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CONFORMAL CHANGE OF THE METRIC ON ALMOST ANTI-HERMITIAN MANIFOLDS

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A b s t r a c t. Some conformally invariant algebraic curvature tensors for the almost anti-Hermitian manifolds are found. It is proved that they are related such that the equation (7.4) holds.

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1. *Intoduction*

An almost anti-Hermitian manifold (M, g, J) is a differentiable manifold M, $\dim M = 2n$, endowed with complex structure J and anti-Hermitian metric g, i.e.

$$J^2 = -Id., \quad g(JX, JY) = -g(X, Y) \tag{1.1}$$

for any vector fields X, Y of the tangent vector space $T(M)$. Then

$$F(X, Y) = g(JX, Y) = g(X, JY) = F(X, Y) \quad \text{and} \quad F(JX, JY) = -F(X, Y).$$

If $\{e_i\}$ $i=1,2,\dots,2n$ is an orthonormal basis of $T_p(M)$, $p\in M$, and $R(X,Y,Z,W)$ is the Riemannian curvature tensor, then the first and the second Ricci tensors are respectively

$$\rho(Y,Z)=\sum_{i=1}^{2n}R(e_i,Y,Z,e_i),\quad \widetilde{\rho}(Y,Z)=\sum_{i=1}^{2n}R(Je_i,Y,Z,e_i),$$

while the first and the second scalar curvatures are

$$\kappa=\sum_{i=1}^{2n}\rho(e_i,e_i),\quad \widetilde{\kappa}=\sum_{i=1}^{2n}\widetilde{\rho}(e_i,e_i).$$

We note that, like the first, the second Ricci tensor is symmetric, that is $\widetilde{\rho}(Y,Z)=\widetilde{\rho}(Z,Y)$.

Let ∇ is the Levi-Civita connection with respect to the metric g. If besides (1.1), the condition $\nabla J=0$ is also satisfied, we have

$$R(X,Y,JZ,JW)=-R(X,Y,Z,W). \tag{1.2}$$

It is well known that the condition $R(X,Y,JZ,JW)=R(X,Y,Z,W)$ is characteristic for the Kähler manifolds. Thus, the anti-Hermitian manifold satisffying the condition $\nabla J=0$, is called <u>anti-Kähler manifold</u> or <u>the Kähler manifold with Norden metric.</u> Such manifolds were first investigated by A.P.Norden [7] in the case dim $M=4$. He named them $B-$manifolds to distinguish them from the Kähler spaces ($A-$manifolds in [7]).

The relation (1.2) implies [8]:

$$R(JX,JY,JZ,JW)=R(X,Y,Z,W) \tag{1.3}$$

and

$$R(JX,Y,Z,W)=R(X,JY,Z,W)=R(X,Y,JZ,W)=R(X,Y,Z,JW). \tag{1.4}$$

The Ricci tensors of the anti-K̈ahler manifold satisfy the conditions

$$\begin{aligned}\rho(JX,Y)&=\widetilde{\rho}(X,Y), & \rho(X,Y)&=-\widetilde{\rho}(JX,Y),\\ \rho(JX,JY)&=-\rho(X,Y), & \widetilde{\rho}(JX,JY)&=-\widetilde{\rho}(X,Y).\end{aligned} \tag{1.5}$$

Indeed, in view of (1.4), we have

$$J(JX,Y)=\sum_{i=1}^{2n}R(e_i,JX,Y,e_i)=\sum_{i=1}^{2n}R(Je_i,X,Y,e_i)=\widetilde{\rho}(X,Y),$$

$$\begin{aligned}\widetilde{\rho}(JX,Y) &= \sum_{i=1}^{2n} R(Je_i, JX, Y, e_i) = \sum_{i=1}^{2n} R(J^2 e_i, X, Y, e_i)\\ &= -\sum_{i=1}^{2n} R(e_i, X, Y, e_i) = -\rho(X,Y),\end{aligned}$$

and similarly for other relations (1.5). Also,

$$\begin{aligned}\sum_i \rho(Je_i, e_i) &= \sum_i \widetilde{\rho}(e_i, e_i) = \widetilde{\kappa},\\ \sum_i \widetilde{\rho}(Je_i, e_i) &= -\sum_i \rho(e_i, e_i) = -\kappa.\end{aligned} \tag{1.6}$$

The anti-Kähler and anti-Hermitian manifolds have been investigated relatively intensively for last ten to fiftien years ([1],[2],[4],[5],[8],[9],[10]). The object of the present paper is to investigate the effects of the conformal change of the metric. This will be done by using the tensor defined in the section 2. The first results are exhibited in the section 3. In the sections 4, 5 and 6 we obtain conformally invariant algebraic curvature tensors, and in the section 7 we prove that they are related such that the equation (7.4) holds.

2. *Algebraic curvature tensor satisfying the condition of type (1.2)*

If $\nabla J \neq 0$, the anti-Kähler condition (1.2) does not hold. Yet, for any almost anti-Hermitian manifold there exists the algebraic curvature tensor satisfying the condition of the type (1.2). It is

$$\begin{aligned}H(X,Y,Z,W) &= \tfrac{1}{8}\left[R(X,Y,Z,W) + R(JX,JY,JZ,JW)\right.\\ &-R(JX,JY,Z,W) - R(JX,Y,JZ,W) - R(JX,Y,Z,JW)\\ &\left.-R(X,JY,JZ,W) - R(X,JY,Z,JW) - R(X,Y,JZ,JW)\right].\end{aligned} \tag{2.1}$$

By the direct calculation, we can see that

$$\begin{aligned}H(X,Y,Z,W) = -H(Y,X,Z,W) = -H(X,Y,W,Z) = H(Z,W,X,Y),\\ H(X,Y,Z,W) + H(Y,Z,X,W) + H(Z,X,Y,W) = 0,\end{aligned} \tag{2.2}$$

as well as

$$\begin{aligned}H(X,Y,JZ,JW) &= -H(X,Y,Z,W)\\ H(JX,Y,Z,W) = H(X,JY,Z,W) &= H(X,Y,JZ,W) = H(X,Y,Z,JW).\end{aligned} \tag{2.3}$$

The relations (2.2) show that the tensor $H(X,Y,Z,W)$ is the algebraic curvature tensor (that is, has all algebraic properties as the Riemannian curvature tensor $R(X,Y,Z,W)$) while (2.3) show that is satisfies the condition of type (1.2).

