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CONTRIBUTIONS TO LOCAL AND MICROLOCAL ANALYSIS, AN OVERVIEW

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A b s t r a c t. *My public lecture given in the Serbian Academy of Sciences and Arts under the title "Generalized functions, Equations, Microlocal Analysis" was consisted of two parts. The first part was devoted to the explanations of generalized function and microlocal theory within the theory of PDE, while the second part is related to the presentation of results which I have obtained in collaboration with my colleagues, J. Toft, J. Vindas, N. Teofanov, K. Johansson and D. Rakic. Here I will recall the extended version of the second part of my talk.*

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1. *Introduction*

Results of this paper are not new; they are already published or will be published. Also, notions used in this paper are known and can be find in a very reach literature a part of which is given in the references. Note that the references contain more titles than we need in this paper.

Concerning local properties, we characterize asymptotic properties of tempered distributions in terms of integral transforms arising from regularizations. We provide Abelian and Tauberian theorems for regularization transforms of tempered distributions, that is, transforms of the form $M^f_\varphi(x,y)=(f*\varphi_y)(x)$, where φ is a test function and $\varphi_y(\cdot)=y^{-n}\varphi(\cdot/y)$. If the first moment of φ vanishes it is a wavelet type transform; otherwise, we say it is a non-wavelet type transform. We investigate asymptotic properties at finite points and infinity. The results are applicable to distributions with values in a Banach space as well [23].

Concerning the microlocal properties of distributions we discuss the wave-front sets for Fourier Lebesgue and modulation space. The wave-front sets of Fourier Lebesgue and modulation space types agree with each others, and the usual wave-front sets with respect to smoothness (cf. [14, Sections 8.1–8.3]) is a special case of wave-front sets of Fourier Lebesgue types. We refer to [25] and[26] for the analysis of pseudo-differential operators with non-smooth symbols and the microlocal analysis of convolution, multiplication and semi-linear equations.

Here we recall a discrete versions of wave-front sets of Fourier Lebesgue and modulation space types, and show that they coincide with corresponding continuous versions. With that respect we emphasize that in the process of analysis and synthesis of a signal, Gabor frame coefficients also give information on micro-local properties of the signal.

2. *Notation*

We denote by $\mathbb{N}$ the set of positive integers including zero, $\mathbb{R}_+$ is the set of positive real numbers and $\mathbb{H}^{d+1}=\mathbb{R}^d\times\mathbb{R}_+$. If $x\in\mathbb{R}^d$ and $\alpha\in\mathbb{N}^d$, then $|x|=\sqrt{x_1^2+\ldots+x_d^2}$, $x^\alpha=x_1^{\alpha_1}\ldots x_n^{\alpha}$, $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\ldots\partial_{x_d}^{\alpha_d}$ and $|\alpha|=\alpha_1+\ldots+\alpha_d$.

The Schwartz spaces of smooth compactly supported and rapidly decreasing test functions are denoted by $\mathcal{D}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$. Their dual spaces, the scalar spaces of distributions and tempered distributions, are $\mathcal{D}'(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, respectively. We refer to [28] for their well known properties. We will use the following Fourier transform, $\varphi\in\mathcal{S}(\mathbb{R}^d)$,

$$\hat{\varphi}(u)=\int_{\mathbb{R}^d}\varphi(t)e^{-iu\cdot t}\mathrm{dt},\quad \mathrm{u}\in\mathbb{R}^{\mathrm{d}}.$$

Following [12], we define $\mathcal{S}_0(\mathbb{R}^d)$ as those elements of $\mathcal{S}(\mathbb{R}^d)$ for which all the moments vanish, i.e., $\varphi \in \mathcal{S}_0(\mathbb{R}^d)$ if and only if

$$\int_{\mathbb{R}^d} x^m \varphi(x)\, dx = 0, \tag{2.1}$$

for all $m \in \mathbb{N}^d$. It is provided with the relative topology inhered from $\mathcal{S}(\mathbb{R}^n)$. Observe that $\mathcal{S}_0(\mathbb{R}^d)$ is a closed subspace of $\mathcal{S}(\mathbb{R}^d)$ and that condition (2.1) is equivalent with $\hat{\varphi}^{(n)}(0) = 0$, for every $n \in \mathbb{N}$. Space $\mathcal{S}(\mathbb{H}^{d+1})$ consists of those $\Phi \in C^\infty(\mathbb{H}^{d+1})$ for which

$$\sup_{(x,y)\in\mathbb{H}^{d+1}} \left(y + \frac{1}{y}\right)^{k_1} (1+|x|)^{k_2} \left|\frac{\partial^l}{\partial y^l}\frac{\partial^m}{\partial x^m}\Phi(x,y)\right| < \infty \ ,$$

for all $k_1, k_2, l \in \mathbb{N}$ and $m \in \mathbb{N}^d$. The canonical topology of this space is defined in the standard way [12].

