Bulletin T.CXLV de l'Académie serbe des sciences et des arts − 2013
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 38

(a, k)-REGULARIZED (C_1, C_2)-EXISTENCE AND UNIQUENESS FAMILIES*

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(Presented at the 9th Meeting, held on December 21, 2012)

A b s t r a c t. In this paper, we introduce and analyze the class of (a, k)-regularized (C_1, C_2)-existence and uniqueness families in the setting of sequentially complete locally convex spaces. The classes of (a, k)-regularized C_1-existence families and (a, k)-regularized C_2-uniqueness families are also defined and considered. The subordination principle as well as many other structural characterizations of (local) exponentially equicontinuous (a, k)-regularized (C_1, C_2)-existence and uniqueness families are proved.

AMS Mathematics Subject Classification (2010): 47D03, 47D06, 47D99
Key Words: (a, k)-regularized (C_1, C_2)-existence and uniqueness families, (a, k)-regularized C-resolvent families, abstract Volterra equations

1. *Introduction and Preliminaries*

In recent years, considerable interest in fractional calculus has been stimulated by the applications in many fields of science and technology, including

* This research was supported by grant 174024 of Ministry of Science and Technological Development, Republic of Serbia.

physics and chemistry. It is well known (E. Bazhlekova [2], 2001) that abstract time-fractional equations with Caputo fractional derivatives can be studied by converting them into equivalent abstract Volterra equations (J. Prüss [16], 1993). On the other hand, R. deLaubenfels ([5], 1991) generalized the notion of C-regularized semigroups by introducing the classes of C_1-existence families and C_2-uniqueness families. For controlling the second order abstract differential equations, the notions of C_1-cosine existence families and C_2-cosine uniqueness families were introduced by J. Z. Zhang ([20], 2002). It is also worthwhile to mention that the ideas from [5] play a crucial role in the papers of S. W. Wang ([17], 1997) and T.-J. Xiao, J. Liang ([19], 2003). The purpose of this paper is to develop the corresponding theory for abstract Volterra equations and abstract time-fractional equations in locally convex spaces ([7]-[11]).

Now we will collect the material needed later on. By E is denoted a complex Hausdorff sequentially complete locally convex space, SCLCS for short; the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of E, and by $L(E)$ is denoted the space which consists of all continuous linear mappings from E into E. The domain, range and resolvent set of a closed linear operator A on E are denoted by $D(A)$, $R(A)$ and $\rho(A)$, respectively. Suppose F is a linear subspace of E. Then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$. Let $L(E) \ni C$ be injective. Then the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(E)\}$. The space $D_\infty(A) := \bigcap_{n\in\mathbb{N}} D(A^n)$, topologized by the following system of seminorms $p_n(x) := \sum_{j=0}^{n} p(A^j x)$ ($p \in \circledast$, $n \in \mathbb{N}$), becomes a SCLCS. The notion of local Hölder continuity of a function $f : [0,\infty) \to E$ is understood in the sense of [7]. In the case that E is a Banach space, we denote by $[D(A)]$ the Banach space $D(A)$ equipped with the graph norm.

Given $s \in \mathbb{R}$ in advance, set $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$ and $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $0^\alpha := 0$ and $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ ($\alpha > 0$, $t > 0$). If $\delta \in (0, \pi]$ and $d \in (0, 1]$, then we define $\Sigma_\delta := \{\lambda \in \mathbb{C} : \lambda \neq 0, \ |\arg \lambda| < \delta\}$ and $B_d := \{z \in \mathbb{C} : |z| \leq d\}$. Denote by $\mathcal{L}$ and $\mathcal{L}^{-1}$ the Laplace transform and its inverse transform, respectively.

Let $\alpha > 0$, let $\beta \in \mathbb{R}$ and let the Mittag-Leffler function $E_{\alpha,\beta}(z)$ be defined by $E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} z^n/\Gamma(\alpha n + \beta)$, $z \in \mathbb{C}$. In this place, we assume that $1/\Gamma(\alpha n + \beta) = 0$ if $\alpha n + \beta \in -\mathbb{N}_0$. Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$. The Wright function $\Phi_\gamma(t)$ is defined by $\Phi_\gamma(t) := \mathcal{L}^{-1}(E_\gamma(-\lambda))(t)$,

$t \geq 0$, and $\mathbf{D}_t^\alpha$ denotes the Caputo fractional derivative of order α ([2]).

Definition 1.1. ([6]-[7]) Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$ and $\beta \in (0,\pi]$.

(i) Let A be a closed linear operator on E. Then a strongly continuous operator family
$(R(t))_{t\in[0,\tau)} \subseteq L(E)$ is called a (local, if $\tau < \infty$) (a,k)-regularized C-resolvent family having A as a subgenerator iff the following holds:

(i.1) $R(t)A \subseteq AR(t),\ t \in [0,\tau)$, $R(0) = k(0)C$ and $CA \subseteq AC$,

(i.2) $R(t)C = CR(t),\ t \in [0,\tau)$ and

(i.3) $R(t)x = k(t)Cx + \int_0^t a(t-s)AR(s)x\,ds,\ t \in [0,\tau),\ x \in D(A)$;

$(R(t))_{t\in[0,\tau)}$ is said to be non-degenerate if the condition $R(t)x = 0,\ t \in [0,\tau)$ implies $x = 0$, and $(R(t))_{t\in[0,\tau)}$ is said to be locally equicontinuous if, for every $t \in (0,\tau)$, the family $\{R(s) : s \in [0,t]\}$ is equicontinuous. In the case $\tau = \infty$, $(R(t))_{t\geq 0}$ is said to be exponentially equicontinuous (equicontinuous) if there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t}R(t) : t \geq 0\}$ is equicontinuous. Furthermore, $(R(t))_{t\geq 0}$ is said to be quasi-exponentially equicontinuous (q-exponentially equicontinuous, for short) (a,k)-regularized C-resolvent family iff, for every $p \in \circledast$, there exist $M_p \geq 1$, $\omega_p \geq 0$ and $q_p \in \circledast$ such that:

$$p(R(t)x) \leq M_p e^{\omega_p t} q_p(x),\ t \geq 0,\ x \in E. \tag{1}$$

(ii) Let A be a subgenerator of a global (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$. Then it is said that $(R(t))_{t\geq 0}$ is an analytic (a,k)-regularized C-resolvent family of angle β, if there exists a function $\mathbf{R} : \Sigma_\beta \to L(E)$ satisfying that, for every $x \in E$, the mapping $z \mapsto \mathbf{R}(z)x$, $z \in \Sigma_\beta$ is analytic as well as that:

(ii.1) $\mathbf{R}(t) = R(t),\ t > 0$ and

(ii.2) $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{R}(z)x = k(0)Cx$ for all $\gamma \in (0,\beta)$ and $x \in E$;

