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SPINORIAL MATTER AND GENERAL RELATIVITY THEORY

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A b s t r a c t. World spinors, the spinorial matter (particles, p-branes and fields) in a generic curved space is considered. Representation theory, as well as the basic algebraic and topological properties of relevant symmetry groups are presented. Relations between spinorial wave equations that transform respectively w.r.t. the tangent flat-space (anholonomic) Affine symmetry group and the world generic-curved-space (holonomic) group of Diffeomorphisms are presented. World spinor equations and certain basic constraints that yield a viable physical theory are discussed. A geometric construction based on an infinite-component generalization of the frame fields (e.g. tetrads) is outlined. The world spinor field equation in $3D$ *is treated in more details.*

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1. *Introduction*

Point-like matter.

The Dirac equation turned out to be one of the most successful equations of the XX century physics - it describes the basic matter constituents (both particles and fields), and very significantly, it paved a way to develop the concept of gauge theories thus playing an important role in description of the basic interactions (forces) as well. It is a Poincaré covariant linear field equation which describes relativistic spin $\frac{1}{2}$ matter objects, that a coupled to fundamental interactions in Nature through the gauge (minimal coupling) prescription.

The first natural step towards a generalization of the Poincaré invariant theories to those defined in a generic curved spacetime is to study spinorial and tensorial matter representations and generalizations of the Dirac equation in the Affine invariance framework. Subsequently, a genuine generalization that will describe world spinors, spinorial matter in a generic curved spacetime (L_n, g) characterized by arbitrary torsion and general-linear curvature, is in order. Note that descriptions of the spinorial matter in theories based on higher-dimensional orthogonal-type groups that generalize the Lorentz group in 4-dimensions (non-affine generalizations of GR) are mathematically possible for special spacetime configurations only, and thus fail to extend to the generic curved spacetime.

The finite-dimensional world tensor fields of Einstein's General Relativity theory in R^n (n-dimensional spacetime) are characterized by non-unitary irreducible representations of the general linear subgroup $GL(n, \mathbb{R})$ of the General Coordinate Transformations (GCT) group, i.e. the Diffeomorphism group $Diff(n, \mathbb{R})$. In the flat-space limit they split up into the $SO(1, n-1)$ ($SL(2, C)/Z_2$ for $n = 4$) group irreducible pieces. The corresponding particle states are defined in the tangent flat-space only. They are characterized by the unitary irreducible representations of the (inhomogeneous) Poincaré group $P(n) = T_n \wedge SO(1, n-1)$, and they are defined by the labels of a relevant "little" group unitary irreducible representations.

In the first step towards generalization to world spinors, the double covering group, $\overline{SO}(1, n-1) = Spin(n)$, of the $SO(1, n-1)$ one, that characterizes a Dirac-type fields in $D = n$ dimensions, is enlarged to the $\overline{SL}(n, \mathbb{R}) \subset \overline{GL}(n, \mathbb{R})$ group,

$$\overline{SO}(1, n-1) \quad \mapsto \quad \overline{SL}(n, \mathbb{R}) \subset \overline{GL}(n, \mathbb{R})$$

while the special Affine group $SA(n, \mathbb{R}) = T_n \wedge \overline{SL}(n, \mathbb{R})$ is to replace the

Poincaré group itself.

$$P(n) \quad \mapsto \quad \overline{SA}(n, \mathbb{R}) = T_n \wedge \overline{SL}(n, \mathbb{R})$$

Now, affine "particles" are characterized by the unitary irreducible representations of the $\overline{SA}(n, \mathbb{R})$ group, that are actually nonlinear unitary representations over an appropriate "little" group. E.g. for $m \neq 0$:

$$T_{n-1} \otimes \overline{SL}(n-1, \mathbb{R}) \supset T_{n-1} \otimes \overline{SO}(n)$$

Affine fields are, in its turn, characterized by infinite-dimensional $\overline{SL}(n, \mathbb{R})$ representations. A mutual particle–field correspondence is achieved by requiring (i) that fields have appropriate mass (Klein-Gordon-like equation condition, for $m \neq 0$), and (ii) that the subgroup of the field-defining homogeneous group, which is isomorphic to the homogeneous part of the "little" group, is represented unitarily. Furthermore, one has to project away all representations except the one that characterizes the particle states. A physically correct picture, in the affine case, is obtained by making use of the $\overline{SA}(n, \mathbb{R})$ group unitary (irreducible) representations for "affine" particles. The affine-particle states are characterized by the unitary (irreducible) representations of the $T_{n-1} \otimes \overline{SL}(n-1, \mathbb{R})$ "little" group. The intrinsic part of these representations is necessarily infinite-dimensional (c.f. [1, 8] for the $n = 3$ case) due to non-compactness of the $SL(n, \mathbb{R})$ group. The corresponding affine fields should be described by the non-unitary infinite-dimensional $\overline{SL}(n, \mathbb{R})$ representations, that are unitary when restricted to the homogeneous "little" subgroup $\overline{SL}(n-1, \mathbb{R})$. Therefore, as already stated, the first step towards world spinor fields is a construction of infinite-dimensional non-unitary $\overline{SL}(n, \mathbb{R})$ representations, that are unitary when restricted to the $\overline{SL}(n-1, \mathbb{R})$ group. These fields reduce to an infinite sum of (non-unitary) finite-dimensional $\overline{SO}(1, n-1)$ fields.

Extended objects matter.

