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EXTREMELY IRREGULAR TREES

G. H. FATH–TABAR, I. GUTMAN, R. NASIRI

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A b s t r a c t. The irregularity of a graph G is defined as $irr(G) = \sum |d(x) - d(y)|$ *where* $d(x)$ *is the degree of vertex* x *and the summation embraces all pairs of adjacent vertices of* G*. We characterize the trees with the five smallest and five largest* irr*-values.*

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1. *Introduction*

In this paper we are concerned with graphs without directed, multiple, or weighted edges, and without loops. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. An edge of G, connecting the vertices u and v will be denoted by uv. The degree of a vertex v of the graph G will be denoted by $d(v|G)$ or, when misunderstanding is avoided, by $d(v)$.

As well known, a graph whose all vertices have mutually equal degrees is said to be regular. Then, a graph in which not all vertices have equal degrees can be viewed as somehow deviating from regularity. In the mathematical literature several measures of such *irregularity* were proposed; for details and additional references see [2, 3, 5, 7, 8]. One of such measures was put forward by Albertson [1], who considered the quantity

$$irr(G) = \sum_{uv \in E(G)} |d(u) - d(v)| \ .$$

This quantity is sometimes referred to as the *Albertson index* [5, 6] and in a recent work [4] it was called the *third Zagreb index*. In what follows, we shall say that $irr(G)$ is the *irregularity* of the graph G. Evidently, $irr(G) = 0$ if and only if every component of G is a regular graph.

For $uv \in E(G)$, define $irr(uv) = |d(u) - d(v)|$ and call it the *irregularity of the edge* uv. Then, of course,

$$irr(G) = \sum_{e \in E(G)} irr(e) \ .$$

In this note we characterize the most and least irregular trees (with regard to the measure irr).

Let $n \geq 3$. As usual, by P_n is denoted the n-vertex path (the tree in which the maximal vertex degree is two) and by S_n the n-vertex star (the tree possessing a vertex of degree $n-1$).

Denote by $\mathcal{T}$ the set of all trees of order greater than two.

Let $\mathcal{P} = \{\mathcal{P}_{\ni}, \mathcal{P}_{\triangle}, \mathcal{P}_{\bigtriangledown}, \ldots\}$. It is an elementary task to verify that if $T \in \mathcal{P}$, then $irr(T) = 2$.

Lemma 1.1. *In the set $\mathcal{T}$, the elements of $\mathcal{P}$, and only these, have the smallest irregularity, equal to 2.*

P r o o f. Albertson [1] proved that the irregularity of any graph is an even number. Therefore, for any tree T of order greater than two, $irr(T) \geq 2$. Consequently, the paths P_n , $n \geq 3$, are the least irregular trees.

It remains to show that these are the only trees with minimal irregularity, namely that if $T \in \mathcal{T} \setminus \mathcal{P}$, then $irr(T) > 2$.

Indeed, if $T \in \mathcal{T} \setminus \mathcal{P}$, then T possesses at least one vertex of degree greater than two. Then T possesses at least three pendent edges. Let uv be a pendent edge, such that $d(u) = 1$ and $d(v) \geq 2$. Then $irr(uv) \geq 1$ and therefore $irr(T) \geq 3$. □

For an edge e of an n-vertex graph, the maximal value that $irr(e)$ can assume is $n-2$, which happens if e is connecting a pendent vertex with a vertex of degree $n-1$. If $irr(e) = n-2$ would hold for all edges of a graph, then this graph would have maximal irregularity.

In the case of trees, this condition is obeyed by the star (and only by it). Thus we arrive at the following simple result:

Lemma 1.2. *Among trees of order n, the star S_n is the unique tree with greatest irregularity, satisfying:*

$$irr(S_n) = (n-1)(n-2) \ . \tag{1}$$

In what follows we characterize the trees with second–, third–, fourth–, and fifth-extremal (minimal and maximal) irr-values. For this we need some preparations.