Wavelet transform $\mathcal{W}_\varphi f$ of function $f \in \mathcal{S}_0(\mathbb{R}^d)$, related to the wavelet $\varphi \in \mathcal{S}_0(\mathbb{R}^d)$, is defined by

$$\mathcal{W}_\varphi f(b,a) = \langle f(x), \frac{1}{a^d}\bar{\varphi}(\frac{x-b}{a})\rangle = \int_{\mathbb{R}^d} f(x)\,\frac{1}{a^d}\bar{\varphi}(\frac{x-b}{a})\,dx. \tag{2.2}$$

If we suppose that φ is an admissible wavelet, i.e. if

$$C_\varphi = \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^2 |\xi|^{-d}\, d\xi \in \mathbb{R}_+,$$

then the reconstruction formula holds:

$$f(x) = C_\varphi^{-1}(\mathcal{M}_\varphi(\mathcal{W}_\varphi f))(x),\ x \in \mathbb{R}^d,$$

where the wavelet synthesis operator $\mathcal{M}_\varphi$ is defined as

$$(\mathcal{M}_\varphi \Phi)(x) = \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \Phi(b,a)\,\frac{1}{a^d}\bar{\varphi}(\frac{x-b}{a})\,db\,\frac{da}{a},\ x \in \mathbb{R}^d,\ \Phi \in \mathcal{S}(\mathbb{H}^{d+1}).$$

If for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the first moment of φ differs from zero then by (2.2) is defined a non-wavelet transform. In particular, if $\int_{\mathbb{R}^d}\varphi(x)\,dx = 1$ we obtain the mollifying transform and φ is called mollifier.

The following two theorems are proved in [24].

Theorem 1. *Let $g \in \mathcal{S}_0(\mathbb{R}^d)$. Then $\mathcal{W}_g$ is a continuous map from $\mathcal{S}_0(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{H}^{d+1})$.*

Theorem 2. *Let $g \in \mathcal{S}_0(\mathbb{R}^d)$ is an admissible wavelet. Then $\mathcal{M}_g$ is continuous map from $\mathcal{S}(\mathbb{H}^{d+1})$ to $\mathcal{S}_0(\mathbb{R}^d)$.*

3. *Abelian and Tauberian theorems*

Recall, a measurable, positive real valued function, defined on an interval of the form $(0, A]$ (resp. $[A, \infty)$), $A > 0$, is called Karamata's function or a *slowly varying* at the origin (resp. at infinity) [1, 29] if

$$\lim_{\varepsilon\to 0^+} \frac{L(a\varepsilon)}{L(\varepsilon)} = 1 \quad \left(\text{resp. } \lim_{\lambda\to\infty} \frac{L(a\lambda)}{L(\lambda)} = 1 \right).$$

Recall the definition of quasiasymptotic behaviour. Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and let L be slowly varying at the origin (resp. at infinity). We say that f has quasiasymptotic behavior of degree $\alpha \in \mathbb{R}$ at the point $x_0 \in \mathbb{R}^d$ (resp. at infinity) with respect to L in $\mathcal{S}'(\mathbb{R}^d)$ if there exists $g \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \langle f(x_0 + \varepsilon x), \varphi(x)\rangle = \langle g(x), \varphi(x)\rangle, \ \forall\varphi \in \mathcal{S}(\mathbb{R}^d) \tag{3.1}$$

$$\left(\text{resp. } \lim_{\lambda\to\infty} \frac{1}{\lambda^\alpha L(\lambda)} \langle f(\lambda x), \varphi(x)\rangle = \langle g(x), \varphi(x)\rangle, \ \forall\varphi \in \mathcal{S}(\mathbb{R}^d)\right). \tag{3.2}$$

In this case we write

$$f(x_0 + \varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) g(x) \quad \text{as } \varepsilon \to 0^+ \ \text{ in } \mathcal{S}'(\mathbb{R}^d)$$

$$\left(\text{resp. } f(\lambda x) \sim \lambda^\alpha L(\lambda) g(x) \quad \text{as } \lambda \to \infty \ \text{ in } \mathcal{S}'(\mathbb{R}^d)\right)$$

or

$$f(x_0 + \varepsilon x) = \varepsilon^\alpha L(\varepsilon) g(x) + o(\varepsilon^\alpha L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+ \ \text{ in } \mathcal{S}'(\mathbb{R}^n)$$

$$f(\lambda x) = \lambda^\alpha L(\lambda) g(x) + o(\lambda^\alpha L(\lambda)) \quad \text{as } \lambda \to \infty \ \text{ in } \mathcal{S}'(\mathbb{R}^d).$$

The next theorem is proved in [37]:

Theorem 3. *Let $f \in \mathcal{S}'(\mathbb{R})$ and $\alpha > 0$, $\alpha \notin \mathbb{N}$. Suppose that $\psi \in \mathcal{S}_0(\mathbb{R})$ is an admissible wavelet. The existence of the limits*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathcal{W}_\psi f(x_0 + \varepsilon b, \varepsilon a) = M_{b,a} < \infty, \quad a^2 + b^2 = 1, \ a > 0, \tag{3.3}$$

and the existence of $m \in \mathbb{R}$ such that

$$\limsup_{\varepsilon\to 0^+} \sup_{a^2+b^2=1,\, a>0} \frac{a^m}{\varepsilon^\alpha L(\varepsilon)} |\mathcal{W}_\psi f(x_0 + \varepsilon b, \varepsilon a)| < \infty, \tag{3.4}$$

are necessary and sufficient conditions for the existence of a polynomial p *of degree less than* α *such that* $f-p$ *has quasiasymptotic behavior of degree* α *with respect to* L *at the point* $x=x_0$. *In such a case there is a homogeneous distribution* g *of degree* α *such that* $M_{b,a}=\mathcal{W}_\psi g(b,a)$.