The subject of extended objects was initiated in the particle/field theory framework by the Dirac action for a closed relativistic membrane as the $(2+1)$-dimensional world-volume swept out in spacetime. It evolved and become one of the central topics following the Nambu-Goto action for a closed relativistic string, as the $(1+1)$-dimensional world-sheet area swept out in spacetime. An important step was the Polyakov action for a closed relativistic string, with auxiliary metric, that enabled consequent formulations of the Green-Schwarz superstring, and the bosonic, and super p-branes with manifest spacetime supersymmetry. In this work, we follow the original path of the Nambu-Goto-like formulation of the bosonic p-brane and address the question of spinors of the brane world-volume symmetries. For $p = 1$, these spinors are well known, and represent an important ingredient of the spinning string formulation and the Neveu-Schwarz-Ramond infinite algebras.

It is interesting to point out that there is a direct analogy between the spinors of the p-brane action symmetry, that are considered in this work, and the so world spinors that describe the spinorial matter fields of the Metric-Affine [2] and Gauge-Affine theories of gravity formulated in a generic (non)Riemannian spacetime of arbitrary torsion and curvature. This is due to common geometric and group-theoretic structures of p-brane theories and affine generalizations of Einstein's gravity theory. The global symmetry of these matter fields in 4 dimensions is the affine $\overline{SA}(4, \mathbb{R})$ group, that generalizes the $\overline{P}(4)$ Poincaré group of the conventional gauge approach to the theory of gravity. When gauging the affine group, one has a complete parallel of both anholonomic (local) and holonomic (world) description of bosonic and fermionic matter fields. In contradistinction to the gauge Poincaré theory, where spinors are scalars of the General Coordinate Transformations group, in the affine case there are both local (tangent) and world (curved) spinors. Analogously, the spinors of the spinning string theory are just the faithful world-sheet spinors.

It was shown by Ogievetsky that the infinite algebra of the General Coordinate Transformation group in 4 dimensions arises upon a Lie algebra closure of the finite algebras of the $SL(3, \mathbb{R})$ group and the 4-dimensional Conformal group. This result paved a way for various approaches to gravity theory, especially those that utilized the nonlinear representations techniques. In particular, it was proven that Einstein's theory of gravity is obtained by simultaneous nonlinear realizations of the affine and Conformal symmetries. The affine and its linear subgroup are nonlinearly realized w.r.t. the Poincaré and Lorentz subgroups, respectively. Thus, the General Coordinate Transformations group is realized over its Poincaré and/or Lorentz subgroup is a nonlinear manner, resulting in a loss of the world (curved space) spinors that demand linear representations over its affine and/or linear subgroup. The spinorial representations of the infinite algebras of the General Coordinate Transformations in three and four dimensions were studied in [3].

Consider a bosonic p-brane embedded in a D-dimensional flat Minkowski spacetime $M^{1,D-1}$. The classical Dirac-Nambu-Goto-like action for p-brane is given by the volume of the world volume swept out by the extended object in the course of its evolution from some initial to some final configuration:

$$S = -\frac{1}{\kappa} \int d^{p+1}\xi \ \sqrt{-det\partial_i X^m \partial_j X^n \eta_{mn}} \ ,$$

where $i = 0, 1, \ldots, p$ labels the coordinates $\xi^i = (\tau, \sigma_1, \sigma_2, \ldots)$ of the brane world volume with metric $\gamma_{ij}(\xi)$, and $\gamma = \det(\gamma_{ij})$; $m = 0, 1, \ldots, D-1$ labels the target space coordinates $X^m(\xi^i)$ with metric η_{mn}. The world volume metric $\gamma_{ij} = \partial_i X^m \partial_j X^n \eta_{mn}$ is induced from the spacetime metric η_{mn}.

The Poincaré $P(1, D-1)$ group, i.e. its homogeneous Lorentz subgroup $SO(1, D-1)$, are the physically relevant spacetime symmetries, while the $(p+1)$-dimensional brane world volume is preserved by the homogeneous volume preserving subgroup $SDiff_0(p+1, \mathbb{R})$ of the General Coordinate Transformation group $Diff(p+1, \mathbb{R})$.

The $sdiff_0(p+1, \mathbb{R})$ algebra operators, that generate the $SDiff_0(p+1, \mathbb{R})$ group, are given as follows [12],

$$sdiff_0(p+1, \mathbb{R}) = \left\{ L^{i_1 i_2 \ldots i_{n-1}}_{(n)k} = \xi^{i_1} \xi^{i_2} \ldots \xi^{i_{n-1}} \frac{\partial}{\partial \xi^k} \quad | \quad n = 2, 3, \ldots \infty \right\}.$$

Preservation of the world volume requires the $L_{(2)}$ operator to be traceless as achieved by subtracting the dilation operator, i.e. $L^i_{(2)k} = \xi^i \frac{\partial}{\partial \xi^k} - \frac{1}{p+1} \delta^i_k \xi^j \frac{\partial}{\partial \xi^j}$. The $L_{(n)}$, $n = 2, 3, \ldots \infty$, operators are irreducible tensor operators of the $SL(p+1, \mathbb{R})$ subgroup, and therfore naturally labeled by the $SL(p+1, \mathbb{R})$ irreducible representations given by the Young tableaux $[\lambda_1, \lambda_2, \ldots, \lambda_p]$ with $\lambda_1 = 2, 3, \ldots \infty$, and $\lambda_2 = \lambda_3 = \ldots = \lambda_p = 1$.