We note that for anti-Kähler manifolds, i.e. if $\nabla J = 0$, it holds

$$H(X,Y,Z,W) = R(X,Y,Z,W). \tag{2.4}$$

The first Ricci tensor, corresponding to the tensor $H(X,Y,Z,W)$ is

$$\begin{aligned}\rho(H)(Y,Z) = & \sum_i H(e_i,Y,Z,e_i) \\ = & \frac{1}{8}\{\rho(Y,Z) - \rho(JY,JZ) \\ & -\widetilde{\rho}(JY,Z) - \widetilde{\rho}(Y,JZ) + \rho(Y,Z) \\ & -\rho(JY,JZ) - \widetilde{\rho}(Z,JY) - \widetilde{\rho}(JZ,Y)\}.\end{aligned}$$

Thus, and in view of the symmetry of the tensor $\widetilde{\rho}(Y,Z)$, we have

$$\rho(H)(Y,Z) = \frac{1}{4}\{\rho(Y,Z) - \rho(JY,JZ) - \widetilde{\rho}(Y,JZ) - \widetilde{\rho}(JY,Z)\}. \tag{2.5}$$

In the similar way we obtain

$$\widetilde{\rho}(H)(Y,Z) = \frac{1}{4}\{\widetilde{\rho}(Y,Z) - \widetilde{\rho}(JY,JZ) + \rho(Y,JZ) + \rho(JY,Z)\}. \tag{2.6}$$

Therefore

$$\begin{aligned}&\rho(H)(JY,Z) = \widetilde{\rho}(H)(Y,Z), && \widetilde{\rho}(H)(JY,Z) = -\rho(H)(Y,Z),\\ &\rho(H)(JY,JZ) = -\rho(H)(Y,Z), && \widetilde{\rho}(H)(JY,JZ) = -\widetilde{\rho}(H)(Y,Z),\end{aligned} \tag{2.7}$$

that is, $\rho(H)$ and $\widetilde{\rho}(H)$ satisfy the conditions analogous to the conditions (1.5).

Finally,

$$\begin{aligned}\kappa(H) &= \frac{1}{2}\left[\kappa - \sum_i \widetilde{\rho}(e_i,Je_i)\right],\\ \widetilde{\kappa}(H) &= \frac{1}{2}\left[\widetilde{\kappa} + \sum_i \rho(e_i,Je_i)\right].\end{aligned} \tag{2.8}$$

3. *Conformal change of the metric*

Now, let us consider the conformal change of the metric

$$\bar{g} = e^{2f} g,$$

where f is a scalar function. Denoting by $\bar{\nabla}$ the Levi-Civita connection with respect to the metric $\bar{g}$, we have

$$(\bar{\nabla} - \nabla)(X, Y) = \theta(X)(Y) + \theta(Y)(X) - g(X, Y)U,$$

for any vector fields $X, Y \in T(M)$, where θ is l-form defined by $\theta = df$, and U is the vector field such that $g(U, X) = \theta(X)$.

From now on, all geometric objects in $(M, \bar{g}, J)$ will be denoted by analogous letters as in (M, g, J), but with "bar".

It is well known that the Riemanian curvature tensors $\bar{R}$ and R of the metrics $\bar{g}$ and g respectively, are related as follows (see for ex. [6]).

$$\begin{gathered} e^{-2f}\bar{R}(X, Y, Z, W) = R(X, Y, Z, W)+ \\ +g(X, W)\sigma(Y, Z) + g(Y, Z)\sigma(X, W) - g(X, Z)\sigma(Y, W) - g(Y, W)\sigma(X, Z), \end{gathered} \tag{3.1}$$

where σ is the tensor field of type (0,2) defined by

$$\sigma(X, Y) = (\nabla_X \theta)(Y) - \theta(X)\theta(Y) + \frac{1}{2}\theta(U)g(X, Y).$$

We note that $\sigma(X, Y) = \sigma(Y, X)$ because θ is a gradient.

It follows from (3.1)

$$\bar{\rho}(Y, Z) = \rho(Y, Z) + 2(n-1)\sigma(Y, Z) + g(Y, Z)\sum_i \sigma(e_i, e_i).$$

Therefore

$$\sum_i \sigma(e_i, e_i) = \frac{e^{2f}\bar{\kappa} - \kappa}{2(2n-1)}, \tag{3.2}$$

such that

$$\begin{aligned} \sigma(Y, Z) = & \frac{1}{2(n-1)}\left[\bar{\rho}(Y, Z) - \frac{\bar{\kappa}}{2(2n-1)}\bar{g}(Y, Z)\right] \\ & -\frac{1}{2(n-1)}\left[\rho(Y, Z) - \frac{\kappa}{2(2n-1)}g(Y, Z)\right]. \end{aligned} \tag{3.3}$$

This relation, together with

$$\begin{aligned}\sigma(JY,JZ) = \quad & \frac{1}{2(n-1)}\left[\bar\rho(JY,JZ)+\frac{\bar\kappa}{2(2n-1)}\bar g(Y,Z)\right]\\ & -\frac{1}{2(n-1)}\left[\rho(Jy,JZ)+\frac{\kappa}{2(2n-1)}g(Y,Z)\right],\end{aligned}$$

yields

$$\begin{aligned}S(Y,Z) \quad &= \frac{1}{2(n-1)}\left[\bar\rho(Y,Z)-\bar\rho(JY,JZ)\right]-\frac{\bar\kappa}{2(n-1)(2n-1)}\bar g(Y,Z)\\ &-\frac{1}{2(n-1)}\left[\rho(Y,Z)-\rho(JY,Jz)\right]+\frac{\kappa}{2(n-1)(2n-1)}g(Y,Z),\end{aligned} \tag{3.4}$$

where we have put

$$S(Y,Z)=\sigma(Y,Z)-\sigma(JY,JZ). \tag{3.5}$$

We note that $S(JY,JZ)=-S(Y,Z)$, and thus $S(Y,JZ)=S(JY,Z)$.