Now we recall a vector-valued version of the quasiasymptotic behaviour given by Drozhinov and Zavialov. Let $r\in\mathbb{N}$, K be a compact set in $\mathbb{R}^n$, $f\in\mathcal{S}'(\mathbb{R}^n,\mathcal{C}^r(K))$ and let L be slowly varying at the origin (resp. at infinity). We say that f has quasiasymptotic behavior of degree $\alpha\in\mathbb{R}$ with the value in $\mathcal{C}^r(K)$ with respect to L if there exists $g\in\mathcal{S}'(\mathbb{R}^d,\mathcal{C}^r(K))$ such that for each test function $\varphi\in\mathcal{S}(\mathbb{R}^d)$ the following limit holds

$$\lim_{\varepsilon\to 0^+}\|\left\langle\frac{f\,(t+\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)}-g(x,t),\varphi(x)\right\rangle\|_{\mathcal{C}^r(K)}\to 0. \tag{3.5}$$

$$\left(\text{resp. }\lim_{\lambda\to\infty}\|\left\langle\frac{f\,(\lambda(x+t))}{\lambda^\alpha L(\lambda)}-g(x,t),\varphi(x)\right\rangle\|_{\mathcal{C}^r(K)}\to 0\right). \tag{3.6}$$

If for every $\varphi\in\mathcal{S}(\mathbb{R}^d)$

$$\|\left\langle\frac{f\,(t+\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)},\varphi(x)\right\rangle\|_{\mathcal{C}^r(K)}<\infty\ \text{ as } \varepsilon\to 0^+, \tag{3.7}$$

$$\left(\text{resp. }\|\left\langle\frac{f\,(\lambda(x+t))}{\lambda^\alpha L(\lambda)},\varphi(x)\right\rangle\|_{\mathcal{C}^r(K)}<\infty\ \text{ as } \lambda\to\infty\right), \tag{3.8}$$

then we say that f is quasiasymptotically bounded with respect to $\varepsilon^\alpha L(\varepsilon)$ with values in $\mathcal{C}^r(K)$.

In this case we use the following notations

$$f\,(t+\varepsilon x)\sim\varepsilon^\alpha L(\varepsilon)g(x,t)\quad\text{as } \varepsilon\to 0^+\ \text{ in } \mathcal{S}'(\mathbb{R}^d,\mathcal{C}^r(K))$$

$$\left(\text{resp. } f\,(\lambda(x+t))\sim\lambda^\alpha L(\lambda)g(x,t)\quad\text{as }\lambda\to\infty\ \text{ in } \mathcal{S}'(\mathbb{R}^d,\mathcal{C}^r(K))\right)$$

or

$$f\,(t+\varepsilon x)=\varepsilon^\alpha L(\varepsilon)g(x,t)+o\,(\varepsilon^\alpha L(\varepsilon))\quad\text{as } \varepsilon\to 0^+\ \text{ in } \mathcal{S}'(\mathbb{R}^d,\mathcal{C}^r(K))$$

$$\left(\text{resp. } f\,(\lambda(x+t))=\lambda^\alpha L(\lambda)g(x,t)+o\,(\lambda^\alpha L(\lambda))\ \text{ as }\lambda\to\infty\ \text{ in } \mathcal{S}'(\mathbb{R}^d,\mathcal{C}^r(K))\right).$$

For the quasiasymptotic boundedness (3.7) and (3.8) we use notation

$$f\,(t+\varepsilon x)=O\,(\varepsilon^\alpha L(\varepsilon))\quad\text{as } \varepsilon\to 0^+\ \text{ in } \mathcal{S}'(\mathbb{R}^d,\mathcal{C}^r(K))$$

$$\left(\text{resp. } f\left(\lambda(x+t)\right) = O\left(\lambda^{\alpha}L(\lambda)\right) \quad \text{as } \lambda \to \infty \;\text{ in } \mathcal{S}'(\mathbb{R}^d, \mathcal{C}^r(K))\right).$$

Our next two theorems are given in [24].

Theorem 4. *Let $f \in \mathcal{S}'(\mathbb{R}^d, \mathcal{C}^r(K))$, let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be an admissible wavelet, and let L be slowly varying at the origin (resp. at infinity). Suppose that the estimate*

$$\limsup_{\varepsilon\to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\varepsilon^{\alpha}L(\varepsilon)} \|\mathcal{W}_{\psi} f\left(t+\varepsilon x, \varepsilon y\right)\|_{\mathcal{C}^r(K)} < \infty \tag{3.9}$$

$$\left(\textit{resp. } \limsup_{\lambda\to\infty} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\lambda^{\alpha}L(\lambda)} \|\mathcal{W}_{\psi} f_t\left(\lambda x, \lambda y\right)\|_{\mathcal{C}^r(K)} < \infty\right)$$

holds for some $k \in \mathbb{N}$, and suppose that for each $(x, y) \in \mathbb{H}^{d+1} \cap \mathbb{S}^d$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{\alpha}L(\varepsilon)} \|\mathcal{W}_{\psi} f(t+\varepsilon x, \varepsilon y) - W_{x,y}\|_{\mathcal{C}^r(K)} = 0 \tag{3.10}$$

$$\left(\textit{resp. } \lim_{\lambda\to\infty} \frac{1}{\lambda^{\alpha}L(\lambda)} \|\mathcal{W}_{\psi} f_t(\lambda x, \lambda y) - W_{x,y}\|_{\mathcal{C}^r(K)} = 0\right).$$