The $SDiff_0(p+1, \mathbb{R})$ commutation relations read:

$$\begin{aligned}
&[L^{i_1 i_2 \ldots i_{m-1}}_{(m)k}, L^{j_1 j_2 \ldots j_{n-1}}_{(n)l}] \\
&= \delta^{j_1}_k L^{i_1 i_2 \ldots i_{m-1} j_2 j_3 \ldots j_{n-1}}_{(m+n-2)l} + \delta^{j_2}_k L^{i_1 i_2 \ldots i_{m-1} j_1 j_3 \ldots j_{n-1}}_{(m+n-2)l} \\
&+ \ldots + \delta^{j_{n-1}}_k L^{i_1 i_2 \ldots i_{m-1} j_1 j_2 \ldots j_{n-2}}_{(m+n-2)l} \\
&- \delta^{i_1}_l L^{i_2 i_3 \ldots i_{m-1} j_1 j_2 \ldots j_{n-1}}_{(m+n-2)k} - \delta^{i_2}_l L^{i_1 i_3 \ldots i_{m-1} j_1 j_2 \ldots j_{n-1}}_{(m+n-2)k} \\
&- \ldots - \delta^{i_{m-1}}_l L^{i_1 i_2 \ldots i_{m-2} j_1 j_2 \ldots j_{n-1} m}_{(m+n-2)k}.
\end{aligned}$$

The above symmetry considerations are purely classical. In the quantum case, the corresponding classical symmetry is modified, up to eventual anomalies, in two ways: (i) the classical group is replaced by its universal covering group, and (ii) the group is minimally extended by the $U(1)$ group of phase factors. The corresponding Lie algebra remains unchanged in the first case, while in the second one, it can have additional central charges.

The feasible ways how to extend the Dirac-Nambu-Goto bosonic p-brane action by the fermionic degrees of freedom are determined by the universal covering group $\overline{SDiff}_0(p+1,\mathbb{R})$ of the $SDiff_0(p+1,\mathbb{R})$ group and the form of its spinorial representations. In the following we address at first the topological issues that define the type of the universal covering of the $SDiff_0(p+1,\mathbb{R})$ group, and subsequently, we face the problem of the $\overline{SDiff}_0(p+1,\mathbb{R})$ group spinorial representations construction.

2. $\overline{GL}(n,\mathbb{R})$ *and* $Diff(n,\mathbb{R})$ *double-coverings*

Let us state first some relevant mathematical results.

Theorem 1: Let $g_0 = k_0 + a_0 + n_0$ be an Iwasawa decomposition of a semisimple Lie algebra g_0 over R. Let G be any connected Lie group with Lie algebra g_0, and let K, A, N be the analytic subgroups of G with Lie algebras k_0,a_0 and n_0 respectively. The mapping $(k,a,n) \to kan$ $(k \in K, a \in A, n \in N)$ is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G, and *the groups* A *and* N *are simply connected.*

Any semisimple Lie group can be decomposed into the product of the maximal compact subgroup K, an Abelian group A and a nilpotent group N. As a result of Theorem 1, only K is not guaranteed to be simply-connected. There exists a universal covering group $\overline{K}$ of K, and thus also a universal covering of G: $\overline{G} \simeq \overline{K} \times A \times N$.

For the group of diffeomorphisms, let $Diff(n,\mathbb{R})$ be the group of all homeomorphisms f or R^n such that f and f^{-1} are of class C^1. Stewart proved the decomposition $Diff(n,\mathbb{R}) = GL(n,\mathbb{R}) \times H \times R_n$, where the subgroup H is contractible to a point. Thus, as $O(n)$ is the compact subgroup of $GL(n,\mathbb{R})$, one finds

Theorem 2: $O(n)$ is a deformation retract of $Diff(n,\mathbb{R})$.

As a result, there exists a universal covering of the Diffeomorphism group $\overline{Diff}(n,\mathbb{R}) \simeq \overline{GL}(n,\mathbb{R}) \times H \times R_n$.

Summing up, we note that $SL(n,\mathbb{R})$ as well as $GL(n,\mathbb{R})$ and $Diff(n,\mathbb{R})$ all have double coverings, defined by $\overline{SO}(n)$ and $\overline{O}(n)$ respectively, the double-coverings of the $SO(n)$ and $O(n)$ maximal compact subgroups.

Let us consider now the question of the universal, i.e. double, covering of the $SL(n,\mathbb{R})$ and $SDiff_0(n,\mathbb{R})$ groups themselves. The universal covering group $\overline{G}$ of a given group G is a group with the same Lie algebra and with a simply-connected group manifold. A finite dimensional covering $\overline{SL}(n,\mathbb{R})$,i.e. $\overline{SDiff}_0(n,\mathbb{R})$, exists provided one can embed $\overline{SL}(n,\mathbb{R})$ into a group of finite complex matrices that contain $Spin(n)$ as subgroup. A scan of the semi-simple classical algebras, as given by the Cartan classification, points at first to the $SL(n,C)$ groups as a natural candidates for the $SL(n,\mathbb{R})$ groups coverings. However, there is no match whatsoever of the defining dimensionalities of the $SL(n,C)$ and $Spin(n)$ groups for $n \geq 3$,

$$dim(SL(n,C)) = n \quad < \quad 2^{\left[\frac{n-1}{2}\right]} = dim(Spin(n)),$$

except for $n = 8$. In the $n = 8$ case, one finds that the orthogonal subgroup of the $SL(8,\mathbb{R})$ and $SL(8,C)$ groups is $SO(8)$ and not $Spin(8)$. For a detailed account of the $D = 4$ case cf. [6]. Thus, we conclude that there are no covering groups of the $SL(n,\mathbb{R})$, i.e $\overline{SDiff}_0(n,\mathbb{R})$ groups for any $n \geq 3$ that are given by finite matrices (defined in finite-dimensional complex spaces). An explicit construction of all spinorial, unitary and nonunitary multiplicity-free and unitary non-multiplicity-free [1], $SL(3,\mathbb{R})$ representations shows that they are indeed all defined in infinite-dimensional spaces.