$(R(t))_{t\geq 0}$ is said to be an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family, resp. equicontinuous analytic (a,k)-regularized C-resolvent family of angle β, if for every $\gamma \in (0,\beta)$, there exists $\omega_\gamma \geq 0$, resp. $\omega_\gamma = 0$, such that the set $\{e^{-\omega_\gamma |z|}\mathbf{R}(z) : z \in \Sigma_\gamma\}$ is equicontinuous. Furthermore, $(R(t))_{t\geq 0}$ is said to be a q-exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family of angle β, if for every $p \in \circledast$ and $\epsilon \in (0,\beta)$, there exist $M_{p,\epsilon} \geq 1$, $\omega_{p,\epsilon} \geq 0$ and $q_{p,\epsilon} \in \circledast$ such that:

$$p(R(z)x) \leq M_{p,\epsilon} e^{\omega_{p,\epsilon}|z|} q_{p,\epsilon}(x),\ z \in \Sigma_{\beta-\epsilon},\ x \in E.$$

Since there is no risk for confusion, we will identify $R(\cdot)$ and $\mathbf{R}(\cdot)$.

Henceforth we shall assume that the function $k(t)$ is a scalar-valued kernel. Consider now the following condition:

(P_0): $a(t)$ is a kernel, or $a(t)$, $k(t)$ satisfy (P1) and A is a subgenerator of a non-degenerate q-exponentially bounded (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$.

In the case that (P_0) holds, we are in a position to define the integral generator $\hat{A}$ of $(R(t))_{t\in[0,\tau)}$ by setting

$$\hat{A} := \Big\{(x,y) \in E \times E : R(t)x - k(t)Cx = \int_0^t a(t-s)R(s)y\,ds,\ t \in [0,\tau)\Big\}. \tag{2}$$

The integral generator $\hat{A}$ of $(R(t))_{t\in[0,\tau)}$ is a linear operator in E which extends any subgenerator of $(R(t))_{t\in[0,\tau)}$ and satisfies $C^{-1}\hat{A}C = \hat{A}$. The local equicontinuity of $(R(t))_{t\in[0,\tau)}$ guarantees that $\hat{A}$ is a closed linear operator in E; if, additionally,

$$A\int_0^t a(t-s)R(s)x\,ds = R(t)x - k(t)Cx,\ t \in [0,\tau),\ x \in E, \tag{3}$$

then $R(t)R(s) = R(s)R(t)$, t, $s \in [0,\tau)$, $\hat{A}$ itself is a subgenerator of $(R(t))_{t\in[0,\tau)}$ and $\hat{A} = C^{-1}AC$. For further information on subgenerators of (a,k)-regularized C-resolvent families, we refer the reader to [6]-[7] and [9]-[10].

The following definition of a (local) (a,k)-regularized C-resolvent family is motivated by the recent researches of C. Chen, M. Li [3] and C. Lizama, F. Poblete [15].

Definition 1.2. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$ and $a \neq 0$. Then a strongly continuous operator family $(R(t))_{t\in[0,\tau)}$ is called a (local, if $\tau < \infty$) (a,k)-regularized C-resolvent family iff the following conditions hold:

(i) $R(0)=k(0)C$, $R(t)C=CR(t)$, $t\in[0,\tau)$ and $R(t)R(s)=R(s)R(t)$, $t,\ s\in[0,\tau)$.

(ii) $R(s)(a*R)(t)-(a*R)(s)R(t)=k(s)(a*R)(t)C-k(t)(a*R)(s)C$, $t,\ s\in[0,\tau)$.

The notions of integral generator and local equicontinuity of $(R(t))_{t\in[0,\tau)}$, as well as the notions of (exponential, q-exponential) equicontinuity of $(R(t))_{t\geq 0}$ and (exponential, q-exponential) analyticity of $(R(t))_{t\geq 0}$ are understood in the sense of the previous definition. By a subgenerator of $(R(t))_{t\in[0,\tau)}$ we mean any closed linear operator A on E satisfying $CA\subseteq AC$, $R(t)A\subseteq AR(t)$, $t\in[0,\tau)$ and the condition (i.3) stated above.

Now we would like to compare Definition 1.1 and Definition 1.2. Suppose that A is a subgenerator of a non-degenerate, locally equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ in the sense of Definition 1.1 and that (3) holds. Using the proof of [15, Theorem 3.1] (cf. also [3]), we infer that $(R(t))_{t\in[0,\tau)}$ is an (a,k)-regularized C-resolvent family in the sense of Definition 1.2. Furthermore, if (P_0) holds, then the operator $\hat{A}$, defined by (2), equals $C^{-1}AC$ and is a subgenerator (the integral generator, in fact) of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ in the sense of Definition 1.2. Suppose, conversely, that $(R(t))_{t\in[0,\tau)}$ is a non-degenerate, locally equicontinuous (a,k)-regularized C-resolvent family in the sense of Definition 1.2, and that (P_0) holds. Then the operator $\hat{A}$ is a subgenerator (the integral generator) of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ in the sense of Definition 1.1, and (3) holds with A replaced by $\hat{A}$ therein.

2. *The Main Structural Properties of (a,k)-Regularized (C_1,C_2)-Existence and Uniqueness Families*

We start this section with the following definition.

Definition 2.1. Suppose $0<\tau\leq\infty$, $k\in C([0,\tau))$, $k\neq 0$, $a\in L^1_{loc}([0,\tau))$, $a\neq 0$ and A is a closed linear operator on E.

(i) Then it is said that A is a subgenerator of a (local, if $\tau<\infty$) mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)}\subseteq L(E)\times L(E)$ iff the mapping $t\mapsto(R_1(t)x,R_2(t)x)$, $t\in[0,\tau)$ is continuous for every fixed $x\in E$ and if the following conditions hold:

(a) $R_i(0) = k(0)C_i$, $i = 1, 2$,

(b) C_2 is injective,

(c)

$$A \int_0^t a(t-s)R_1(s)x\,ds = R_1(t)x - k(t)C_1x,\ t \in [0,\tau),\ x \in E \text{ and} \tag{4}$$

$$\int_0^t a(t-s)R_2(s)Ax\,ds = R_2(t)x - k(t)C_2x,\ t \in [0,\tau),\ x \in D(A). \tag{5}$$

(ii) Let $(R_1(t))_{t\in[0,\tau)} \subseteq L(E)$ be strongly continuous. Then it is said that A is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ iff $R_1(0) = k(0)C_1$ and (4) holds.

(c) Let $(R_2(t))_{t\in[0,\tau)} \subseteq L(E)$ be strongly continuous. Then it is said that A is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ iff $R_2(0) = k(0)C_2$, C_2 is injective and (5) holds.