On the other hand, putting into (3.1), $X=Je_i$, $W=e_i$ and summing up, we obtain

$$\bar{\widetilde\rho}(Y,Z)=\widetilde\rho(Y,Z)+g(Y,Z)\sum_i\sigma(e_i,Je_i)-\left[\sigma(Y,JZ)+\sigma(JY,Z)\right], \tag{3.6}$$

where from it follows

$$\sum_i\sigma(e_i,Je_i)=\frac{e^{2f}\bar{\widetilde\kappa}-\widetilde\kappa}{2(n-1)}, \tag{3.7}$$

such that (3.6) becomes

$$\begin{aligned}\sigma(Y,Z)-\sigma(JY,JZ)=\bar{\widetilde\rho}(JY,Z)-\frac{\bar{\widetilde\kappa}}{2(n-1)}\bar F(Y,Z)\\ -\left[\widetilde\rho(JY,Z)-\frac{\widetilde\kappa}{2(n-1)}F(Y,Z)\right].\end{aligned}$$

The symmetric part of this relation, in view of (3.5), is

$$\begin{aligned}S(Y,Z) \quad &= \frac{1}{2}\left[\bar{\widetilde\rho}(Y,JZ)+\bar{\widetilde\rho}(JY,Z)\right]-\frac{\bar{\widetilde\kappa}}{2(n-1)}\bar F(Y,Z)\\ &-\frac{1}{2}\left[\widetilde\rho(Y,JZ)+\widetilde\rho(JY,Z)\right]+\frac{\widetilde\kappa}{2(n-1)}F(Y,Z).\end{aligned} \tag{3.8}$$

Comparating (3.4) and (3.8), we find

$$\bar{\rho}(Y,Z) - \bar{\rho}(JY,JZ) - (n-1)\left[\bar{\widetilde{\rho}}(Y,JZ) + \bar{\widetilde{\rho}}(JY,Z)\right] - \frac{\bar{\kappa}}{2n-1}\bar{g}(Y,Z) + \bar{\widetilde{\kappa}}\bar{F}(Y,Z)$$
$$= \rho(Y,Z) - \rho(JY,JZ) - (n-1)\left[\widetilde{\rho}(Y,JZ) + \widetilde{\rho}(JY,Z)\right] - \frac{\kappa}{2n-1}g(Y,Z) + \widetilde{\kappa}F(Y,Z). \tag{3.9}$$

Thus, we can state

Proposition 3.1 *For an almost anti-Hermitian manifold, the tensor*

$$\begin{aligned} P(Y,Z) = \rho(Y,Z) - \rho(JY,JZ) - (n-1)\left[\widetilde{\rho}(Y,JZ) + \widetilde{\rho}(JY,Z)\right] \\ -\frac{\kappa}{2n-1}g(Y,Z) + \widetilde{\kappa}F(Y,Z) \end{aligned} \tag{3.10}$$

is conformally invariant.

For the later use, we present still two other forms of the tensor (3.10). Putting into (3.9) $Y = Z = e_i$ and summing up, we obtain

$$e^{2f}\left[\frac{\bar{\kappa}}{2n-1} - \sum_i \bar{\widetilde{\rho}}(e_i, Je_i)\right] = \frac{\kappa}{2n-1} - \sum_i \widetilde{\rho}(e_i, Je_i), \tag{3.11}$$

while putting $Y = e_i$, $Z = Je_i$, we get

$$e^{2f}\left[\bar{\widetilde{\kappa}} - \sum_i \bar{\rho}(e_i, Je_i)\right] = \widetilde{\kappa} - \sum_i \rho(e_i, Je_i). \tag{3.12}$$

The relation (3.12) implies

$$\bar{\widetilde{\kappa}}\bar{F}(Y,Z) - \bar{F}(Y,Z)\sum_i \bar{\rho}(e_i, Je_i) = \widetilde{\kappa}F(Y,Z) - F(Y,Z)\sum_i \rho(e_i, Je_i).$$

Substituting this into (3.9), we find

$$\bar{\rho}(Y,Z) - \bar{\rho}(JY,JZ) - (n-1)\left[\bar{\widetilde{\rho}}(Y,JZ) + \bar{\widetilde{\rho}}(JY,Z)\right] - \frac{\bar{\kappa}}{2n-1}\bar{g}(Y,Z) + \bar{F}(Y,Z)\sum_i \bar{g}(e_i, Je_i)$$

$$= \rho(Y,Z) - \rho(JY,JZ) - (n-1)\left[\widetilde{\rho}(Y,JZ) + \widetilde{\rho}(JY,Z)\right] - \frac{\kappa}{2n-1}g(Y,Z) + F(Y,Z)\sum_i \rho(e_i, Je_i).$$

Similarly, the relation (3.11) yields

$$-\frac{\bar{\kappa}}{2n-1}\bar{g}(Y,Z) = -\bar{g}(Y,Z)\sum_i \bar{\widetilde{\rho}}(e_i, Je_i) - \frac{\kappa}{2n-1}g(Y,Z) + g(Y,Z)\sum_i \widetilde{\rho}(e_i, Je_i).$$

Substituting this into (3.9), we obtain

$$\begin{aligned}&\bar\rho(Y,Z)-\bar\rho(JY,JZ)-(n-1)\left[\bar{\widetilde\rho}(Y,JZ)+\bar{\widetilde\rho}(JY,Z)\right]\\&-\bar g(Y,Z)\sum_i\bar{\widetilde\rho}(e_i,Je_i)+\bar{\widetilde\kappa}\bar F(Y,Z)\\=&\ \rho(Y,Z)-\rho(JY,JZ)-(n-1)\left[\widetilde\rho(Y,JZ)+\widetilde\rho(JY,Z)\right]\\&-g(Y,Z)\sum_i\widetilde\rho(e_i,Je_i)+\widetilde\kappa F(Y,Z).\end{aligned}$$

Thus, we can state

Proposition 3.2 *The tensors*

$$\begin{aligned}P'(Y,Z)=&\ \rho(Y,Z)-\rho(JY,JZ)-(n-1)\left[\widetilde\rho(Y,JZ)+\widetilde\rho(JY,Z)\right]\\&-\frac{\kappa}{2n-1}g(Y,Z)+F(Y,Z)\sum_i\rho(e_i,Je_i).\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}P''(Y,Z)=&\ \rho(Y,Z)-\rho(JY,JZ)-(n-1)\left[\widetilde\rho(Y,JZ)+\widetilde\rho(JY,Z)\right]\\&-g(Y,Z)\sum_i\widetilde\rho(e_i,Je_i)+\widetilde\kappa F(Y,Z).\end{aligned}\tag{3.14}$$

are conformally invariant.