2. *Auxiliary results*

Lemma 2.1. *Let G be a graph possessing a pendent edge uv, such that $d(u|G) = 1$ and $d(v|G) \geq 2$. Construct the graph G^* by inserting a new vertex x on the edge uv. Then $irr(G^*) = irr(G)$.*

P r o o f. The graph G^* has edges ux and xv, and $d(u|G^*) = 1$, $d(x|G^*) = 2$, $d(v|G^*) = d(v|G)$. Then

$$\begin{aligned} irr(G^*) - irr(G) = \ & |d(x|G^*) - d(u|G^*)| + |d(v|G^*) \\ & - d(x|G^*)| - |d(v|G) - d(u|G)| \\ = \ & [2-1] + [d(v) - 2] - [d(v) - 1] = 0 \ . \qquad \Box \end{aligned}$$

In a fully analogous manner we can demonstrate the validity of the following result:

Lemma 2.2. *Let G be a graph possessing a vertex u of degree 2 adjacent to the vertices v and v'. Construct the graph G^{**} by inserting a new vertex x on the edge uv. Then $irr(G^{**}) = irr(G)$.*

Let G be any graph. Denote by $\Gamma(G)$ the set consisting of the graph G and of the graphs obtained from G by repeated application of the transformations $G \to G^*$ and $G \to G^{**}$, specified in Lemmas 2.1 and 2.2. Then the results of the Lemmas 2.1 and 2.2 can be summarized as follows:

Theorem 2.3. *For any graph G, the irregularities of all elements of $\Gamma(G)$ are mutually equal.*

At this point it is worth noting that $\mathcal{P} \equiv -(\mathcal{P}_{\ni})$.

Theorem 2.4. *For any positive even integer k, except $k = 4$, there exist infinitely many trees whose irregularity is equal to k.*

P r o o f. The case $k = 2$ is settled by Lemma 1.1.

Let $k \geq 6$. Choose a tree $T(k)$ possessing $n_3 = k/2 - 2$ vertices of degree 3 and no vertices of degree greater than 3. This can be done for any $n_3 = 1, 2, 3, \ldots$, i.e., for any $k = 6, 8, 10, \ldots$.

The tree $T(k)$ has $k/2$ pendent edges, each with irregularity 2. All other edges of $T(k)$ connect two vertices of degree 3 and thus their irregularity is zero. Consequently, $irr(T(k)) = k$. Then by Theorem 2.3, all the infinitely many elements of $\Gamma(T(k))$ have irregularity k.

It remains to show that there are no trees with irregularity 4. By Lemma 1.1, any tree T with irregularity greater than 2 must possess a vertex of degree greater than 2. Let v be such a vertex, $d(v) \geq 3$. Then T possesses at least $d(v)$ pendent vertices. Let u be a pendent vertex and let the vertices $u, x_1, \ldots, x_p, v$ form the (unique) path connecting u and v. The edges in this path contribute to $irr(T)$ by:

$$\begin{aligned} & |d(x_1) - d(u)| + |d(x_2) - d(x_1)| + \cdots + |d(v) - d(x_p)| \\ \geq\ & [d(x_1) - d(u)] + [d(x_2) - d(x_1)] + \cdots + [d(v) - d(x_p)] \\ =\ & d(v) - d(u) \geq 3 - 1 = 2 \ . \end{aligned}$$

Consequently, $irr(T) \geq 2\, d(v) \geq 6$. □

3. *Trees with small irregularities*

Trees with the smallest irregularity are characterized in Lemma 1.1. We now characterize the next four classes of trees with smallest irr.

From Theorem 2.4 we know that for any small value of k, there will be infinitely many trees whose irregularity is equal to k. Therefore we seek for the smallest representatives of such trees. In view of Lemmas 2.1 and 2.2, such representatives must have the following properties:

(a) no pendent vertex is attached to a vertex of degree 2, and

(b) no two vertices of degree 2 are adjacent.

