chapter III, 3). But equation (3.31) is in reality Volterra's equation (because of the kernel $(\bar{K})_p$). So, equation (3.31) is a Volterra's equation with a bounded kernel. Therefore we can follow the proof of Theorem 4.3. The difference is only in the fact that $y(t)$ in this case is $y(t) = {}_0I_t^{\alpha_m-\varepsilon}v(t)$ and

$${}_0D_t^{\alpha_m}y(t) = {}_0D_t^{\varepsilon}v(t) = D\,{}_0I_t^{1-\varepsilon}v(t) = {}_0I_t^{1-\varepsilon}v'(t) = \frac{t^{\varepsilon}}{\Gamma(1-\varepsilon)} * v'(x).$$

Thus we have to prove that $v'(t) \in \mathcal{J}_0([0,b])$, then ${}_0D_t^{\alpha_m} y(t) \in \mathcal{J}_0([0,b])$ as the composition of two functions belonging to $\mathcal{J}_0([0,b])$. Consequently we have to repeat the procedure we applied for $u'(t)(t, c_1)$ (in Theorem 4.3) now on the solution v'.

5. *Conclusion*

In this paper we studied the existence of the solution to linear fractional differential equations with variable coefficients that are bounded functions. Other interesting problems related to such equations as initial conditions, number of linearly independent solutions, and the case when α_m is an integer will be discussed in the next paper as part II.

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Department of Mechanics
Faculty of Techical Sciences
University of Novi Sad
Serbia

Department of Mathematics and Informatics
Faculty of Sciences
University of Novi Sad
Serbia