The notions of (q-)exponential equicontinuity, analyticity and (q-)exponential analyticity of mild (a,k)-regularized C_1-existence families (C_2-uniqueness families) are understood in the sense of Definition 1.1. For a global mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\geq 0}$ having A as subgenerator, it is said that is (q-)exponentially equicontinuous (analytic, (q-)exponentially analytic) iff both $(R_1(t))_{t\geq 0}$ and $(R_2(t))_{t\geq 0}$ are.

If $a(t) = g_\alpha(t)$ for some $\alpha > 0$, then it is not difficult to prove that, for every (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ with subgenerator A, the following holds: $\bigcup_{t\in[0,\tau)} \overline{R(R_1(t))} \subseteq \overline{D(A)}$. In the case that A is a subgenerator of a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)}$, we have intuitively that $R_1(t) = E_\alpha(t^\alpha A)C_1$ and $R_2(t) = C_2E_\alpha(t^\alpha A)$ for $t \in [0,\tau)$. Further on, it is clear that the notion of a mild (a,k)-regularized C_2-uniqueness family is more general than that of an (a,k)-regularized C-resolvent family. Observe also that the notion of a mild (a,k)-regularized C_1-existence family extends the notion of a global n times integrated C-existence family ($n \in \mathbb{N}_0$), introduced by S. S. Wang [17, Definition 3.1] in the Banach space setting (for the exponential case, cf. [5, Definition 4.1]). It could be interesting to transfer the assertion of

[17, Theorem 3.3] to mild (a,k)-regularized (C_1,C_2)-existence and uniqueness families (cf. also [11, Section 2] for some other recent results in this direction).

Notice that (4)-(5) together imply that, for every $0 \le t,\ s < \tau$ and $x \in E$,

$$\begin{aligned}(a*R_2)(s)R_1(t)x &= (a*R_2)(s)[A(a*R_1)(t)x + k(t)C_1x]\\ &= k(t)(a*R_2)(s)C_1x + (a*R_2)(s)A(a*R_1)(t)\\ &= k(t)(a*R_2)(s)C_1x + R_2(s)(a*R_1)(t)x - k(s)C_2(a*R_1)(t)x.\end{aligned}$$

This motivates the introduction of the following definition of a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family, slightly different from the previous one.

Definition 2.2. Suppose $0 < \tau \le \infty$, $k \in C([0,\tau))$, $k \ne 0$, $a \in L^1_{loc}([0,\tau))$ and $a \ne 0$. Then it is said that a strongly continuous operator family $(R_1(t),R_2(t))_{t\in[0,\tau)} \subseteq L(E)\times L(E)$ is a (local, if $\tau < \infty$) mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family iff the following conditions hold:

(i) $R_i(0) = k(0)C_i$, $i = 1,2$,

(ii) C_2 is injective,

(iii) for every $0 \le t,\ s < \tau$ and $x \in E$, the following equality holds:

$$\begin{aligned}&(a*R_2)(s)R_1(t)x - R_2(s)(a*R_1)(t)x\\ &\qquad = k(t)(a*R_2)(s)C_1x - k(s)C_2(a*R_1)(t)x.\end{aligned} \tag{6}$$

A closed linear operator A acting on E is said to be a subgenerator of $(R_1(t),R_2(t))_{t\in[0,\tau)}$ iff (4)-(5) hold.

We shall occasionally use the following condition

(P): $a(t)$ is a kernel, or $a(t)$, $k(t)$ satisfy (P1) and $(R_2(t))_{t\ge 0}$ is a non-degenerate operator family satisfying (1) with $R(\cdot)$ replaced by $R_2(\cdot)$ therein.

No matter which one of the introduced definitions of (a,k)-regularized (C_1,C_2)-existence and uniqueness family one uses, the validity of condition (P) implies that we can define the integral generator $\hat{A}$ of $(R_1(t),R_2(t))_{t\in[0,\tau)}$ by setting

$$\hat{A} := \Big\{(x,y) \in E \times E : R_2(t)x - k(t)C_2x = \int_0^t a(t-s)R_2(s)y\,ds,\ t \in [0,\tau)\Big\}. \tag{7}$$

Certainly, $\hat{A}$ is a linear operator and the local equicontinuity of $(R_2(t))_{t\in[0,\tau)}$ implies that $\hat{A}$ is closed. Moreover, if $R_2(t)C_2 = C_2R_2(t)$, $t \in [0,\tau)$, then $C_2^{-1}\hat{A}C_2 = \hat{A}$; the notion of integral generator of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ can be also understood in the sense of (7). Suppose now that $(R_1(t), R_2(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized (C_1, C_2)-existence and uniqueness family in the sense of Definition 2.2. If, additionally, $(R_2(t))_{t\in[0,\tau)}$ is locally equicontinuous and (P) holds, then it readily follows from (6) that the integral generator $\hat{A}$ is a maximal subgenerator of $(R_1(t), R_2(t))_{t\in[0,\tau)}$ with respect to the set inclusion.

Remark 2.3. Suppose $a(t) \equiv t$, $k(t) \equiv 1$, E is a Banach space and A is a subgenerator of a mild (a,k)-regularized (C_1, C_2)-existence and uniqueness family in the sense of Definition 2.1. Then the proof of implication (b) $\Rightarrow$ (a) of [20, Theorem 1.8] implies that

$$\begin{aligned} 2R_2(t)R_1(s) &= C_2[R_1(t+s) + R_1(|t-s|)] \\ &= [R_2(t+s) + R_2(|t-s|)]C_1, \quad 0 \le t,\ s,\ t+s < \tau. \end{aligned} \tag{8}$$

In particular, $(R_1(t), R_2(t))_{t\in[0,\tau)}$ is a mild (C_1, C_2)-regularized cosine existence and uniqueness family in the sense of [20, Definition 1.1], provided that $\tau = \infty$. Suppose, conversely, that $(R_1(t), R_2(t))_{t\in[0,\tau)}$ is a strongly continuous operator family, $R_i(0) = C_i$, $i = 1, 2$, C_2 is injective and (8) holds. Then we may define the infinitesimal generator $\breve{A}$ of $(R_1(t), R_2(t))_{t\in[0,\tau)}$ by

$$\breve{A} := \Big\{(x,y) \in E \times E : \lim_{t\to 0+} \frac{2}{t^2}(R_2(t)x - C_2x) = C_2y\Big\}.$$

Using the proof of [20, Theorem 1.6], we get that $\breve{A}$ is a subgenerator of a mild (a,k)-regularized (C_1, C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\in[0,\tau)}$ in the sense of Definition 2.1 (Definition 2.2); moreover, $\breve{A}$ coincides with the integral generator of $(R_1(t), R_2(t))_{t\in[0,\tau)}$. The previous conclusions can be reformulated in the case that $a(t) \equiv k(t) \equiv 1$ (cf. [4, Section XVI]) or that E is a general SCLCS.