4. *The first conformally invariant algebraical curvature tensor*

The algebraic curvature tensor (2.1) with respect to the metric $\bar g$ is

$$\begin{aligned}8e^{-2f}\bar H(X,Y,Z,W)=&\ e^{-2f}\left[\bar R(X,Y,Z,W)+\bar R(JX,JY,JZ,JW)\right.\\&-\bar R(JX,JY,Z,W)-\bar R(JX,Y,JZ,W)-\bar R(JX,Y,Z,JW)\\&\left.-\bar R(X,JY,JZ,W)-\bar R(X,JY,Z,JW)-\bar R(X,Y,JZ,JW)\right]\end{aligned}$$

Substituting (3.1) and using the notation (3.5), we get

$$\begin{aligned}&e^{-2f}\bar H(X,Y,Z,W)=H(X,Y,Z,W)\\&+\frac14\left[g(X,W)S(Y,Z)+g(Y,Z)S(X,W)-g(X,Z)S(Y,W)\right.\\&-g(Y,W)S(X,Z)-F(X,W)S(Y,JZ)-F(Y,Z)S(X,JW)\\&\left.+F(X,Z)S(Y,JW)+F(Y,W)S(X,JZ)\right]\end{aligned}\tag{4.1}$$

Putting $X = W = e_i$, we find

$$\begin{aligned} 4\rho(\bar{H})(Y,Z) &= 4\rho(H)(Y,Z) + 2(n-2)S(Y,Z) \\ &+ g(Y,Z)\sum_i S(e_i,e_i) - F(Y,Z)\sum_i S(e_i,Je_i), \end{aligned} \tag{4.2}$$

where from, putting $Y = Z = e_i$, we obtain

$$\sum_i S(e_i,e_i) = \frac{e^{2f}\kappa(\bar{H})}{n-1} - \frac{\kappa(H)}{n-1}.$$

To find $\sum_i S(e_i,Je_i)$, we put into (4.2) $Y = Je_i$, $Z = e_i$, and get

$$\sum_i S(e_i,Je_i) = \frac{e^{2f}\widetilde{\kappa}(\bar{H})}{n-1} - \frac{\widetilde{\kappa}(H)}{n-1}.$$

Thus, the relation (4.2), for $n > 2$, yields

$$\begin{aligned} \frac{1}{4}S(Y,Z) &= \frac{1}{n-2}\left[\frac{1}{2}\rho(\bar{H})(Y,Z) - \frac{\kappa(\bar{H})}{8(n-1)}\bar{g}(Y,Z) + \frac{\widetilde{\kappa}(\bar{H})}{8(n-1)}\bar{F}(Y,Z)\right] \\ &- \frac{1}{n-2}\left[\frac{1}{2}\rho(H)(Y,Z) - \frac{\kappa(H)}{8(n-1)}g(Y,Z) + \frac{\widetilde{\kappa}(H)}{8(n-1)}F(Y,Z)\right]. \end{aligned} \tag{4.3}$$

Finally, substituting (4.3) into (4.1), we have, for $n > 2$,

$$e^{-2f}\,\underset{1}{\bar{\mathrm{B}}}(X,Y,Z,W) = \underset{1}{\mathrm{B}}(X,Y,Z,W),$$

where

$$
\begin{aligned}
&\underset{1}{\mathrm{B}}(X,Y,Z,W) = H(X,Y,Z,W)\\
-&\frac{1}{2(n-2)}\,[g(X,W)\rho(H)(Y,Z) + g(Y,Z)\rho(H)(X,W)\\
&\quad -g(X,Z)\rho(H)(Y,W) - g(Y,W)\rho(H)(X,Z)\\
&\quad -F(X,W)\rho(H)(Y,JZ) - F(Y,Z)\rho(H)(X,JW)\\
&\quad +F(X,Z)\rho(H)(Y,JW) + F(Y,W)\rho(H)(X,JZ)]\\
+&\frac{\kappa(H)}{4(n-1)(n-2)}\,[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\\
&\quad -F(X,W)F(Y,Z) + F(X,Z)F(Y,W)]\\
-&\frac{\widetilde{\kappa}(H)}{4(n-1)(n-2)}\,[g(X,W)F(Y,Z) + g(Y,Z)F(X,W)\\
&\quad -g(X,Z)F(Y,W) - g(Y,W)F(X,Z)]\,,
\end{aligned}
\tag{4.4}
$$

and $\underset{1}{\bar{\mathrm{B}}}(X,Y,Z,W)$ is constructed in the same way, but with respect to the metric $\bar{g}$.

The tensor (4.4) is the algebraic curvature tensor. Also, it satisfies the condition of type (1.2). We say that the tensor (4.4) is *the first conformally invariant algebraic curvature tensor of the almost anti-Hermitian manifold* (M,g,J).