3. *The deunitarizing automorphism*

The unitarity properties, that ensure correct physical characteristics of the affine fields, can be achieved by combining the unitary (irreducible) representations and the so called "deunitarizing" automorphism [6] of the $\overline{SL}(n,\mathbb{R})$ group.

The commutation relations of the $\overline{SL}(n,\mathbb{R})$ generators Q_{ab}, $a,b = 0,1,\ldots,n-1$ are

$$[Q_{ab}, Q_{cd}] = i(\eta_{bc}Q_{ad} - \eta_{ad}Q_{cb}),$$

taking $\eta_{ab} = diag(+1,-1,\ldots,-1)$. The important subalgebras are as follows.

(i) $so(1,n-1)$: The $M_{ab} = Q_{[ab]}$ operators generate the Lorentz-like subgroup $\overline{SO}(1,n-1)$ with $J_{ij} = M_{ij}$ (angular momentum) and $K_i = M_{0i}$ (the boosts) $i,j = 1,2,\ldots,n-1$.

(ii) $so(n)$: The J_{ij} and $N_i = Q_{\{0i\}}$ operators generate the maximal compact subgroup $\overline{SO}(n)$.

(iii) $sl(n-1)$: The J_{ij} and $T_{ij} = Q_{\{ij\}}$ operators generate the subgroup $\overline{SL}(n-1,\mathbb{R})$ - the "little" group of the massive particle states.

The $\overline{SL}(n,\mathbb{R})$ commutation relations are invariant under the "deunitarizing" automorphism,

$$J'_{ij} = J_{ij} \ , \quad K'_i = iN_i \ , \quad N'_i = iK_i \ ,$$
$$T'_{ij} = T_{ij} \ , \quad T'_{00} = T_{00} \ (= Q_{00}) \ ,$$

so that $(J_{ij},\ iK_i)$ generate the new compact $\overline{SO}(n)'$ and $(J_{ij},\ iN_i)$ generate $\overline{SO}(1,n-1)'$.

For the massive (spinorial) particle states we use the basis vectors of the unitary irreducible representations of $\overline{SL}(n,\mathbb{R})'$, so that the compact subgroup finite multiplets correspond to $\overline{SO}(n)'$: $(J_{ij},\ iK_i)$ while $\overline{SO}(1,n-1)'$: $(J_{ij},\ iN_i)$ is represented by unitary infinite-dimensional representations. We now perform the inverse transformation and return to the unprimed $\overline{SL}(n,\mathbb{R})$ for our physical identification: $\overline{SL}(n,\mathbb{R})$ is represented non-unitarily, the compact $\overline{SO}(n)$ is represented by non-unitary infinite representations while the Lorentz group is represented by non-unitary finite representations. These finite-dimensional non-unitary Lorentz group representations are necessary in order to ensure a correct particle interpretation (i.g. boosted proton remains proton). Note that $\overline{SL}(n-1,\mathbb{R})$, the stability subgroup of $\overline{SA}(n,\mathbb{R})$, is represented unitarily.

4. *World spinor field transformations*

The world spinor fields transform w.r.t. $\overline{Diff}(n,\mathbb{R})$ as follows [7]
$(D(a,\bar{f})\Psi_M)(x) = (D_{\overline{Diff}_0(n,\mathbb{R})})^N_M(\bar{f})\Psi_N(f^{-1}(x-a))$,
$(a,\bar{f}) \in T_n \wedge \overline{Diff}_0(n,\mathbb{R})$,
where $\overline{Diff}_0(n,\mathbb{R})$ is the homogeneous part of $\overline{Diff}(n,\mathbb{R})$, and $D_{\overline{Diff}_0(n,\mathbb{R})} \supset \sum^{\oplus} D_{\overline{SL}(n,\mathbb{R})}$ is the corresponding representation in the space of world spinor field components. As a matter of fact, we consider here those representations of $\overline{Diff}_0(n,\mathbb{R})$ that are nonlinearly realized over the maximal linear subgroup $\overline{SL}(n,\mathbb{R})$ (here given in terms of infinite matrices). Due to these more complex field transformation properties, a question of generalizing the Dirac equation to the Affine case is rather subtle [9].

The affine "particle" states transform according to the following representation

$$D(a,\bar{s}) \to e^{i(sp)\cdot a} D_{\overline{SL}(n,\mathbb{R})}(L^{-1}(sp)\bar{s}L(p)), \quad (a,\bar{s}) \in T_n \wedge \overline{SL}(n,\mathbb{R}),$$

where $L \in \overline{SL}(n,\mathbb{R})/\overline{SL}(n-1,\mathbb{R})$, and p is the n-momentum. The unitarity properties of various representations in these expressions are as described in the previous section.