The proof of following theorem is left to the reader as an easy exercise.

Theorem 2.4. *Suppose A is a closed linear operator on E, $C_1,\ C_2 \in L(E)$, C_2 is injective, $\omega_0 \geq 0$, $a(t)$, $k(t)$ satisfy* (P1) *and $\omega \geq \max(\omega_0, abs(a), abs(k))$.*

(i) *Let $(R_1(t),R_2(t))_{t\geq 0}$ be strongly continuous and let the family $\{e^{-\omega t}R_i(t) : t \geq 0\}$ be equicontinuous for $i=1,2$.*

  (a) *Suppose $(R_1(t),R_2(t))_{t\geq 0}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator A. Then, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator $I-\tilde{a}(\lambda)A$ is injective, $R(C_1) \subseteq R(I-\tilde{a}(\lambda)A)$,*

$$\tilde{k}(\lambda)(I-\tilde{a}(\lambda)A)^{-1}C_1x = \int_0^\infty e^{-\lambda t}R_1(t)x\,dt,\ x \in E, \tag{9}$$

$$\Big\{\frac{1}{\tilde{a}(z)} : \Re z > \omega,\ \tilde{k}(z)\tilde{a}(z) \neq 0\Big\} \subseteq \rho_{C_1}(A) \tag{10}$$

  and

$$\tilde{k}(\lambda)C_2x = \int_0^\infty e^{-\lambda t}[R_2(t)x-(a*R_2)(t)Ax]\,dt,\ x \in D(A). \tag{11}$$

  (b) *Let $R_2(0) = k(0)C_2x$, $x \in E \setminus \overline{D(A)}$, let* (10) *hold, and let* (9) *and* (11) *hold for any $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$. Then $(R_1(t),R_2(t))_{t\geq 0}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator A.*

(ii) *Let $(R_1(t))_{t\geq 0}$ be strongly continuous, and let the family $\{e^{-\omega t}R_1(t) : t \geq 0\}$ be equicontinuous. Then $(R_1(t))_{t\geq 0}$ is a mild (a,k)-regularized C_1-existence family with a subgenerator A iff for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, one has $R(C_1) \subseteq R(I-\tilde{a}(\lambda)A)$ and*

$$\tilde{k}(\lambda)C_1x = (I-\tilde{a}(\lambda)A)\int_0^\infty e^{-\lambda t}R_1(t)x\,dt, \quad x \in E.$$

(iii) *Let $(R_2(t))_{t\geq 0}$ be strongly continuous, let $R_2(0) = k(0)C_2x$, $x \in E \setminus \overline{D(A)}$, and let the family $\{e^{-\omega t}R_2(t) : t \geq 0\}$ be equicontinuous. Then $(R_2(t))_{t\geq 0}$ is a mild (a,k)-regularized C_2-uniqueness family with a subgenerator A iff for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator $I-\tilde{a}(\lambda)A$ is injective and* (11) *holds.*

The subsequent theorem can be shown following the lines of the proof of [10, Theorem 2.1] (cf. also Theorem 2.4(b), [7, Theorem 3.9] and [2, Section 3]).

Theorem 2.5. *Assume $k_\beta(t)$ satisfies* (P1), $0<\alpha<\beta$, $\gamma=\alpha/\beta$ *and A is a subgenerator of an exponentially equicontinuous (g_β,k_β)-regularized C_1-existence family $(R_{1,\beta}(t))_{t\geq 0}$, resp. a q-exponentially equicontinuous (g_β,k_β)-regularized C_2-uniqueness family $(R_{2,\beta}(t))_{t\geq 0}$ satisfying that the family $\{e^{-\omega t}R_{1,\beta}(t) : t\geq 0\}$ is equicontinuous for some $\omega\geq 0$, resp.* (1) *with $R(\cdot)$ replaced by $R_{2,\beta}(\cdot)$ therein. Assume that there exist a continuous function $k_\alpha(t)$ satisfying* (P1) *and a number $\upsilon>0$ such that $k_\alpha(0)=k_\beta(0)$ and*

$$\widetilde{k_\alpha}(\lambda)=\lambda^{\gamma-1}\widetilde{k_\beta}(\lambda^\gamma),\ \lambda>\upsilon.$$

Then A is a subgenerator of an exponentially equicontinuous, resp. a q-exponentially equicontinuous, mild (g_α,k_α)-regularized C_1-existence family $(R_{1,\alpha}(t))_{t\geq 0}$, resp. mild (g_α,k_α)-regularized C_2-uniqueness family $(R_{2,\alpha}(t))_{t\geq 0}$, given by $R_{i,\alpha}(0):=k_\alpha(0)C_i,\ i=1,2$ and

$$R_{i,\alpha}(t)x:=\int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})R_{i,\beta}(s)x\,ds,\ x\in E,\ t>0,\ i=1,2.$$

Furthermore,

$$p(R_{2,\alpha}(t)x)\leq c_\gamma M_p\exp(\omega_p^{1/\gamma}t)q_p(x),\ p\in\circledast,\ t\geq 0,\ x\in E.$$

Let $p\in\circledast$. Then the following estimate holds

$$p(R_{2,\alpha}(t)x)\leq c_\gamma M_p\exp(\omega_p^{1/\gamma}t)q_p(x),\ t\geq 0,\ x\in E,$$

and the condition

$$p(R_{2,\beta}(t)x)\leq M_p(1+t^{\xi_p})e^{\omega_p t}q_p(x),\ t\geq 0,\ x\in E\ (\xi_p\geq 0),$$

resp.,

$$p(R_{2,\beta}(t)x)\leq M_p t^{\xi_p}e^{\omega_p t}q_p(x),\ t\geq 0,\ x\in E,$$

implies that there exists $M_p'\geq 1$ such that

$$p(R_{2,\alpha}(t)x)\leq M_p'(1+t^{\xi_p\gamma})(1+\omega_p t^{\xi_p(1-\gamma)})\exp(\omega_p^{1/\gamma}t)q_p(x),\ t\geq 0,\ x\in E,$$

resp.,

$$p(R_{2,\alpha}(t)x)\leq M_p' t^{\xi_p\gamma}(1+\omega_p t^{\xi_p(1-\gamma)})\exp(\omega_p^{1/\gamma}t)q_p(x),\ t\geq 0,\ x\in E.$$

Furthermore, in the above inequalities we may replace $R_{2,\alpha}(\cdot)$ and ω_p by $R_{1,\alpha}(\cdot)$ and ω respectively. We also have the following:

(i) *The mapping* $t \mapsto R_{i,\alpha}(t)$, $t>0$ *admits an extension to* $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ *and, for every* $x \in E$, *the mapping* $z \mapsto R_{i,\alpha}(z)x$, $z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ *is analytic* ($i=1,2$).