For the first Ricci tensor corresponding to the tensor (4.4), we have

$$
\begin{aligned}
\rho(\underset{1}{\mathrm{B}})(Y,Z) &= \sum_i \underset{1}{\mathrm{B}}(e_i,Y,Z,e_i)\\
&= \frac{1}{2(n-2)}\left[\sum_i \rho(H)(e_i,Je_i) - \widetilde{\kappa}(H)\right] F(Y,Z).
\end{aligned}
$$

But, according (2.7),

$$
\sum_i \rho(H)(e_i,Je_i) = \sum_i \widetilde{\rho}(H)(e_i,e_i) = \widetilde{\kappa}(H).
$$

Therefore, $\rho(\underset{1}{\mathrm{B}})(Y,Z) = 0$. In the similar way we prove that $\widetilde{\rho}(\underset{1}{\mathrm{B}})(Y,Z) = 0$. Thus, we can state

Theorem 4.1. *For an almost anti-Hermitian manifold* (M,g,J)*, dim* $M >$

4, *the tensor (4.4) is the first conformally invariant algebraic curvature tensor. Both its Ricci tensors vanish.*

The Ricci tensor of the generalized Bochner curvature tensor of an almost Hermitian manifold vanishes. Thus, we can say that for an almost anti-Hermitian manifold, the tensor (4.4) is the tensor corresponding to the generalized Bochner curvature tensor.

For an anti-Kähler manifold, the relation (2.4) holds, such that the tensor $\underset{1}{\mathrm{B}}$ has the form

$$
\begin{aligned}
&B(X,Y,Z,W) = R(X,Y,Z,W)\\
&-\frac{1}{2(n-2)}\left[g(X,W)\rho(Y,Z) + g(Y,Z)\rho(X,W) - g(X,Z)\rho(Y,W)\right.\\
&-g(Y,W)g(X,Z) - F(X,W)\rho(Y,JZ) - F(Y,Z)\rho(X,JW)\\
&\left.+F(X,Z)\rho(Y,JW) + F(Y,W)\rho(X,JZ)\right]\\
&+\frac{\kappa}{4(n-1)(n-2)}\left[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\right.\\
&\left.-F(X,W)F(Y,Z) + F(X,Z)F(Y,W)\right]\\
&-\frac{\widetilde{\kappa}}{4(n-1)(n-2)}\left[g(X,W)F(Y,Z) + g(Y,Z)F(X,W)\right.\\
&\left.-g(X,Z)F(Y,W) - g(Y,W)F(X,Z)\right].
\end{aligned}
\tag{4.5}
$$

But this is just the tensor obtained in [8] using the pseudoconformal correspondence $\bar{g} = \alpha g + \beta F$, where α and β are some scalar function.

5. *The second conformally invariant algebraic curvature tensor*

Substituting (3.4) into (4.1) and putting

$$
\begin{aligned}
&\underset{2}{\mathrm{B}}(X,Y,Z,W) = H(X,Y,Z,W)\\
&-\frac{1}{8(n-1)}\left\{g(X,W)\left[\rho(Y,Z)-\rho(JY,JZ)\right]+g(Y,Z)\left[\rho(X,W)-\rho(JX,JW)\right]\right.\\
&\quad -g(X,Z)\left[\rho(Y,W)-\rho(JY,JW)\right]-g(Y,W)\left[\rho(X,Z)-\rho(JX,JZ)\right]\\
&\quad -F(X,W)\left[\rho(Y,JZ)+\rho(JY,Z)\right]-F(Y,Z)\left[\rho(X,JW)+\rho(JX,W)\right]\\
&\quad \left.+F(X,Z)\left[\rho(Y,JW)+\rho(JY,W)\right]+F(Y,W)\left[\rho(X,JZ)+\rho(JX,Z)\right]\right\}\\
&\quad +\frac{\kappa}{4(n-1)(n-2)}\left[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\right.\\
&\quad \left.-F(X,W)F(Y,Z)+F(X,Z)F(Y,W)\right],
\end{aligned}
\tag{5.1}
$$

we see at once that

$$e^{-2f}\ \underset{2}{\bar{\mathrm{B}}}(X,Y,Z,W) = \underset{2}{\mathrm{B}}(X,Y,Z,W).$$

The tensor (5.1) is the algebraic curvature tensor and it satisfies the condition of type (1.2). We say that (5.1) is *the second conformall invariant algebraic curvature tensor of the almost anti-Hermitian manifold.*

The tensor (5.1) can also be obtained in the following way.

It is well known that the Weyl tensor of conformal curvature for a Riemannian manifold (M,g), $\dim M = 2n$, is (see for ex. [6])

$$
\begin{aligned}
&C(X,Y,Z,W) = R(X,Y,Z,W)\\
&-\frac{1}{2(n-1)}\left[g(X,W)\rho(Y,Z)+g(Y,Z)\rho(X,W)-g(X,Z)\rho(Y,W)-g(Y,W)\rho(X,Z)\right]\\
&+\frac{\kappa}{2(n-1)(2n-1)}\left[g(Y,W)g(Y,Z)-g(X,Z)f(Y,W)\right].
\end{aligned}
$$

This can be rewritten in the form

$$C = R - \frac{1}{2(n-1)}\varphi + \frac{\kappa}{2(n-1)(2n-1)}\pi \tag{5.2}$$

where

$$
\begin{aligned}
\varphi(X,Y,Z,W) &= g(X,W)\rho(Y,Z)+g(Y,Z)\rho(X,W)\\
&\quad -g(X,Z)\rho(Y,W)-g(Y,W)\rho(X,Z),
\end{aligned}
\tag{5.3}
$$

$$\pi(X,Y,Z,W) = g(X,W)g(Y,Z)-g(X,Z)g(Y,W).$$

In [11], G. Stanilov used the holomorphic curvature tensor of an almost Hermitian manifold and applied it as an operator to the tensor (5.2). Here, we use the tensor (2.1). Applying it to the tensor (5.2) instead to the tensor R, we get

$$H(C) = H(R) - \frac{1}{16(n-1)} H(\varphi) + \frac{\kappa}{16(n-1)(2n-1)} H(\pi). \tag{5.4}$$