Provided the relevant $\overline{SL}(n,\mathbb{R})$ representations are known, one can first define the corresponding general/special Affine spinor fields in the tangent to R^n, and than make use of the infinite-component pseudo-frame fields $E^A_M(x)$, "alephzeroads", that generalize the tetrad fields of R^4. Let us define a pseudo-frame $E^A_M(x)$ [5] s.t.

$$\Psi_M(x) = E^A_M(x)\Psi_A(x),$$

where $\Psi_M(x)$ and $\Psi_A(x)$ are the world (holonomic), and general/special Affine spinor fields respectively. The $E^A_M(x)$ (and their inverses $E^M_A(x)$) are thus infinite matrices related to the quotient $\overline{Diff}_0(n,\mathbb{R})/\overline{SL}(n,\mathbb{R})$. Their infinitesimal transformations are

$$\delta E^A_M(x) = i\epsilon^a_b(x)\{Q^b_a\}^A_B E^B_M(x) + \partial_\mu \xi^\nu e^a_\nu e^\mu_b \{Q^a_b\}^A_B E^B_M(x),$$

where ϵ^a_b and ξ^μ are group parameters of $\overline{SL}(n,\mathbb{R})$ and $\overline{Diff}(n,\mathbb{R})/\overline{Diff}_0(n,\mathbb{R})$ respectively, while e^a_ν are the standard n-bine fields.

The infinitesimal transformations of the world spinor fields themselves are given as follows:

$$\delta\Psi^M(x) = i\{\epsilon^a_b(x)E^M_A(x)(Q^b_a)^A_B E^B_N(x) + \xi^\mu[\delta^M_N\partial_\mu + E^M_B(x)\partial_\mu E^B_N(x)]\}\Psi^N(x).$$

The $(Q^b_a)^M_N = E^M_A(x)(Q^b_a)^A_B E^B_N(x)$ is the holonomic form of the $\overline{SL}(n,\mathbb{R})$ generators given in terms of the corresponding anholonomic ones in the space of spinor fields $\Psi_M(x)$ and $\Psi_A(x)$ respectively.

The above outlined construction allows one to define a fully $\overline{Diff}(n,\mathbb{R})$ covariant Dirac-like wave equation for the corresponding world spinor fields provided a Dirac-like wave equation for the $\overline{SL}(n,\mathbb{R})$ group is known [10]. In other words, one can lift an $\overline{SL}(n,\mathbb{R})$ covariant equation of the form

$$(ie^\mu_a(X^a)^B_A\partial_\mu - M)\Psi_B(x) = 0,$$

to a $\overline{Diff}(n,\mathbb{R})$ covariant equation

$$(ie^\mu_a E^N_B(X^a)^B_A E^A_M\partial_\mu - M)\Psi_N(x) = 0,$$

provided a spinorial $\overline{SL}(n,\mathbb{R})$ representation for the Ψ field is given, with the corresponding representation Hilbert space invariant w.r.t. X^a action. Thus, the crucial step towards a Dirac-like world spinor equation is a construction of the corresponding $\overline{SL}(n,\mathbb{R})$ wave equation.

5. $\overline{SL}(n,\mathbb{R})$ *vector operator* X

For the construction of a Dirac-type equation, which is to be invariant under (special) affine transformations, there are two possible approaches to derive the matrix elements of the generalized Dirac matrices X_a.

One can consider the defining commutation relations of a $\overline{SL}(n,\mathbb{R})$ vector operator X_a,

$$[M_{ab}, X_c] = i\eta_{bc}X_a - i\eta_{ac}X_b$$
$$[T_{ab}, X_c] = i\eta_{bc}X_a + i\eta_{ac}X_b,$$

and obtain the matrix elements of the generalized Dirac matrices X_a by solving these relations for X_a in the Hilbert space of some suitable representation of $\overline{SL}(n,\mathbb{R})$.

Alternatively, one can embed [11] $\overline{SL}(n,\mathbb{R})$ into $\overline{SL}(n+1,\mathbb{R})$. Let the generators of $\overline{SL}(n+1,\mathbb{R})$ be $Q_A{}^B$, $A, B = 0, ..., n$. Now, there are two natural $\overline{SL}(5,\mathbb{R})$ vectors X_a, and Y_a defined by

$$X_a = Q_{an}, \quad Y_a = Q_{na}, \qquad a = 0, 1, \dots n.$$

The operator X_a (Y_a) obtained in this way fulfills the required $SL(n,\mathbb{R})$ vector commutation relations by construction. It is interesting to point out that the operator $G_a = \frac{1}{2}(X_a - Y_a)$ satisfies

$$[G_a, G_b] = -iM_{ab},$$

thereby generalizing a property of Dirac's γ-matrices. Since X_a, M_{ab} and T_{ab} form a closed algebra, the action of X_a on the $\overline{SL}(n,\mathbb{R})$ states does not move out of the $\overline{SL}(n+1,\mathbb{R})$ representation Hilbert space.