(ii) *Let* $\varepsilon \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi))$. *If, for every* $p \in ⊛$, *one has* $\omega_p = 0$, *then* $(R_{1,\alpha}(t))_{t\geq 0}$ *is an equicontinuous analytic* (g_α,k_α)*-regularized* C_1*-existence family of angle* $\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)$, *resp.* $(R_{2,\alpha}(t))_{t\geq 0}$ *is an equicontinuous analytic* (g_α,k_α)*-regularized* C_2*-uniqueness family of angle* $\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)$.

(iii) *If* $\omega_p > 0$ *for some* $p \in ⊛$, *then* $(R_{1,\alpha}(t))_{t\geq 0}$ *is an exponentially equicontinuous, analytic* (g_α,k_α)*-regularized* C_1*-existence family of angle* $\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})$, *and* $(R_{2,\alpha}(t))_{t\geq 0}$ *is a q-exponentially equicontinuous, analytic* (g_α,k_α)*-regularized* C_2*-uniqueness family of angle* $\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})$.

Notice that it is not clear how one can prove an analogue of Theorem 2.5 for a general q-exponentially equicontinuous (g_β,k_β)-regularized C_1-existence family $(R_\beta(t))_{t\geq 0}$.

The main objective in the subsequent theorem, whose standard proof is omitted, is to transfer the assertion of subordination principle [7, Theorem 2.11] to mild exponentially equicontinuous (a,k)-regularized (C_1,C_2)-existence and uniqueness families. The notions of completely positive, creep and log-convex functions are understood in the sense of [16] and the n-th convolution power of the kernel $a(t)$ is denoted by $a^{*n}(t)$.

Theorem 2.6. *Suppose* $C_1,\ C_2 \in L(E)$ *and* C_2 *is injective.*

(i) *Let* $a(t)$, $b(t)$ *and* $c(t)$ *satisfy* (P1), *and let* $\int_0^\infty e^{-\beta t}|b(t)|\,dt < \infty$ *for some* $\beta \geq 0$. *Let*

$$\alpha = \tilde{c}^{-1}\Big(\frac{1}{\beta}\Big) \text{ if } \int\limits_0^\infty c(t)\,dt > \frac{1}{\beta},\ \alpha = 0 \text{ otherwise,}$$

and let $\tilde{a}(\lambda) = \tilde{b}(\frac{1}{\tilde{c}(\lambda)})$, $\lambda \geq \alpha$. *Let* A *be a subgenerator of a* (b,k)*-regularized* C_1*-existence family* $(R_1(t))_{t\geq 0}$ *((*(b,k)*-regularized* C_2*-uniqueness family* $(R_2(t))_{t\geq 0}$*) satisfying that the family* $\{e^{-\omega_b t}R_1(t) : t \geq 0\}$ *(*$\{e^{-\omega_b t}R_2(t) : t \geq 0\}$*) is equicontinuous for some* $\omega_b \geq 0$. *Assume, further, that* $c(t)$ *is completely positive and that there exists a* *scalar-valued continuous kernel* $k_1(t)$ *satisfying* (P1) *and*

$$\tilde{k_1}(\lambda)=\frac{1}{\lambda\tilde{c}(\lambda)}\tilde{k}\Bigl(\frac{1}{\tilde{c}(\lambda)}\Bigr),\ \lambda>\omega_0,\ \tilde{k}\Bigl(\frac{1}{\tilde{c}(\lambda)}\Bigr)\neq 0,\ \textit{for some } \omega_0>0.$$

Let

$$\omega_a=\tilde{c}^{-1}\Bigl(\frac{1}{\omega_b}\Bigr)\ \textit{if}\ \int\limits_0^\infty c(t)\,dt>\frac{1}{\omega_b},\ \omega_a=0\ \textit{otherwise.}$$

Then, for every $r\in(0,1]$, A *is a subgenerator of a global* (a,k_1*g_r)*-regularized* C_1*-existence family* $(R_{r,1}(t))_{t\geq 0}$ *(*(a,k_1*g_r)*-regularized* C_2*-uniqueness family* $(R_{r,2}(t))_{t\geq 0}$*) such that the family* $\{e^{-\omega_a t}R_{r,i}(t):t\geq 0\}$ *is equicontinuous and that the mapping* $t\mapsto R_{r,i}(t)$, $t\geq 0$ *is locally Hölder continuous with exponent* r, *if* $\omega_b=0$ *or* $\omega_b\tilde{c}(0)\neq 1$ *(*$i=1,2$*), resp., for every* $\varepsilon>0$, *there exists* $M_\varepsilon\geq 1$ *such that the family* $\{e^{-\varepsilon t}R_{r,i}(t):t\geq 0\}$ *is equicontinuous and that the mapping* $t\mapsto R_{r,i}(t)$, $t\geq 0$ *is locally Hölder continuous with exponent* r, *if* $\omega_b>0$ *and* $\omega_b\tilde{c}(0)=1$ *(*$i=1,2$*). Furthermore, if* A *is densely defined, then* A *is a subgenerator of a global* (a,k_1)*-regularized* C_1*-existence family* $(R_1(t))_{t\geq 0}$ *(*(a,k_1)*-regularized* C_2*-uniqueness family* $(R_2(t))_{t\geq 0}$*) such that the family* $\{e^{-\omega_a t}R_i(t):t\geq 0\}$ *is equicontinuous, resp., for every* $\varepsilon>0$, *the family* $\{e^{-\varepsilon t}R_i(t):t\geq 0\}$ *is equicontinuous (*$i=1,2$*).*