In view of (5.3) we find

$$\begin{aligned}
&H(\varphi)(X,Y,Z,W) = \\
&2\{g(X,W)[\rho(Y,Z) - \rho(JY,JZ)] + g(Y,Z)[\rho(X,W) - \rho(JX,JW)] \\
&-g(X,Z)[\rho(Y,W) - \rho(JY,JW)] - g(Y,W)[\rho(X,Z) - \rho(JX,JZ)] \\
&-F(X,W)[\rho(Y,JZ) + \rho(JY,Z)] - F(Y,Z)[\rho(X,JW) + \rho(JX,W)] \\
&+F(X,Z)[\rho(Y,JW) + \rho(JY,W)] + F(Y,W)[\rho(X,JZ) + \rho(JX,Z)]\}
\end{aligned}$$

$$\begin{aligned}
H(\pi)(X,Y,Z,W) = 2[&g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\
&-F(X,W)F(Y,Z) + F(X,Z)F(Y,W)].
\end{aligned}$$

Thus, the right hand side of the relation (5.4) is just the right hand side of the relation (5.1), that is, we have

$$H(C)(X,Y,Z,W) = \underset{2}{B}(X,Y,Z,W). \tag{5.5}$$

The first Ricci tensor corresponding to the tensor (5.1) is

$$\begin{aligned}
\rho(\underset{2}{B})(Y,Z) &= \sum_i \underset{2}{B}(e_i,Y,Z,e_i) \\
&= \rho(H)(Y,Z) - \frac{n-2}{4(n-1)}[\rho(Y,Z) - \rho(JY,JZ)] \\
&- \frac{\kappa}{4(n-1)(2n-1)} g(Y,Z) + \frac{1}{4(n-1)} F(Y,Z) \sum_i \rho(e_i,Je_i)
\end{aligned}$$

or, in view of (2.5),

$$\begin{aligned}
\rho(\underset{2}{B})(Y,Z) = \frac{1}{4(n-1)} &\{\rho(Y,Z) - \rho(JY,JZ) - (n-1)[\widetilde{\rho}(Y,JZ) + \widetilde{\rho}(JY,JZ)] \\
&- \frac{\kappa}{2n-1} g(Y,Z) + F(Y,Z) \sum_i \rho(e_i,Je_i)\Big\}
\end{aligned} \tag{5.6}$$

Comparing the relations (5.6) and (3.13), we see that

$$\rho(\underset{2}{\mathrm{B}})(Y,Z) = \frac{1}{4(n-1)} P'(Y,Z). \tag{5.7}$$

Setting into (5.1) $X = Je_i$, $W = e_i$, and using (2.6), we get

$$\begin{aligned}\widetilde{\rho}(\underset{2}{\mathrm{B}})(Y,Z) = \frac{1}{4(n-1)} \{&\rho(Y,JZ) + \rho(JY,Z) + (n-1)\,[\widetilde{\rho}(Y,Z) - \widetilde{\rho}(JY,JZ)] \\ &-g(Y,Z)\sum_i \rho(e_i,Je_i) - \frac{\kappa}{2n-1}F(Y,Z)\Bigg\}.\end{aligned} \tag{5.8}$$

The relations (5.6) and (5.8) show that

$$\rho(\underset{2}{\mathrm{B}})(Y,JZ) = \widetilde{\rho}(\underset{2}{\mathrm{B}})(Y,Z). \tag{5.9}$$

Thus, we can state

Theorem 5.1 *For almost anti-Hermitian manifold, the tensor (5.1) is the second conformally invariant algebraic curvature tensor. It satisfies the condition (5.5). The relations (5.6) and (5.8) determine its Ricci tensors and (5.7) and (5.9) hold.*

We remark that the relation (5.9) is also the consequence of the fact that the tensor (5.1) satisfies the condition of the type (1.2).

It (M,g,J) is anti-Kähler manifold, then (1.5) and (2.4) hold, such that

$$\begin{aligned}\underset{2}{\mathrm{B}}(X,Y,Z,W) &= R(X,Y,Z,W) \\ &-\frac{1}{4(n-1)}[g(X,W)\rho(Y,Z) + g(Y,Z)\rho(X,W) \\ &\quad -g(X,Z)\rho(Y,W) - g(Y,W)\rho(X,Z) \\ &\quad -F(X,W)\rho(Y,JZ) - F(Y,Z)\rho(X,JW) \\ &\quad +F(X,Z)\rho(Y,JW) + F(Y,W)\rho(X,JZ)] \\ &+\frac{\kappa}{4(n-1)(2n-1)}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &\quad -F(X,W)F(Y,Z) + F(X,Z)F(Y,W)],\end{aligned} \tag{5.10}$$

and

$$\rho(\underset{2}{\mathrm{B}}) = \frac{n}{2(n-1)}\rho(Y,Z) - \frac{1}{4(n-1)}\left[\frac{\kappa}{2n-1}g(Y,Z) - \widetilde{\kappa}F(Y,Z)\right] \tag{5.11}$$

6. *The third conformally invariant algebraic curvature tensor*

Substituting (3.8) into (4.1), we obtain

$$e^{-2f}\ \underset{3}{\bar{\mathrm{B}}}(X,Y,Z,W) = B(X,Y,Z,W)$$

where

$$\begin{aligned}
\underset{3}{\mathrm{B}}(X,Y,Z,W) &= H(X,Y,Z,W)\\
-\frac{1}{8}\,\{&g(X,W)[\widetilde{\rho}(Y,JZ)+\widetilde{\rho}(JY,JZ)] + g(Y,Z)[\widetilde{\rho}(X,JW)+\widetilde{\rho}(JX,W)]\\
&-g(X,Z)[\widetilde{\rho}(Y,JW)+\widetilde{\rho}(JY,W)] - g(Y,W)[\widetilde{\rho}(X,JZ)+\widetilde{\rho}(JX,Z)]\\
&+F(X,W)[\widetilde{\rho}(Y,Z)-\widetilde{\rho}(JY,JZ)] + F(Y,Z)[\widetilde{\rho}(X,W)-\widetilde{\rho}(JX,JW)]\\
&-F(X,Z)[\widetilde{\rho}(Y,W)-\widetilde{\rho}(JY,JW)] - F(Y,W)[\widetilde{\rho}(X,Z)-\widetilde{\rho}(JX,JZ)]\}\\
&+\frac{\widetilde{\kappa}}{4(n-1)}[g(X,W)F(Y,Z)+g(Y,Z)F(X,W)\\
&\quad -g(X,Z)F(Y,W)-g(Y,W)F(X,Z)].
\end{aligned} \tag{6.1}$$

It is easy to see that the tensor (6.1) is the algebraic curvature tensor and that it satisfies the condition of type (1.2).