In order to illustrate the general structure of the operator X_a action, let us consider the following embedding of three finite-dimensional tensorial $SL(n,\mathbb{R})$ irreducible representations into the corresponding $SL(n+1,\mathbb{R})$ representation ,

$$\begin{array}{ccc} SL(n+1,\mathbb{R}) & \supset & SL(n,\mathbb{R}) \\ \underbrace{\overset{(n+1)(n+2)/2}{\square\square}}_{\varphi_{AB}} & \supset & \underbrace{\overset{n(n+1)/2}{\square\square}}_{\varphi_{ab}} \oplus \underbrace{\overset{n}{\boxtimes\square}}_{\varphi_a} \oplus \underbrace{\overset{1}{\boxtimes\boxtimes}}_{\varphi}, \end{array}$$

where "box" is the Young tableau for an irreducible vector representation of $SL(m,\mathbb{R})$, $m = n, n+1$. The effect of the action of the $SL(n,\mathbb{R})$ vector X_a on the fields φ, φ_a and φ_{ab} is as follows,

$$\overset{X_a}{\square} \otimes \overset{\varphi}{\boxtimes\boxtimes} \mapsto \overset{\varphi_a}{\square}, \qquad \overset{X_a}{\square} \otimes \overset{\varphi_a}{\boxtimes\square} \mapsto \overset{\varphi_{ab}}{\square\square}, \qquad \overset{X_a}{\square} \otimes \overset{\varphi_{ab}}{\square\square} \mapsto 0\,.$$

Other possible Young tableaux do not appear due to invariance of the representation Hilbert space. Gathering these fields in a vector $\varphi_M = (\varphi, \varphi_a, \varphi_{ab})^{\mathrm{T}}$, we can read off the structure of X_a, which in the case $n = 4$ reads:

$$X_a = \left[\begin{array}{c|c|c} 0 & & \\ \hline \begin{matrix}\mathrm{x}\\\mathrm{x}\\\mathrm{x}\\\mathrm{x}\end{matrix} & \mathbf{0}_4 & \\ \hline & \begin{matrix}\mathrm{xxxx}\\\mathrm{xxxx}\\\mathrm{xxxx}\\\mathrm{xxxx}\\\mathrm{xxxx}\\\mathrm{xxxx}\\\mathrm{xxxx}\\\mathrm{xxxx}\\\mathrm{xxxx}\\\mathrm{xxxx}\end{matrix} & \mathbf{0}_{10} \end{array}\right].$$

It is a significant result that X_a has zero matrices on the block-diagonal which implies that the mass operator κ of an affine invariant equation must vanish.

This can be proven for a general finite representation of $SL(4,\mathbb{R})$. Let us consider the action of a vector operator on an arbitrary irreducible representation $D(g)$ of $SL(4,\mathbb{R})$ labeled by $[\lambda_1, \lambda_2, \dots \lambda_{n-1}]$, λ_i being the number of boxes in the i-th raw,

$$\begin{aligned} &[\lambda_1, \lambda_2, \dots \lambda_{n-1}] \otimes [1, 0, \dots 0] \\ &= [\lambda_1 + 1, \lambda_2, \dots \lambda_{n-1}] \oplus [\lambda_1, \lambda_2 + 1, \dots \lambda_{n-1}] \oplus \\ &\quad [\lambda_1, \lambda_2, \dots \lambda_{n-1} + 1] \oplus [\lambda_1 - 1, \lambda_2 - 1, \dots \lambda_{n-1} - 1]\,. \end{aligned}$$

None of the resulting representations coincides neither with the representation $D(g)$ itself nor with its contragradient representation $D^c(g) = D^{\mathrm{T}}(g^{-1})$ given by

$$[\lambda_1, \lambda_2, \lambda_3]^{\mathrm{c}} = [\lambda_1, \lambda_1 - \lambda_{n-1}, \dots \lambda_1 - \lambda_2]\,.$$

For a general (reducible) representation this implies, by a similar argumentation, that the vector operator X_a has null matrices on the block-diagonal positions. Let the representation space be spanned by $\Phi = (\varphi_1, \varphi_2, ...)^{\mathrm{T}}$ with φ_i irreducible. Now we consider the Dirac-type equation in the rest frame, i.e. for $p_\mu = (E, 0, \dots 0)$, restricted to the subspaces spanned by φ_i $(i = 1, 2, \dots)$,

$$E < \varphi_i, X^0 \varphi_j > = < \varphi_i, M \varphi_i > = m_i \delta_{ij},$$

where we assumed the mass operator M to be diagonal. Due to the above properties of the X_a operator (vanishing block diagonal parts), it follows that the mass eigenvalues m_i vanish and the entire mass operator M equals zero since $< \varphi_i, X^0 \varphi_i >= 0$. Therefore, for an affine invariant Dirac-type wave equation the mass generation can only be dynamical, i.e. a result of an interaction. This agrees with the fact that the Casimir operator of the special affine group $\overline{SA}(4,\mathbb{R})$ vanishes leaving the masses unconstrained. It is natural to expect that this result holds also for infinite representations of the $\overline{SL}(n,\mathbb{R})$ symmetry group.

6. *Lie group decontraction and $SL(n,\mathbb{R})$ (unitary) irreducible representations*

The key issue of the affine and/or world matter description, as well as of the corresponding physical applications, is a detailed knowledge of the $\overline{SL}(n,\mathbb{R})$ group (unitary) irreducible spinorial and tensorial representations. We present briefly a new approach to this problem based on an inverse procedure to the Lie algebra/group contraction.