(ii) *Suppose* $\alpha\geq 0$, A *is a subgenerator of a global exponentially equicontinuous* $(1,g_{\alpha+1})$*-regularized* C_1*-existence family (*$(1,g_{\alpha+1})$*-regularized* C_2*-uniqueness family),* $a(t)$ *is completely positive and satisfies* (P1), $k(t)$ *satisfies* (P1) *and* $\tilde{k}(\lambda)=\tilde{a}(\lambda)^\alpha$, λ *sufficiently large. Then, for every* $r\in(0,1]$, A *is a subgenerator of a locally Hölder continuous (with exponent* r*), exponentially equicontinuous* $(a,k*g_r)$*-regularized* C_1*-existence family (*$(a,k*g_r)$*-regularized* C_2*-uniqueness family); if* $\alpha=n\in\mathbb{N}$, *resp.* $\alpha=0$, *then* A *is a subgenerator of a locally Hölder continuous (with exponent* r*), exponentially equicontinuous* $(a,a^{*n}*g_r)$*-regularized* C_1*-existence family (*$(a,a^{*n}*g_r)$*-regularized* C_2*-uniqueness family) if* $\alpha=n\in\mathbb{N}$, *resp.* (a,g_{r+1})*-regularized* C_1*-existence family (*(a,g_{r+1})*-regularized* C_2*-uniqueness family). If, additionally,* A *is densely defined, then* A *is a subgenerator of an exponentially equicontinuous* $(a,1*k)$*-regularized* C_1*-existence family (*$(a,1*k)$*-regularized* C_2*-uniqueness family); if* $\alpha=n\in\mathbb{N}$, *resp.,* $\alpha=0$, *then* A *is a subgenerator of an exponentially equicontinuous* $(a,1*a^{*n})$*-regularized* C_1*-existence family (*$(a,1*a^{*n})$*-regularized* C_2*-uniqueness* *family), resp.* $(a,1)$*-regularized* C_1*-existence family ((*$a,1$*)-regularized* C_2*-uniqueness family).*

(iii) *Suppose* $\alpha \geq 0$ *and* A *is a subgenerator of an exponentially equicontinuous* $(t,g_{\alpha+1})$*-regularized* C_1*-existence family ((*$t,g_{\alpha+1}$*)-regularized* C_2*-uniqueness family). Let* $L^1_{loc}([0,\infty)) \ni c$ *be completely positive and let* $a(t)=(c*c)(t)$, $t\geq 0$. *(Given* $L^1_{loc}([0,\infty)) \ni a$ *in advance, such a function* $c(t)$ *always exists provided* $a(t)$ *is completely positive or* $a(t)\neq 0$ *is a creep function and* $a_1(t)$ *is log-convex.) Assume* $k(t)$ *satisfies* (P1) *and* $\tilde{k}(\lambda)=\tilde{c}(\lambda)^{\alpha}/\lambda$, λ *sufficiently large. Then, for every* $r\in(0,1]$, A *is a subgenerator of a locally Hölder continuous (with exponent* r*), exponentially equicontinuous* $(a,k*g_r)$*-regularized* C_1*-existence family ((*$a,k*g_r$*)-regularized* C_2*-uniqueness family); if* $\alpha=n\in\mathbb{N}$, *resp.* $\alpha=0$, *then* A *is a subgenerator of a locally Hölder continuous (with exponent* r*), exponentially equicontinuous* $(a,c^{*n}*g_r)$*-regularized* C_1*-existence family ((*$a,c^{*n}*g_r$*)-regularized* C_2*-uniqueness family), resp.* (a,g_{r+1})*-regularized* C_1*-existence family ((*a,g_{r+1}*)-regularized* C_2*-uniqueness family). If, additionally,* A *is densely defined, then* A *is a subgenerator of an exponentially equicontinuous* $(a,1*k)$*-regularized* C_1*-existence family ((*$a,1*k$*)-regularized* C_2*-uniqueness family); if* $\alpha=n\in\mathbb{N}$, *resp.* $\alpha=0$, *then* A *is a subgenerator of an exponentially equicontinuous* $(a,1*c^{*n})$*-regularized* C_1*-existence family ((*$a,1*c^{*n}$*)-regularized* C_2*-uniqueness family), resp.* $(a,1)$*-regularized* C_1*-existence family ((*$a,1$*)-regularized* C_2*-uniqueness family).*

Proposition 2.7. *Suppose* $(R_1(t),R_2(t))_{t\in[0,\tau)}$ *is a mild* (a,k)*-regularized* (C_1,C_2)*-existence and uniqueness family with a subgenerator* A, *the family* $\{R_2(t) : t\in[0,\tau)\}$ *is locally equicontinuous, and the following condition holds:*

(P_1): *the function* $(a*k)(t)$ *is a kernel, or* $(a*k)(t)$ *satisfies* (P1), $(C_2R_1(t))_{t\geq 0}$ *and* $(R_2(t)C_1)_{t\geq 0}$ *satisfy* (1) *with* $R(\cdot)$ *replaced by* $C_2R_1(\cdot)$ $(R_1(\cdot)C_2)$ *therein* $(i=1,2)$*.*

Then $C_2R_1(t)=R_2(t)C_1$, $t\in[0,\tau)$.

P r o o f. Let $x\in E$ be fixed. Then the mapping $t\mapsto (a*R_2)(t)x$, $t\in[0,\tau)$ is continuous. Due to the local equicontinuity of the family $\{R_2(t) : t\in[0,\tau)\}$, the mappings $t\mapsto (R_2*(a*R_1))(t)x$, $t\in[0,\tau)$ and $t\mapsto ((a*$

$R_2) * R_1)(t)x$, $t \in [0, \tau)$ are also continuous and coincide. Therefore, for every $0 \leq t < \tau$,

$$\begin{aligned} R(t,x) := \; & -[(a * R_2) * (R_1(\cdot) - k(\cdot)C_1)](t)x \\ & +[(k(0) - k(\cdot)) * C_2(a * R_1)](t)x + (R_2 * (a * R_1))(t)x \\ = \; & [(k(0) - k(\cdot)) * C_2(a * R_1)](t)x - [(a * R_2) * k(\cdot)C_1](t)x. \end{aligned}$$

On the other hand, a trivial computation involving the equalities (4)-(5) shows that

$$R(t,x) = [k(0) * C_2(a * R_1)](t)x, \quad 0 \leq t < \tau.$$

The above implies $(a * k * C_2R_1)(t)x = (a * k * R_2C_1)(t)x$, $t \in [0, \tau)$, which completes the proof by (P_1).

Of importance is the following abstract Volterra equation:

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds, \; t \in [0, \tau), \tag{12}$$

where $f \in C([0, \tau) : E)$. A function $u \in C([0, \tau) : E)$ is called a *mild solution*, resp. a *strong solution*, of (12) iff $(a * u)(t) \in D(A)$, $t \in [0, \tau)$ and $A(a * u)(t) = u(t) - f(t)$, $t \in [0, \tau)$, resp. $u(t) \in D(A)$, $t \in [0, \tau)$, the mapping $t \mapsto Au(t)$, $t \in [0, \tau)$ is continuous and (12) holds. Suppose $(R_1(t), R_2(t))_{t \in [0,\tau)}$ is a mild (a, k)-regularized (C_1, C_2)-existence and uniqueness family with a subgenerator A. Then it is clear that the function $t \mapsto R_1(t)x$, $t \in [0, \tau)$, resp. $t \mapsto R_2(t)x$, $t \in [0, \tau)$, is a mild solution of (12) with $f(t) = k(t)C_1x$, $t \in [0, \tau)$ $(x \in E)$, resp. a strong solution of (12) with $f(t) = k(t)C_2x$, $t \in [0, \tau)$ $(x \in D(A))$, provided additionally in the last case that $R_2(t)x \in D(A)$, $t \in [0, \tau)$ and $AR_2(t)x = R_2(t)Ax$, $t \in [0, \tau)$. Every strong solution of (12) is a mild solution of (12), while the converse statement is not true, in general. It would be a rather long to consider various types of the (exponential) C-wellposedness of the problem (12); for further information concerning the cases $a(t) \equiv 1$ and $a(t) \equiv t$, the interested reader may consult [5] and [20].