Theorem 3.1. *Let* T_1 , T_2 , T_3 , T_4 , T_5 , T_7 , T_8 , *and* T_{17} *be the trees depicted in Fig. 1. Let* T *be a tree. Then*
(a) $irr(T) = 6$ *if and only if* $T \in \Gamma(T_1)$*;*
(b) $irr(T) = 8$ *if and only if* $T \in \Gamma(T_2)$*;*
(c) $irr(T) = 10$ *if and only if* $T \in \Gamma(T_3) \cup \Gamma(T_4)$*;*
(d) $irr(T) = 12$ *if and only if* $T \in \Gamma(T_5) \cup \Gamma(T_7) \cup \Gamma(T_8) \cup \Gamma(T_{17})$*.*

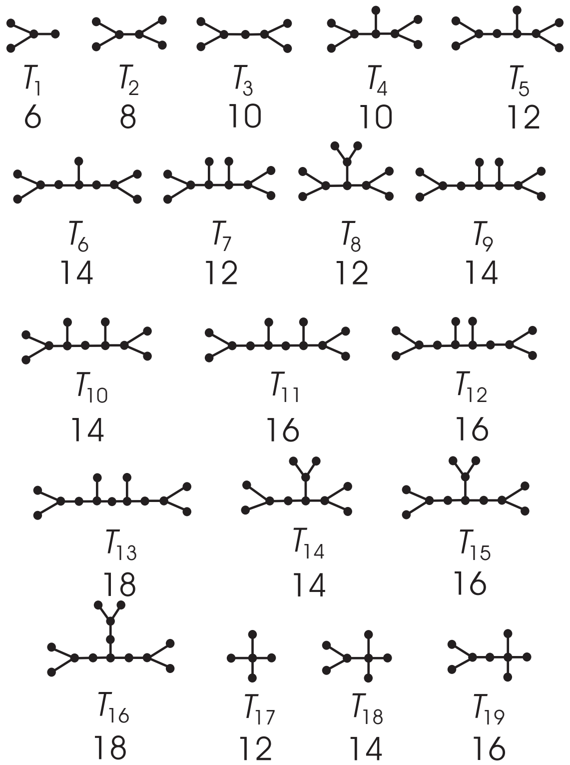

Fig. 1. Trees used in the statement and proof of Theorem 3.1.
with their *irr*-values indicated

P r o o f. Denote by n_i the number of vertices of degree i. In Fig. 1 are depicted all trees satisfying conditions (a) and (b), having either $n_3 \leq 4$ and $n_i = 0$ for all $i \geq 4$ or $n_3 \leq 1$, $n_4 = 1$ and $n_i = 0$ for all $i \geq 5$. From the calculated *irr*-values (indicated in Fig. 1) and from Theorem 2.3, the "if" part of Theorem 3.1 is evident.

The "only if" claims in Theorem 3.1 follow from the fact that if $n_3 \geq 5$ and/or $n_4 \geq 2$ and/or $n_i > 0$ for any $i \geq 5$, then $irr(T) \geq 14$. Therefore, all trees satisfying conditions (a) and (b) and having $6 \leq irr \leq 12$ are among T_1–T_{19} in Fig. 1. □

Remark. Continuing the same line of reasoning, it would be possible to characterize further trees with small irregularity.

4. *Trees with large irregularities*

In view of Theorem 2.4, trees with maximal irregularity can be determined only for fixed values of the number of vertices n. From Lemma 1.2 we already know that this is the star S_n. This result was obtained by requiring that the tree has a vertex of as large as possible degree, and as many as possible pendent vertices.

Denote by $\Delta = \Delta(T)$ the maximal degree of a vertex of the tree T. As before, let $n_1 = n_1(T)$ be the number of pendent vertices of T. Recall that $\Delta(S_n) = n - 1$ and $n_1(S_n) = n - 1$.

In order to determine the second–maximal, third–maximal, etc. trees of order n we need to examine cases different from S_n, where both Δ and n_1 are as large as possible. The trees of order n with $\Delta = n - 2$, $\Delta = n - 3$, and $\Delta = n - 4$ are depicted in Fig. 2.