The "decontraction" or Gell-Mann formula, advocated by Hermann, is an expression designed to play a role of an "inverse" to the well known Lie algebra/group contraction notion introduced by Inönü and Wigner. Let $\mathcal{A}$ be a symmetric Lie algebra $\mathcal{A} = \mathcal{M} + \mathcal{T}$ with a subalgebra $\mathcal{M}$ such that:

$$[\mathcal{M},\mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M},\mathcal{T}] \subset \mathcal{T}, \quad [\mathcal{T},\mathcal{T}] \subset \mathcal{M}.$$

Further, let $\mathcal{A}'$ be its Inönü-Wigner contraction algebra w.r.t its subalgebra $\mathcal{M}$, i.e. $\mathcal{A}' = \mathcal{M} + \mathcal{U}$, where

$$[\mathcal{M},\mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M},\mathcal{U}] \subset \mathcal{U}, \quad [\mathcal{U},\mathcal{U}] = \{0\}.$$

The Gell-Mann formula states that the elements $T \in \mathcal{T}$ can be obtained in terms of the contracted algebra elements $M \in \mathcal{M}$ and $U \in \mathcal{U}$ by the following rather simple expression:

$$T = i \frac{\alpha}{\sqrt{U \cdot U}} [C_2(\mathcal{M}), U] + \sigma U,$$

Where $C_2(\mathcal{M})$ denotes the second order Casimir operator of the $\mathcal{M}$ subalgebra, α is a normalization constant and σ is an arbitrary parameter.

This formula is valid on the algebraic level in the case of $\mathcal{A} = so(p,q)$ contracted w.r.t. $\mathcal{M} = so(p-1,q)$ and/or $\mathcal{M} = so(p,q-1)$ subalgebras, while it has a rather limited validity in the case of $sl(n,\mathbb{R})$ contracted w.r.t. its maximal compact $so(n)$ subalgebra. In the latter case, the original Gell-Mann formula holds only for the algebra representation spaces that are multiplicity free when reduced under the $so(n)$ subalgebra representations. Recently, we generalized the Gell-Mann formula to hold for an arbitrary $sl(n,\mathbb{R})$ irreducible representation, thus paving an efficient way of constructing explicitly, in a closed form, all matrix elements of the (unitary) infinite-dimensional irreducible $sl(n,\mathbb{R})$ representations [13].

In the case of the $sl(4,\mathbb{R})$ algebra, the 9 noncompact algebra operators, as given in the basis of its maximal compact subalgebra $so(4) = so(3)\oplus so(3) \supset so(2)\oplus so(2)$ are given by the generalized Gell-Mann formula as follows

$$
\begin{aligned}
&T_{\mu_1\mu_2} = \sigma D^{11}_{00\mu_1\mu_2} + \frac{i}{2}[C_2(so(4)), D^{11}_{00\mu_1\mu_2}] \\
&+\delta_1(D^{11}_{11\mu_1\mu_2} + D^{11}_{-1-1\mu_1\mu_2}) + (D^{11}_{11\mu_1\mu_2} - D^{11}_{-1-1\mu_1\mu_2})(K^{10}_{00} + K^{01}_{00}) \\
&+\delta_2(D^{11}_{-11\mu_1\mu_2} + D^{11}_{1-1\mu_1\mu_2}) + (D^{11}_{-11\mu_1\mu_2} - D^{11}_{1-1\mu_1\mu_2})(K^{10}_{00} - K^{01}_{00}),
\end{aligned}
$$

where $\mu_1, \mu_2 = 0, \pm 1$. As the rank of the $sl(4,\mathbb{R})$ algebra is three, there are precisely three representation labels σ, δ_1, and δ_2 (if complex, only three real are independent).

In the case of the $sl(5,\mathbb{R})$ algebra, the 14 noncompact algebra operators, as given in the basis of its maximal compact subalgebra $so(5) \supset so(4) = so(3)\oplus so(3) \supset so(3)$ are given by the generalized Gell-Mann formula as follows :

$$
\begin{aligned}
&T^{\ j_1 j_2}_{\ \mu_1\mu_2} = \sigma_1 D^{\substack{\overline{11} \\ 00 j_1\, j_2}}_{00\mu_1\mu_2} + [C_2(so(5)), D^{\substack{\overline{11} \\ 00 j_1\, j_2}}_{00\mu_1\mu_2}] \\
&+\sqrt{\tfrac{5}{4}}\Bigg(\sigma_2 D^{\substack{\overline{11} \\ 11 j_1\, j_2}}_{00\mu_1\mu_2} + [C_2(so(4)_K), D^{\substack{\overline{11} \\ 11 j_1\, j_2}}_{00\mu_1\mu_2}] \\
&-D^{\substack{\bar{1}\ \bar{1} \\ 1\ 1\ j_1\, j_2}}_{1-1\mu_1\mu_2}(\delta_1 + K^{\substack{\overline{10} \\ 10}}_{00} - K^{\substack{\overline{10} \\ 01}}_{00}) - D^{\substack{\bar{1}\ \bar{1} \\ 1\ 1 j_1\, j_2}}_{-11\mu_1\mu_2}(\delta_1 - K^{\substack{\overline{10} \\ 10}}_{00} + K^{\substack{\overline{10} \\ 01}}_{00}) \\
&+D^{\substack{\overline{11} \\ 11 j_1\, j_2}}_{11\mu_1\mu_2}(\delta_2 + K^{\substack{\overline{10} \\ 10}}_{00} + K^{\substack{\overline{10} \\ 01}}_{00}) + D^{\substack{\bar{1}\ \bar{1} \\ 1\ 1\ j_1\, j_2}}_{-1-1\mu_1\mu_2}(\delta_2 - K^{\substack{\overline{10} \\ 10}}_{00} - K^{\substack{\overline{10} \\ 01}}_{00})\Bigg),
\end{aligned}
\tag{1}
$$

where $C_2(so(5))$ and $C_2(so(4))$ are the second order Casimir invariants of the $so(5)$ and $so(4)$ algebras, respectively, D are the corresponding matrix elements of the 14-dimensional $SO(5)$ group representation, and the K operators are the $SO(5)$ generators "acting to the left" in the group manifold. The four parameters σ_1, σ_2, δ_1 and δ_2 label the $sl(5,\mathbb{R})$ irreducible representations.