Then, there exist an $g \in \mathcal{S}'(\mathbb{R}^d, \mathcal{C}^r(K))$ such that:

(i) If $\alpha \notin \mathbb{N}$, g is homogeneous of degree α and there exists a polynomial P_t with values in $\mathcal{C}^r(K)$ such that

$$f(t+\varepsilon x) - P_t(\varepsilon x) \sim \varepsilon^{\alpha}L(\varepsilon)g(x,t) \quad \textit{as } \varepsilon \to 0^+ \;\; \textit{in } \mathcal{S}'(\mathbb{R}^d, \mathcal{C}^r(K))$$

$$\left(\textit{resp. } f_t\left(\lambda x\right) - P_t(\lambda x) \sim \lambda^{\alpha}L(\lambda)g(x,t) \quad \textit{as } \lambda \to \infty \;\; \textit{in } \mathcal{S}'(\mathbb{R}^n, \mathcal{C}^r(K))\right).$$

(ii) If $\alpha = k \in \mathbb{N}$, there exist a polynomial P_t with values in $\mathcal{C}^r(K)$ and associated asymptotically homogeneous functions c_m, $|m| = k$, with values in $\mathcal{C}^r(K)$, of degree 0 with respect to L such that f has the following asymptotic expansion

$$f(t+\varepsilon x) = P_t(\varepsilon x) + \varepsilon^k L(\varepsilon)g(x,t) + \varepsilon^k \sum_{|m|=k} x^m c_m(t,\varepsilon) + o\left(\varepsilon^k L(\varepsilon)\right)$$

$$\left(\textit{resp. } f_t\left(\lambda x\right) = P_t(\lambda x) + \lambda^k L(\lambda)g(x,t) + \lambda^k \sum_{|m|=k} x^m c_m(t,\lambda) + o\left(\lambda^k L(\lambda)\right)\right),$$

as $\varepsilon \to 0^+$ (resp. $\lambda \to \infty$) in the space $\mathcal{S}'(\mathbb{R}^n, \mathcal{C}^r(K))$.

Furthermore, $\mathcal{W}_\psi g = W_{x,y}$.

Theorem 5. *Let* $f \in \mathcal{S}'(\mathbb{R}^n, \mathcal{C}^r(K))$, *let* $\psi \in \mathcal{S}(\mathbb{R}^n)$ *be an admissible wavelet, and let* L *be slowly varying at the origin (resp. at infinity). Suppose that there exists* $k \in \mathbb{N}$ *such that*

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} \| \mathcal{W}_\psi f\,(t+\varepsilon x, \varepsilon y)\,\|_{\mathcal{C}^r(K)} < \infty \tag{3.11}$$

$$\left(\textit{resp. } \limsup_{\lambda \to \infty} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\lambda^\alpha L(\lambda)} \| \mathcal{W}_\psi f_t\,(\lambda x, \lambda y)\,\|_{\mathcal{C}^r(K)} < \infty \right). \tag{3.12}$$

Condition (3.11) (resp. (3.12)) is necessary and sufficient for:

(i) If $\alpha \notin \mathbb{N}$, *there exists a polynomial* P_t *with values in* $\mathcal{C}^r(K)$ *such that* $f - P$ *is quasiasymptotically bounded of degree* α *with values in* $\mathcal{C}^r(K)$ *with respect to* L *in the space* $\mathcal{S}'(\mathbb{R}^n, \mathcal{C}^r(K))$.

(ii) If $\alpha = k \in \mathbb{N}$, *there exist a polynomial* P_t *with values in* $\mathcal{C}^r(K)$ *and asymptotically homogeneously bounded functions* c_m, $|m| = k$, *with values in* $\mathcal{C}^r(K)$, *of degree 0 with respect to* L *such that* f *has the following asymptotic expansion*

$$f(t+\varepsilon x) = P_t(\varepsilon x) + \varepsilon^k \sum_{|m|=k} x^m c_m(t,\varepsilon) + O\left(\varepsilon^k L(\varepsilon)\right)$$

$$\left(\textit{resp. } f_t(\lambda x) = P_t(\lambda x) + \lambda^k \sum_{|m|=k} x^m c_m(t,\varepsilon) + O\left(\lambda^k L(\lambda)\right) \right)$$

as $\varepsilon \to 0^+$ *(resp.* $\lambda \to \infty$*) in the space* $\mathcal{S}'(\mathbb{R}^n, \mathcal{C}^r(K))$.