Suppose now that $(R_2(t))_{t \in [0,\tau)}$ is a locally equicontinuous C_2-uniqueness family with a subgenerator A. By the proof of [14, Theorem 2.7], we easily infer that every strong solution $u(t)$ of (12) satisfies the following equality:

$$(R_2 * f)(t) = (kC_2 * u)(t), \quad 0 \leq t < \tau. \tag{13}$$

Since $k(t)$ is a kernel and C_2 is injective, the above equality implies that (12) has at most one strong solution. Now we will prove the uniqueness of mild solutions of the problem (12). Towards this end, suppose $u_1(t)$ and $u_2(t)$ are two such solutions. Put $u(t) := u_1(t) - u_2(t)$, $t \in [0,\tau)$. Then $A(a*u)(t) = u(t)$, $t \in [0,\tau)$ and $(a*A(a*u))(t) = (a*u)(t)$, $t \in [0,\tau)$, which implies that the function $U(t) := (a*u)(t)$, $t \in [0,\tau)$ is a strong solution of (12) with $f(t) \equiv 0$. Therefore, $u(t) = AU(t) = A0 = 0$, $t \in [0,\tau)$ and we have proved the following proposition.

Proposition 2.8. *Suppose $(R_2(t))_{t\in[0,\tau)}$ is a locally equicontinuous C_2-uniqueness family with a subgenerator A. Then every strong solution $u(t)$ of* (12) *satisfies* (13). *Furthermore, the problem* (12) *has at most one strong (mild) solution.*

We continue by stating the following proposition.

Prposition 2.9. (*cf. also* [8, Theorem 2.1.11] *and* [6, Theorem 2.34]) *Assume $\tau \in (0,\infty]$, $L^1_{loc}([0,\tau)) \ni a_1(t)$ is a kernel, $L^1_{loc}([0,\tau)) \ni k(t)$ is a kernel, $a(t) = (a_1 * a_1)(t)$, $t \in [0,\tau)$ and $k_1(t) = (k * a_1)(t)$, $t \in [0,\tau)$. Let A be a closed linear operator on E, let $C_1,\ C_2 \in L(E)$, and let C_2 be injective. Put $\mathcal{A} \equiv \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ and $\mathcal{C}_i \equiv \begin{pmatrix} C_i & 0 \\ 0 & C_i \end{pmatrix}$, $i = 1,2$. Then A is a subgenerator of an (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ ((a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$) iff $\mathcal{A}$ is a subgenerator of an (a_1,k_1)-regularized $\mathcal{C}_1$-existence family $(S_1(t))_{t\in[0,\tau)}$ ((a_1,k_1)-regularized $\mathcal{C}_2$-uniqueness family $(S_2(t))_{t\in[0,\tau)}$).*

The main objective in the following theorem is to clarify a rescaling result for subgenerators of (local) $(1,k)$-convoluted C_1-existence (C_2-uniqueness) families. We omit the proof since it follows from the argumentation given in that of [8, Theorem 2.5.1].

Theorem 2.10. *Suppose $z \in \mathbb{C}$, $K(t)$ and $F(t)$ satisfy* (P1), $\Theta(t) = \int_0^t K(s)\,ds$, *$t \in [0,\tau)$, there exists $\omega > 0$ such that*

$$\frac{\tilde{K}(\lambda) - \tilde{K}(\lambda+z)}{\tilde{K}(\lambda+z)} = \int_0^\infty e^{-\lambda t} F(t)\,dt, \ \ \Re\lambda > \omega,\ \tilde{K}(\lambda+z) \neq 0,$$

and A is a subgenerator of a (local) $(1,\Theta)$-convoluted C_1-existence family $(R_{K,1}(t))_{t\in[0,\tau)}$, resp. $(1,\Theta)$-convoluted C_2-uniqueness family $(R_{K,2}(t))_{t\in[0,\tau)}$. Then $A-z$ is a subgenerator of a (local) $(1,\Theta)$-convoluted C_1-existence family $(R_{K,1,z}(t))_{t\in[0,\tau)}$, resp. $(1,\Theta)$-convoluted C_2-uniqueness family

$(R_{K,2,z}(t))_{t\in[0,\tau)}$, given by

$$R_{K,i,z}(t) := e^{-tz}R_{K,i}(t)+\int_0^t F(t-s)e^{-zs}R_{K,i}(s)\,ds,\ t\in[0,\tau),\ i=1,2. \quad (14)$$

Furthermore, in the case $\tau=\infty$, $(R_{K,i,z}(t))_{t\geq 0}$ is (q-)exponentially equicontinuous provided that $F(t)$ is exponentially bounded and that $(R_{K,i}(t))_{t\geq 0}$ is (q-)exponentially equicontinuous ($i=1,2$).

Before proceeding further, we would like to point out that the assertion of [8, Theorem 2.5.3] (cf. also [8, Remark 2.5.4(iii)] and [7, Theorem 4.2]) can be reformulated for $(1,\Theta)$-convoluted C_1-existence (C_2-uniqueness) families.