The tensor (6.1) is *the third conformally invariant algebraic curvature tensor of an almost anti-Hermitian manifold.* Its first Ricci tensor, in view of (2.5), is

$$\begin{aligned}
\rho(\underset{3}{\mathrm{B}})(Y,Z) &= \sum_i \underset{3}{\mathrm{B}}(e_i,Y,Z,e_i)\\
&= \frac{1}{4}\,\Big\{\rho(Y,Z)-\rho(JY,JZ)-(n-1)[\widetilde{\rho}(Y,JZ)+\widetilde{\rho}(JY,Z)]\\
&\qquad -g(Y,Z)\sum_i \widetilde{\rho}(e_i,Je_i)+\widetilde{\kappa}F(Y,Z)\Big\},
\end{aligned} \tag{6.2}$$

such that, according (3.14), we have

$$\rho(\underset{3}{\mathrm{B}})(Y,Z) = \frac{1}{4}P''(Y,Z). \tag{6.3}$$

As for the second Ricci, we have

$$\begin{aligned}\widetilde{\rho}(\underset{3}{\mathrm{B}})(Y,Z) &= \sum_i \underset{3}{\mathrm{B}}(Je_i,Y,Z,e_i)\\ &= \frac{1}{4}\Big\{\rho(Y,JZ)+\rho(JY,Z)+(n-1)[\widetilde{\rho}(Y,Z)-\widetilde{\rho}(JY,JZ)]\\ &\qquad -\widetilde{\kappa}g(Y,Z)-F(Y,Z)\sum_i\widetilde{\rho}(e_i,Je_i)\Big\},\end{aligned} \tag{6.4}$$

and therefore

$$\widetilde{\rho}(\underset{3}{\mathrm{B}})(Y,Z)=\rho(\underset{3}{\mathrm{B}})(JY,Z)=\frac{1}{4}P''(Y,JZ). \tag{6.5}$$

Thus, we can state

Theorem 6.1 *For an almost anti-Hermitian manifold, the tensor (6.1) is the third conformally invariant algebraic curvature tensor. The relations (6.2) and (6.4) determine its Ricci tensors and the relations (6.3) and (6.5) hold.*

If (M,g,J) is an anti-Kähler manifold, then

$$\begin{aligned}\underset{3}{\mathrm{B}}(X,Y,Z,W) &= R(X,Y,Z,W)\\ &+\frac{1}{4}\{g(X,W)\rho(Y,Z)+g(Y,Z)\rho(X,W)\\ &-g(X,Z)\rho(Y,W)-g(Y,W)\rho(X,Z)\\ &-F(X,W)\rho(Y,JZ)-F(Y,Z)\rho(X,JW)\\ &+F(X,Z)\rho(Y,JW)+F(Y,W)\rho(X,JZ)\}\\ &+\frac{\widetilde{\kappa}}{4(n-1)}[g(X,W)F(Y,Z)+g(Y,Z)F(X,W)\\ &-g(X,Z)F(Y,W)-g(Y,W)F(X,Z)]\end{aligned}$$

and

$$\rho(\underset{3}{\mathrm{B}})(Y,Z)=\frac{n}{2}\rho(Y,Z)+\frac{1}{4}[\kappa g(Y,Z)+\widetilde{\kappa}F(Y,Z)].$$

7. *Linear dependence of the conformally invariant tensors*

In view of (5.1), (6.1), (2.5) and (2.6), we have

$$
\begin{gathered}
\frac{1}{n-2}[(n-1)\underset{2}{\mathrm{B}}(X,Y,Z,W)-\underset{3}{\mathrm{B}}(X,Y,Z,W)]= \\
H(X,Y,Z,W)-\frac{1}{2(n-2)}[g(X,W)\rho(H)(Y,Z) \\
+g(Y,Z)\rho(H)(X,W) \\
-g(X,Z)\rho(H)(Y,W)-g(Y,W)\rho(H)(X,Z) \\
-F(X,W)\widetilde{\rho}(H)(Y,Z)-F(Y,Z)\widetilde{\rho}(H)(X,W) \\
+F(X,Z)\widetilde{\rho}(H)(Y,W)+F(Y,W)\widetilde{\rho}(H)(X,Z)] \\
+\frac{\kappa}{4(n-2)(2n-1)}[g(X,W)g(Y,Z)-g(X,Z)g(Y,W) \\
-F(X,W)F(Y,Z)+F(X,Z)F(Y,W)] \\
-\frac{\widetilde{\kappa}}{4(n-1)(n-2)}[g(X,W)F(Y,Z)+g(Y,Z)F(X,W) \\
-g(X,Z)F(Y,W)-g(Y,W)F(X,Y)].
\end{gathered}
$$

This, taking into account (4.4), can be rewritten in the form

$$
\begin{gathered}
\frac{1}{n-2}[(n-1)\underset{2}{\mathrm{B}}(X,Y,Z,W)-\underset{3}{\mathrm{B}}(X,Y,Z,W)]=\underset{1}{\mathrm{B}}(X,Y,Z,W) \\
+\frac{1}{4(n-1)}\left[-\frac{\kappa(H)}{n-2}+\frac{\kappa}{2n-1}\right][g(X,W)g(Y,Z)-g(X,Z)g(Y,W) \\
-F(X,W)F(Y,Z)+F(X,Z)F(Y,W)[\\
+\frac{1}{4(n-1)(n-2)}[\widetilde{\kappa}(H)-\widetilde{\kappa}][g(X,W)F(Y,Z) \\
+g(Y,Z)F(X,W)-g(X,Z)F(Y,W)-g(Y,W)F(X,Z)],
\end{gathered}
$$

or, according (2.8), in the form

$$\underset{1}{\mathrm{B}}(X,Y,Z,W) = \frac{1}{n-2}[(n-1)\ \underset{2}{\mathrm{B}}(X,Y,Z,W) - \underset{3}{\mathrm{B}}(X,Y,Z,W)]$$
$$+\frac{1}{8(n-1)(n-2)}\left[\frac{\kappa}{2n-1} - \sum_i \widetilde{\rho}(e_i, Je_i)\right]$$
$$[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) - F(X,W)F(Y,Z) + F(X,Z)F(Y,W)]$$
$$+\frac{1}{8(n-1)(n-2)}\left[\widetilde{\kappa} - \sum_i \rho(e_i, Je_i)\right]$$
$$[g(X,W)F(Y,Z) + g(Y,Z)F(X,W) - g(X,Z)F(Y,W) - g(Y,W)F(X,Z)]. \tag{7.1}$$