4. *Applications of Abelian and Tauberian theorems*

Our Tauberian theorem can be applied for the proof of the famous Littlewood's Tauberian theorem: If

$$\lim_{y \to 0^+} \sum_{n=0}^{\infty} c_n e^{-yn} = \gamma \tag{4.1}$$

and if the Tauberian hypothesis $c_n = O(1/n)$ is satisfied, then

$$\sum_{n=0}^{\infty} c_n = \gamma. \tag{4.2}$$

We will not give the details of this proof. We proceed with a now proof of the non-differentiability of the Weirstrass function.

An important special case of quasiasymptotic behavior is the value of distributions at a point in the sense of Łojasiewicz, which is obtained when $\alpha = 0$ and $L = 1$. A distribution f is said to have a (distributional) point value at x_0 in the sense of Łojasiewicz if

$$\lim_{\varepsilon \to 0} f(x_0 + \varepsilon x) = \gamma \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Let $\{\lambda_n\}_{n=0}^{\infty}$ be a lacunary sequence with $\lambda_{n+1}/\lambda_n > \sigma > 1$. Let $f \in \mathcal{S}'(\mathbb{R})$ have a series representation $f(x) = \sum_{n=0}^{\infty} c_n e^{i\lambda_n x}$, where the series is convergent in $\mathcal{S}'(\mathbb{R})$. Furthermore, suppose that at a given x_0 the point value $f(x_0)$ exists in the sense of Łojasiewicz. Then, by selecting $\psi \in \mathcal{S}_0(\mathbb{R})$ with supp $\overline{\hat{\psi}} \subset [\sigma^{-\frac{1}{2}}, \sigma^{\frac{1}{2}}]$ and $\overline{\hat{\psi}}(1) = 1$, the lacunarity of $\{\lambda_n\}_{n \in \mathbb{N}_0}$ implies that for m large enough, $\overline{\hat{\psi}}(\lambda_n/\lambda_m) = 0$ if $m \neq n$. There holds

$$\mathcal{W}_\psi f(x_0, \lambda_m^{-1}) = \sum_{n=0}^{\infty} c_n e^{i\lambda_n x_0} \overline{\hat{\psi}} \left(\frac{\lambda_n}{\lambda_m} \right) = c_m e^{i\lambda_m x_0}.$$

So, the existence of the distributional point value $f(x_0)$ and Abelian theorem (with $\alpha = 0, L = 1, \mathcal{W}_\psi 1(0,1) = 0$) implies that $c_m e^{i\lambda_m x_0} = o(1)$, or,

$$\lim_{m \to \infty} c_m = 0. \tag{4.3}$$

Therefore, (4.3) is a necessary condition for the existence of the distributional point value of f at x_0. On the other hand, we have just shown: *If (4.3) is violated, then f cannot have distributional point values anywhere.* The same argument we can applie to distributions of the form $\sum_{n=0}^{\infty} c_n \cos(\lambda_n x)$ and $\sum_{n=0}^{\infty} c_n \sin(\lambda_n x)$. We observe the Weierstrass function

$$w(x) = \sum_{n=0}^{\infty} \gamma^{-n} \cos(\beta^n x), \quad \beta \geq \gamma > 1$$

and its first derivative

$$w'(x) = - \sum_{n=0}^{\infty} \left(\frac{\beta}{\gamma} \right)^n \sin(\beta^n x).$$

Since $(\beta/\gamma)^n \neq o(1)$, it follows from previous Example that $w'(x_0)$ does not exist in the sense of Łojasiewicz at any $x_0 \in \mathbb{R}$. In particular, w is nowhere differentiable.

5. *Wave fronts in Fourier-Lebesgue and modulation spaces*

First we recall some notions. In what follows we let Γ denote an open cone in $\mathbf{R^d} \setminus 0$. If $\xi \in \mathbf{R^d} \setminus 0$ is fixed, then an open cone which contains ξ is sometimes denoted by Γ_ξ.

Assume that ω and v are positive and measurable functions on $\mathbf{R^d}$. Then ω is called v-moderate weight if

$$\omega(x+y) \leq C\omega(x)v(y) \tag{5.1}$$

for some constant C which is independent of $x, y \in \mathbf{R^d}$. If v in (5.1) can be chosen as a polynomial, then ω is called polynomially moderated. We let $\mathcal{P}(\mathbf{R^d})$ be the set of all polynomially moderated functions on $\mathbf{R^d}$. If $\omega(x,\xi) \in \mathcal{P}(\mathbf{R^{2d}})$ is constant with respect to the x-variable (ξ-variable), then we sometimes write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathcal{P}(\mathbf{R^{2d}})$ or in $\mathcal{P}(\mathbf{R^d})$ depending on the situation.

Assume that $q \in [1,\infty]$ and $\omega \in \mathcal{P}(\mathbf{R^d})$. Then the (weighted) Fourier Lebesgue space $\mathcal{F}L^q_{(\omega)}(\mathbf{R^d})$ is the Banach space which consists of all $f \in \mathcal{S}'(\mathbf{R^d})$ such that

$$\|f\|_{\mathcal{F}L^q_{(\omega)}} \equiv \|\widehat{f} \cdot \omega\|_{L^q}. \tag{5.2}$$

is finite. If $\omega = 1$, then the notation $\mathcal{F}L^q$ is used instead of $\mathcal{F}L^q_{(\omega)}$.