In the remaining part of the paper, we will illustrate the obtained results with some examples. First of all, it is worth noting that there exist examples of exponentially bounded, analytic (g_α,k)-regularized (C_1,C_2)-existence and uniqueness families whose angle of analyticity can be strictly greater than $\pi/2$ $(0<\alpha<1)$. In order to illustrate this, we will make use of the following adaptation of [20, Example 3.1] (cf. also [4]-[5] for the first examples of such kind). Let $E:=\{f\in C(\mathbb{R})\ ;\ \lim_{|x|\to\infty} f(x)e^{x^2}=0\}$. Then E, endowed with the norm $||f||:=\sup_{x\in\mathbb{R}}|f(x)e^{x^2}|$, $f\in E$, is a Banach space. Let $A:=d^2/dx^2$ act on E with its maximal domain and let $(C_if)(x):=e^{-x^2}f(x)$, $x\in\mathbb{R}$, $f\in E$, $i=1,2$. Put, for every $t\geq 0$, $f\in E$ and $x\in\mathbb{R}$:

$$(C_1(t)f)(x):=\frac{1}{2}(e^{-(x+t)^2}f(x+t)+e^{-(x-t)^2}f(x-t))$$

and

$$(C_2(t)f)(x):=\frac{1}{2}e^{-x^2}(f(x+t)+f(x-t)).$$

Then $(C_1(t),C_2(t))_{t\geq 0}$ is a contractive mild (C_1,C_2)-regularized cosine existence and uniqueness family generated by A, which implies by Theorem 2.5(ii) that, for every $\alpha\in(0,2)$, A is the integral generator of an exponentially bounded analytic $(g_\alpha,1)$-regularized (C_1,C_2)-existence and uniqueness family of angle $\min((\frac{2}{\alpha}-1)\frac{\pi}{2},\pi)$. Suppose now that $L^1_{loc}([0,\infty))\ni c$ is completely positive and $a(t)=(c*c)(t)$, $t\geq 0$. By Theorem 2.6(iii), A is the integral generator of an exponentially bounded $(a,1)$-regularized (C_1,C_2)-existence and uniqueness family.

It is also worth noting that the conclusions established in [12, Example 36(iii)] and [8, Example 3.1.35(ii)] are false. In the following example, we shall correct some inconsistencies in the first of two above-mentioned examples, providing in such a way a new application of subordination principles established in Theorem 2.5/Theorem 2.6.

Example 2.11. We deal with the space $L^p_\varrho(\Omega,\mathbb{C})$, where Ω is an open non-empty subset of $\mathbb{R}^n$, $\varrho:\Omega\to(0,\infty)$ is a locally integrable function, m_n is the Lebesgue measure in $\mathbb{R}^n$, $1\leq p<\infty$, and the norm of an element $f\in L^p_\varrho(\Omega,\mathbb{C})$ is given by $||f||_p:=(\int_\Omega|f(\cdot)|^p\varrho(\cdot)\,dm_n)^{1/p}$. Set $|x|:=(x_1^2+\cdot\cdot\cdot+x_n^2)^{1/2}$, $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$, $\varrho(x):=\exp(-|x|)$, $x\in\mathbb{R}^n$, $E:=L^p_\varrho(\mathbb{R}^n,\mathbb{C})$,

$$(T_1(t)f)(x):=e^{-\left(|e^tx|^2+1\right)}f(e^tx),\ t\in\mathbb{R},\ x\in\mathbb{R}^n,\ f\in E,$$

$$(T_2(t)f)(x):=e^{-\left(|x|^2+1\right)}f(e^tx),\ t\in\mathbb{R},\ x\in\mathbb{R}^n,\ f\in E,$$

$C_1=C_2:=T_1(0)$ and

$$C_i(t):=\frac{1}{2}(T_i(t)+T_i(-t)),\ t\in\mathbb{R},\ i=1,2.$$

Let A be the closure of the operator $f\mapsto\sum_{i=1}^n x_i\frac{\partial f}{\partial x_i}$, $f\in D(A)\equiv\{g\in C^1(\mathbb{R}^n:E):\ \text{supp}(g)$ is a compact subset of $\mathbb{R}^n\}$. Then it is straightforward to see that the operator A^2 is the integral generator of a mild (C_1,C_2)-regularized cosine existence and uniqueness family $(C_1(t),C_2(t))_{t\in\mathbb{R}}$. Furthermore, $(C_1(t))_{t\geq0}$ is exponentially bounded and $(C_2(t))_{t\geq0}$ is not exponentially bounded.

We close the paper with the observation that many other examples of (g_α,g_β)-regularized C_1-existence families (C_2-uniqueness families), where $\alpha>0$ and $\beta\geq1$, can be constructed by using the matricial operators (cf. [5, Section 7] and [20]).

REFERENCES

[1] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Birkhäuser Verlag, 2001.

[2] E. Bazhlekova, *Fractional evolution equations in Banach spaces,* PhD Thesis, Eindhoven University of Technology, Eindhoven, 2001.

[3] C. Chen, M. Li, *On fractional resolvent operator functions*, Semigroup Forum **80** (2010), 121–142.

[4] R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations,* Lecture Notes in Mathematics **1570**, Springer, New York, 1994.

[5] R. deLaubenfels, *Existence and uniqueness families for the abstract Cauchy problem,* J. London Math. Soc. **(2) 44** (1991), 310-338.

[6] M. Kostić, *(a, k)-regularized C-resolvent families: regularity and local properties,* Abstr. Appl. Anal. vol. 2009, Art. ID 858242, 27 pages, 2009.

[7] M. Kostić, *Abstract Volterra equations in locally convex spaces,* Sci. China Math. **55** (2012), 1797–1825.

[8] M. Kostić, *Generalized Semigroups and Cosine Functions,* Mathematical Institute Belgrade, 2011.

[9] M. Kostić, *Perturbation theory for abstract Volterra equations,* Abstr. Appl. Anal. 2013, Art. ID 307684, 26 pp.

[10] M. Kostić, *On a class of (a, k)-regularized C-resolvent families,* Electron. J. Qual. Theory Differ. Equ. 2012, No. 94, 27 pp.

[11] M. Kostić, *Some contributions to the theory of abstract Volterra equations*, Int. J. Math. Anal. (Russe) **5** (2011), 1529–1551.

[12] M. Kostić, *Hypercyclic and chaotic integrated C-cosine functions*, Filomat **26** (2012), 1–44.

[13] M. Li, Q. Zheng, J. Zhang, *Regularized resolvent families,* Taiwanese J. Math. **11** (2007), 117–133.

[14] C. Lizama, *Regularized solutions for abstract Volterra equations*, J. Math. Anal. Appl. **243** (2000), 278–292.

[15] C. Lizama, F. Poblete, *On a functional equation associated with (a, k)-regularized resolvent families*, Abstr. Appl. Anal., volume 2012 (2012), Article ID 495487, 23 pages.

[16] J. Prüss, *Evolutionary Integral Equations and Applications,* Birkhäuser-Verlag, Basel, 1993.

[17] S. W. Wang, *Mild integrated C-existence families*, Studia Math. **112** (1995), 251–266.

[18] T.-J. Xiao, J. Liang, *The Cauchy Problem for Higher-Order Abstract Differential Equations,* Springer-Verlag, Berlin, 1998.

[19] T.-J. Xiao, J. Liang, *Higher order abstract Cauchy problems: their existence and uniqueness families,* J. London Math. Soc. **(2) 67** (2003), 149-164.

[20] J. Z. Zhang, *Regularized cosine existence and uniqueness families for second order abstract Cauchy problems*, Studia Math. **152** (2002), 131-145.

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