If we put

$$\underset{4}{\mathrm{B}}(X,Y,Z,W) = \left[\frac{\kappa}{2n-1} - \sum_i \widetilde{\rho}(e_i, Je_i)\right][g(X,W)g(Y,Z) - g(X,Z)g(Y,W)$$
$$-F(X,W)F(Y,Z) + F(X,Z)F(Y,W)], \tag{7.2}$$

and

$$\underset{5}{\mathrm{B}}(X,Y,Z,W) = \left[\widetilde{\kappa} - \sum_i \widetilde{\rho}(e_i, Je_i)\right][g(X,W)F(Y,Z) + g(Y,Z)F(X,W)$$
$$-g(X,Z)F(Y,W) - g(Y,W)F(X,Z)], \tag{7.3}$$

we find that, in view of (3.11) and (3.12), we have

$$e^{-2f}\ \underset{4}{\bar{\mathrm{B}}}(X,Y,Z,W) = \underset{4}{\mathrm{B}}(X,Y,Z,W),$$

and

$$e^{-2f}\ \underset{5}{\bar{\mathrm{B}}}(X,Y,Z,W) = \underset{5}{\mathrm{B}}(X,Y,Z,W).$$

Thus, the relation (7.1) can be expressed as follows

$$\underset{1}{\mathrm{B}} = \frac{1}{n-2}[(n-1)\ \underset{2}{\mathrm{B}} - \underset{3}{\mathrm{B}}] + \frac{1}{8(n-1)(n-2)}(\underset{4}{\mathrm{B}} + \underset{5}{\mathrm{B}}). \tag{7.4}$$

Each of the tensors $\underset{1}{\mathrm{B}}, \dots, \underset{5}{\mathrm{B}}$ is the algebraic curvature tensor, and each satisfies the condition of type (1.2). Thus, we can state the theorem

Theorem 7.1 *The conformally invariant tensors* $\underset{1}{B}, \dots, \underset{5}{B}$ *are linearly* *dependent such that the relation (7.4) holds. Each of this tensors is algebraic curvature tensor and each satisfies the condition of type (1.2).*

If (M, g, J) is and anti-Kähler manifold, then, according (1.6)

$$\frac{\kappa}{2n-1} - \sum_i \widetilde{\rho}(e_i, Je_i) = \frac{2n}{2n-1}\kappa,$$

$$\widetilde{\kappa} - \sum_i \rho(e_i, Je_i) = 0.$$

Thus,

$$\underset{4}{\mathrm{B}}(X,Y,Z,W) = \frac{2n}{2n-1}\kappa[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)$$
$$-F(X,W)F(Y,Z) + F(X,Z)F(Y,W)]$$

and (7.4) reduces to

$$\underset{1}{\mathrm{B}} = \frac{1}{n-2}[(n-1)\underset{2}{\mathrm{B}} - \underset{3}{\mathrm{B}}] + \frac{n}{4(n-1)(n-2)}\underset{4}{\mathrm{B}},$$

where, now, $\underset{1}{\mathrm{B}}$, $\underset{2}{\mathrm{B}}$ and $\underset{3}{\mathrm{B}}$ are given by the relation (4.5), (5.10) and (6.6) respectively.

REFERENCES

[1] A. B o r o v i e c, M. F r a n c o v i g l i a, I. V o l o v i c h, *Anti-Kähler manifolds,* Ar. XIV: math-ph/ 9906012 VI, 13 june 1999

[2] R. D e s z c z, M. P r v a n o v i ć, *Complex hypersurface of a holomorphically conformally flat anti-Kähler manifold,* Analele Stiintifice ale Universitatii "Al.I.Cuza" din Iaşi (N.S.) Matematica, T.LIII, 2007, Supliment, 123–143.

[3] G. G a n c h e v, *On Bochner curvature tensors in almost Hermitian manifolds,* Pliska, Studia math. Bulgarica., **9** (1987), 33–43.

[4] V. O p r o i u, N. P a p a g h i u c, *Some classes of almost anti-Hermitian structures on the tangent bundl,* Mediterranean J.Math., 1 (2004), 269–282.

[5] D. M e k e r o v, M. M a n e v, *On the geometry of quasi-Kähler manifolds with Norden metruc,* Ar.X.IV: 0804.2804 VI math D.G, 17. Apr. 2008

[6] J. M i k e š, A. V a n ž u r o v a, I. H i n t e r l e i t n e r, *Geodesic mapping and some generalization,* Palaský Univ. Olomouc, Faculty of Science, Olomouc, 2009.

[7] A. P. N o r d e n, *On a class of four dimensional A-manifolds*, Izv. Vuzov.Mat. **4** (1960), 145–152 (in Rusian)

[8] E. P a v l o v, *Conformally equivalent generalized B-manifolds,* Com. Rend.Acad. Bulg. Sci. 38 (1985), 1315–1317

[9] M. P r v a n o v i ć, N. P u š i ć, *Some conformally invariant tensors on anti-Kähler manifolds and their geometrical properties,* submitted for the publication

[10] K. S l u k a, *Properties of the Weyl conformal curvature of Kähler-Norden manifolds, Steps in differential geometry,* Proceeding of the colloquium on the differential geometry, 25-30, July, 2000, Debrecen, Hungary, 317–328.

[11] G. S t a n i l o v, *Relation between Weyl's and Bochner's curvature tensprs,* Serdica **86** (1982), 166–169

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