As in [25] we remark that it is sometimes convenient to permit ω to depend on both x and ξ. That is, $\omega = \omega(x,\xi) \in \mathcal{P}(\mathbf{R^{2d}})$ In this case we set

$$\|f\|_{\mathcal{F}L^q_{(\omega)}} = \|f\|_{\mathcal{F}L^q_{(\omega),x}} \equiv \|\widehat{f} \cdot \omega(x,\,\cdot\,)\|_{L^q}.$$

We note that the fact that ω is v-moderate for some $v \in \mathcal{P}(\mathbf{R^{2d}})$ implies that different choices of x give rise to equivalent norms. In that sense, the condition $\|f\|_{\mathcal{F}L^q_{(\omega),x}} < \infty$ is independent of x.

Let $p, q \in [1,\infty]$, $\phi \in \mathcal{S}(\mathbf{R^d})$ and let $\omega \in \mathcal{P}(\mathbf{R^{2d}})$. Then the modulation space $M^{p,q}_{(\omega)}(\mathbf{R^d})$ consists of all $f \in \mathcal{S}'(\mathbf{R^d})$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \Big(\int_{\mathbf{R^d}} \Big(\int_{\mathbf{R^d}} |V_\phi f(x,\xi)\omega(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q} \tag{5.3}$$

is finite. Here $V_\phi f$ is the short-time Fourier transform of f with respect to $\phi \in \mathcal{S}(\mathbf{R^d}) \setminus \prime$, which is equal to $\mathcal{F}(f\,\overline{\phi(\,\cdot\, - x)})$. We note that $V_\phi f$ takes the form

$$V_\phi f(x,\xi) = (2\pi)^{-d/2} \int_{\mathbf{R^d}} f(y)\overline{\phi(y-x)} e^{-i\langle y,\xi\rangle}\, dy$$

when $f \in L^1(\mathbf{R^d})$.

The definition of wave-front sets with respect to Fourier Lebesgue types and modulation space types depends on semi-norms which are defined in a similar way as the norms of these spaces. More precisely, let $f \in \mathcal{S}'(\mathbf{R^d})$ and let $\Gamma \subseteq \mathbf{R^d} \setminus 0$ be an open cone. Set

$$\begin{aligned} |f|_{\mathcal{B}(\Gamma)} &\equiv \Big(\int_\Gamma |\widehat{f}(\xi)\omega(x,\xi)|^q \, d\xi \Big)^{1/q}, \\ |f|_{\mathcal{C}(\Gamma)} &\equiv \Big(\int_\Gamma \Big(\int_{\mathbf{R^d}} |V_\phi f(x,\xi)\omega(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q}, \\ \text{when}\quad \mathcal{B} &= \mathcal{F}L^q_{(\omega)}(\mathbf{R^d}) \quad \text{and} \quad \mathcal{C} = M^{p,q}_{(\omega)}(\mathbf{R^d}). \end{aligned} \tag{5.4}$$

respectively. We set $|f|_{\mathcal{B}(\Gamma)} = +\infty$ and $|f|_{\mathcal{C}(\Gamma)} = +\infty$ when $\widehat{f} \notin L^1_{loc}(\Gamma)$.

The wave-front set $\mathrm{W}F_{\mathcal{B}}(f)$, respectively $\mathrm{W}F_{\mathcal{C}}(f)$, of $f \in \mathcal{D}'(X)$, X is open in $\mathbf{R^d}$, consists of all pairs $(x_0,\xi_0) \in X \times (\mathbf{R^d} \setminus 0)$ such that for each $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$ and each conical neighbourhood Γ of ξ_0 it holds

$$|\varphi\, f|_{\mathcal{B}(\Gamma)} = +\infty, \quad \text{respectively} \quad |\varphi\, f|_{\mathcal{C}(\Gamma)} = +\infty.$$

By [25, Theorem 5.1], for each distribution f,

$$\mathrm{W}F_{\mathcal{B}}(f) = \mathrm{W}F_{\mathcal{C}}(f), \tag{5.5}$$

where

$$\mathcal{B} = \mathcal{F}L^q_{(\omega)}(\mathbf{R^d}) \quad \text{and} \quad \mathcal{C} = M^{p,q}_{(\omega)}(\mathbf{R^d}), \quad p,q \in [1,\infty], \tag{5.6}$$

and ω is an appropriate weight function.

We introduce in [17] discrete versions of $\mathrm{W}F_{\mathcal{F}L^q_{(\omega)}}(f) = \mathrm{W}F_{M^{p,q}_{(\omega)}}(f)$, denoted by $\mathrm{D}F_{\mathcal{F}L^q_{(\omega)}}(f)$ and $\mathrm{D}F_{M^{p,q}_{(\omega)}}(f)$ respectively, and prove that indeed (5.5) extends to

Theorem 6.

$$WF_{\mathcal{B}}(f) = WF_{\mathcal{C}}(f) = DF_{\mathcal{B}}(f) = DF_{\mathcal{C}}(f).$$

6. *Discrete semi-norms in Fourier-Lebesgue spaces*

Assume that $q \in [1,\infty]$, $\omega \in \mathcal{P}(\mathbf{R^d})$ and that $H \subseteq \mathbf{R^d}$ is a discrete set. Then we set

$$|f|^D_{\mathcal{B}(H)} \equiv \Big(\sum_{\{\xi_k\}\in H} |\widehat{f}(\xi_k)\omega(\xi_k)|^q \Big)^{1/q}, \quad (\text{recall} \quad \mathcal{B} = \mathcal{F}L^q_{(\omega)}(\mathbf{R^d})),$$

(with obvious modifications when $q = \infty$). As in the continuous case, we may allow that the weight functions in $\mathcal{P}(\mathbf{R^{2d}})$, i, e. $\omega = \omega(x,\xi)$. In this situation the semi-norm might depends on $x \in \mathbf{R^d}$. However, again we note that the condition

$$|f|^D_{\mathcal{B}(H)} < \infty$$

is independent of $x \in \mathbf{R^d}$. For that reason, from now on we assume that ω is independent of x.