Having these expressions of the noncompact algebra elements, it is rather straightforward to evaluate the required matrix elements for an arbitrary irreducible representation.

7. *Dirac-like world spinor equation in* 3*D*

There are three principal steps in the process of constructing a generalization of the Dirac equation for the $\overline{SL}(3,\mathbb{R})$ spinorial fields: (i) construction of physically relevant spinorial unitary irreducible representations, (ii) their appropriate modification via deunitarizing automorphism, in order to have a viable physical interpretation, and (iii) construction of an $sl(3,\mathbb{R})$ vector operator X_μ, acting it the space of spinorial field components, that generalized Dirac's "gamma" matrices. As stated above, the most efficient method to construct explicitly the X_μ operator is to embed $sl(3,\mathbb{R})$ into $sl(4,\mathbb{R})$ and to chose in the latter algebra a vector operator in the appropriate spinorial representation. Such an equation can subsequently be adapted to be fully $\overline{Diff}(3,\mathbb{R})$ covariant by appropriate modifications making use of the "triad" $e^\mu_a(x)$ and pseudo-frame $E^A_M(x)$ fields.

The $\overline{SL}(4,\mathbb{R})$ commutation relations in the Minkowski space are given by,

$$[Q_{ab}, Q_{cd}] = i\eta_{bc}Q_{ad} - i\eta_{ad}Q_{cb}.$$

where, $a, b, c, d = 0, 1, 2, 3$, and $\eta_{ab} = diag(+1, -1, -1, -1)$. The relevant subgroup chain reads:

$$\begin{array}{ccc} \overline{SL}(4,\mathbb{R}) & \supset & \overline{SL}(3,\mathbb{R}) \\ \cup & & \cup \\ \overline{SO}(4), \overline{SO}(1,3) & \supset & \overline{SO}(3), \overline{SO}(1,2). \end{array}$$

There are three (independent) $\overline{SO}(3)$ vectors in the algebra of the $\overline{SL}(4,\mathbb{R})$ group. They are the $\overline{SO}(3)$ generators themselves and the operators:

$$A_i = Q_{i0}, \quad B_i = Q_{0i}.$$

A_i and B_i are $\overline{SL}(3,\mathbb{R})$ vectors transforming as the 3-dimensional representation $[1, 0]$, and as its contragradient 3-dimensional representation $[1, 1]$, respectively. Either choice $X_i \sim A_i$ and $X_i \sim B_i$ insures that a Dirac-like wave equation $(iX\partial - m)\Psi(x) = 0$ for an infinite-component spinor field is fully $\overline{SL}(3,\mathbb{R})$ covariant.

Due to complexity of the generic unitary irreducible representations of the $\overline{SL}(4,\mathbb{R})$ group, we confine to the multiplicity-free representation case.

In this case, there are just two, mutually contragradient, representations that contain spin $J = \frac{1}{2}$ representation of the $\overline{SO}(3)$ subgroup, and belong to the set of the so called Discrete Series [4] i.e.

$$D^{disc}_{\overline{SL}(4,\mathbb{R})}(\frac{1}{2},0) \supset D^{(\frac{1}{2},0)}_{\overline{SO}(4)} \supset D^{\frac{1}{2}}_{\overline{SO}(3)} \quad D^{disc}_{\overline{SL}(4,\mathbb{R})}(0,\frac{1}{2}) \supset D^{(0,\frac{1}{2})}_{\overline{SO}(4)} \supset D^{\frac{1}{2}}_{\overline{SO}(3)}.$$

The full reduction of these representations to the representations of the $\overline{SL}(3,\mathbb{R})$ subgroup reads [10]:

$$\begin{array}{rcl} D^{disc}_{\overline{SL}(4,\mathbb{R})}(j_0,0) & \rightarrow & \Sigma^{\oplus\infty}_{j=1} D^{disc}_{\overline{SL}(3,\mathbb{R})}(j_0;\sigma_2,\delta_1,j) \\ D^{disc}_{\overline{SL}(4,\mathbb{R})}(0,j_0) & \rightarrow & \Sigma^{\oplus\infty}_{j=1} D^{disc}_{\overline{SL}(3,\mathbb{R})}(j_0;\sigma_2,\delta_1,j) \end{array}$$

By making use of the known expressions of the $\overline{SL}(4,\mathbb{R})$ generators matrix elements for these spinorial representations, we can write down an $\overline{SL}(3,\mathbb{R})$ covariant wave equation in the form

$$\begin{array}{l} (iX^\mu \partial_\mu - M)\Psi(x) = 0, \\ \Psi \ \sim \ D^{disc}_{\overline{SL}(4,\mathbb{R})}(\frac{1}{2},0) \oplus D^{disc}_{\overline{SL}(4,\mathbb{R})}(0,\frac{1}{2}), \\ X^\mu = Q^{\mu 0}, \ Q^{0\mu}. \end{array}$$

Finally, the fully $\overline{Diff}(3,\mathbb{R})$ field equation reads:

$$(ie^\mu_a E^N_B (X^a)^B_A E^A_M \partial_\mu - M)\Psi_N(x) = 0.$$

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