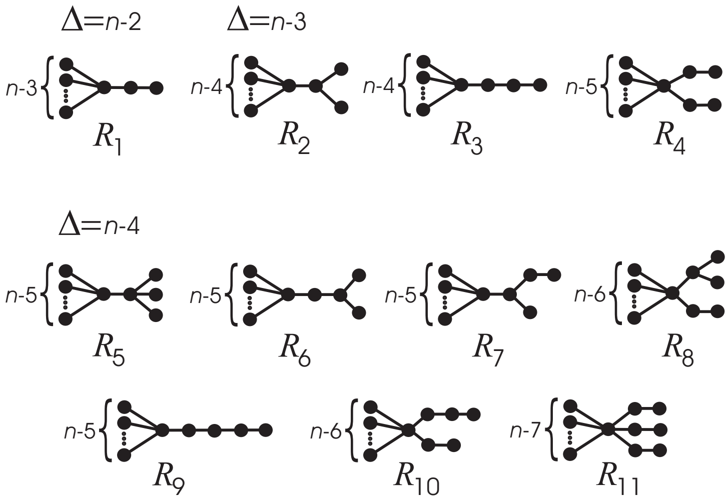

Fig. 2. Trees used in the statement and proof of Theorem 4.1. These are all trees of order n in which the maximal vertex degree Δ is equal to $n - 2$, $n - 3$, and $n - 4$

Theorem 4.1. *Let R_1 , R_2 , R_3 , R_4 , and R_5 be the trees depicted in Fig. 2.*
(a) R_1 is the unique tree of order n , $n \geq 5$, with second–greatest irregularity.
(b) R_2 is the unique tree of order n , $n \geq 6$, with third–greatest irregularity.
(c) R_3 and R_4 are the two trees of order n , $n \geq 6$, with fourth–greatest irregularity. (d) R_5 is the unique tree of order n , $n \geq 8$, with fifth–greatest irregularity.

P r o o f. Bearing in mind Lemma 2.1, we recognize that $irr(R_1) = irr(S_{n-1})$, $irr(R_3) = irr(R_4) = irr(S_{n-2})$, $irr(R_9) = irr(R_{10}) = irr(R_{11}) = irr(S_{n-3})$, and $irr(R_7) = irr(R_8)$. Expressions for the irregularity of the trees from Fig. 2 are now readily obtained, either by using Eq. (1) or by direct calculation:

$$irr(R_1) = n^2 - 5n + 6 \tag{2}$$
$$irr(R_2) = n^2 - 7n + 14 \tag{3}$$
$$irr(R_3) = irr(R_4) = n^2 - 7n + 12 \tag{4}$$
$$irr(R_5) = n^2 - 9n + 27 \tag{5}$$
$$irr(R_6) = n^2 - 9n + 24 \tag{6}$$
$$irr(R_7) = irr(R_8) = n^2 - 9n + 22 \tag{7}$$
$$irr(R_9) = irr(R_{10}) = irr(R_{11}) = n^2 - 9n + 20 \; . \tag{8}$$

In the general case, the expression for the irregularity of a tree of order n and $\Delta = n - k$ is a quadratic polynomial in the variable n of the form $n^2 - (2k+1)n + C$ where C is some constant. Therefore, the trees with the first few greatest irr-vales are among those depicted in Fig. 2. Theorem 3.1 is now straightforwardly deduced from Eqs. (2)–(8).

The unique tree with $\Delta = n - 2$ automatically becomes the unique tree with second maximal irr-value. Among trees with $\Delta = n - 3$, R_2 has the greatest number of pendent vertices, which puts it on the third-maximal position. The remaining two trees with $\Delta = n-3$ have then fourth–maximal irregularities. Finally, among the trees with $\Delta = n - 4$, R_5 has the greatest number of pendent vertices. □

Remark. Continuing the same line of reasoning, it would be possible to characterize further trees with large irregularity.

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Department of Pure Mathematics
Faculty of Mathematical Sciences
University of Kashan
Kashan 87317–51167 I. R. Iran
e-mail: `fathtabar@kashanu.ac.ir`

Department of Mathematics
University of Qom
Qom
I. R. Iran
e-mail: `r.nasiri.82@gmail.com`

Faculty of Science
University of Kragujevac
P. O. Box 60
34000 Kragujevac
Serbia
e-mail: `gutman@kg.ac.rs`