Lemma 1. *Let $f \in \mathcal{E}'(\mathbf{R^d})$, Γ and Γ_0 be open cones in $\mathbf{R^d}$ such that $\overline{\Gamma_0} \subseteq \Gamma$, $q \in [1,\infty]$, and let $\Lambda \subseteq \mathbf{R^d}$ be a lattice. If $|f|_{\mathcal{B}(\Gamma)}$ is finite, then $|f|^D_{\mathcal{B}(\Gamma_0\cap\Lambda)}$ is finite.*

The converse to Lemma 1 holds in the case when the lattice Λ is dense enough. Let $e_1,\dots,e_d$ in $\mathbf{R^d}$ be a basis for Λ, i. e. for some $x_0 \in \Lambda$ we have

$$\Lambda = \{\, x_0 + t_1e_1 + \cdots + t_de_d\,;\, t_1,\dots,t_d \in \mathbf{Z}\,\}.$$

A parallelepiped, spanned by a basis $e_1,\dots,e_d$ for Λ and with corners in Λ, is called a Λ-*parallelepiped*. We let $\mathcal{A}(\Lambda)$ be the set of all Λ-parallelepipeds. If $D_1, D_2 \in \mathcal{A}(*)$, then their volumes $|D_1|$ and $|D_2|$ agree and we let $\|\Lambda\|$ to denote the common value, i. e.

$$\|\Lambda\| = |D_1| = |D_2|.$$

Assume that Λ_1 and Λ_2 are lattices in $\mathbf{R^d}$ with bases $e_1,\dots,e_d$ and $\varepsilon_1,\dots,\varepsilon_d$ respectively. Then the pair (Λ_1,Λ_2) is called an *admissible lattice pair*, if for some $0 < c \le 2\pi$ we have $\langle e_j,\varepsilon_j\rangle = c$ and $\langle e_j,\varepsilon_k\rangle = 0$ when $j \neq k$. If in addition $c < 2\pi$, then (Λ_1,Λ_2) is called a *strongly admissible lattice pair.* If instead $c = 2\pi$, then the pair (Λ_1,Λ_2) is called a *weakly admissible lattice pair.*

Lemma 2. *Let (Λ_1,Λ_2) be an admissible lattice pair, $D \in \mathcal{A}(\Lambda_1)$, and let $f \in \mathcal{E}'(\mathbf{R^d})$ be such that an open neighbourhood of its support is contained* *in D. Also let Γ and Γ_0 be open cones in $\mathbf{R^d}$ such that $\overline{\Gamma_0} \subseteq \Gamma$. If $|f|^D_{\mathcal{B}(\Gamma\cap\Lambda_2)}$ is finite, then $|f|_{\mathcal{B}(\Gamma_0)}$ is finite.*

Corollary 1. *Let (Λ_1, Λ_2) be an admissiblle lattice pair, $D \in \mathcal{A}(\Lambda_1)$, and $f \in \mathcal{E}'(\mathbf{R^d})$ be such that an open neighbourhood of its support is contained in D. Also let Γ and Γ_0 be open cones in $\mathbf{R^d}$ such that $\overline{\Gamma_0} \subseteq \Gamma$. If $|f|^D_{\mathcal{B}(\Gamma\cap\Lambda_2)}$ is finite, then $|\varphi f|^D_{\mathcal{B}(\Gamma_0\cap\Lambda_2)}$ is finite for every $\varphi \in \mathcal{S}(\mathbf{R^d})$.*

7. *Admissible Gabor pairs*

In this section we introduce the notion of admissible Gabor pairs and provide an example which illustrates that conditions in Definition 1 are quite general.

Assume that $e_1, \dots, e_d$ is a basis for Λ_1, and that (Λ_1, Λ_2) is a weakly admissible lattice pair. If $f \in L^2_{loc}$ is periodic with respect to Λ_1, and D is the parallelepiped, spanned by $\{e_1, \dots, e_d\}$, then we may make Fourier expansion of f as

$$f(x) = \sum_{\{\xi_k\}\in\Lambda_2} c_k e^{i\langle x,\xi_k\rangle} \tag{7.1}$$

(with convergence in L^2_{loc}), where the coefficients c_k are given by

$$c_k = \int_\Delta f(y) e^{-i\langle y,\xi_k\rangle}\, dy. \tag{7.2}$$

Here and in what follows we let

$$y = y_1 e_1 + \cdots y_d e_d, \quad dy = dy_1 \cdots dy_d \quad \text{and} \quad \Delta = [0,1]^d. \tag{7.3}$$

For non-periodic functions and distributions we instead make Gabor expansions. Because of the support properties of the involved Gabor atoms and their duals, we are usually forced to change the assumption on the involved lattice pairs. More precisely, instead of assuming that (Λ_1, Λ_2) should be a weakly admissible lattice pair, we assume from now on that (Λ_1, Λ_2) is a strongly admissible lattice pair, with $\Lambda_1 = \{x_j\}_{j\in J}$ and $\Lambda_2 = \{\xi_k\}_{k\in J}$. Also let

$$\begin{aligned} \phi, \psi \in C_0^\infty(\mathbf{R^d}), \quad \phi_{j,k}(x) &= \phi(x - x_j) e^{i\langle x,\xi_k\rangle} \\ \text{and} \quad \psi_{j,k}(x) &= \psi(x - x_j) e^{i\langle x,\xi_k\rangle} \end{aligned} \tag{7.4}$$

be such that $\{\phi_{j,k}\}_{j,k\in J}$ and $\{\psi_{j,k}\}_{j,k\in J}$ are dual Gabor frames (see [10, 11] for the definition and basic properties of Gabor frames and their duals). If $f \in \mathcal{S}'(\mathbf{R^d})$, then, in the sense of convergence in $\mathcal{S}'(\mathbf{R^d})$,

$$f = \sum_{j,k\in J} c_{j,k}\phi_{j,k}, \text{ where} \tag{7.5}$$

$$c_{j,k} = C_{\phi,\psi}(f,\psi_{j,k})_{L^2(\mathbf{R^d})} \tag{7.6}$$

and the constant $C_{\phi,\psi}$ depends on the frames only.

By replacing (Λ_1,Λ_2) here above with $(\varepsilon\Lambda_1,\Lambda_2/\varepsilon)$, and ϕ and ψ with $\phi^\varepsilon = \phi(\,\cdot\,/\varepsilon)$ and $\psi^\varepsilon = \psi(\,\cdot\,/\varepsilon)$ respectively, we still have

$$f = \sum_{j,k\in J} c_{j,k}(\varepsilon)\phi^\varepsilon_{j,k}, \text{ where} \tag*{(7.5)$'$}$$

$$c_{j,k}(\varepsilon) = C_{\phi,\psi}(\varepsilon)(f,\psi^\varepsilon_{j,k})_{L^2(\mathbf{R^d})}, \text{ and} \tag*{(7.6)$'$}$$

$$\phi^\varepsilon_{j,k} = \phi_{j,k}(\,\cdot\,/\varepsilon), \qquad \psi^\varepsilon_{j,k} = \psi_{j,k}(\,\cdot\,/\varepsilon). \tag{7.7}$$

Here the constants $C_{\phi,\psi}(\varepsilon)$ depend on ϕ, ψ and ε.

In some situations it is convenient to play with the parameter ε in $\varepsilon\Lambda_1$, $\phi^\varepsilon = \phi(\,\cdot\,/\varepsilon)$ and $\psi^\varepsilon = \psi(\,\cdot\,/\varepsilon)$, but keeping Λ_2 fixed and independent of ε. A problem is then that (7.5)$'$ and (7.6)$'$ might be violated. In the following we establish sufficient conditions for this to work properly. To that end we introduce the following notion.

Definition 1. *Assume that $\varepsilon \in (0,1]$, $\{x_j\}_{j\in J} = \Lambda_1 \subseteq \mathbf{R^d}$ and $\{\xi_k\}_{k\in J} = \Lambda_2 \subseteq \mathbf{R^d}$ are lattices and let $\Lambda_1(\varepsilon) = \varepsilon\Lambda_1$. Also let $\phi,\psi \in C_0^\infty(\mathbf{R^d})$ be non-negative, and set*

$$\begin{aligned} \phi^\varepsilon &= \phi(\,\cdot\,/\varepsilon), & \psi^\varepsilon &= \psi(\,\cdot\,/\varepsilon),\\ \phi^\varepsilon_{j,k} &= \phi^\varepsilon(\,\cdot\,-\varepsilon x_j)e^{i\langle\,\cdot\,,\xi_k\rangle}, & \psi^\varepsilon_{j,k} &= \psi^\varepsilon(\,\cdot\,-\varepsilon x_j)e^{i\langle\,\cdot\,,\xi_k\rangle}, \end{aligned} \tag{7.8}$$

when $\varepsilon x_j \in \Lambda_1(\varepsilon)$ (i. e. $x_j \in \Lambda_1$) and $\xi_k \in \Lambda_2$. Then the pair

$$(\{\phi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J}) \tag{7.9}$$

is called an admissible Gabor pair *if for each $\varepsilon \in (0,1]$, the sets $\{\phi^\varepsilon_{j,k}\}_{j,k\in J}$ and $\{\psi^\varepsilon_{j,k}\}_{j,k\in J}$ are dual Gabor frames.*

By Definition 1 and Chapters 5–13 in [11] it follows that if $f \in \mathcal{S}'(\mathbf{R^d})$ and if $(\{\phi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$ is an admissible Gabor pair, then (in $\mathcal{S}'(\mathbf{R^d})$)

$$f = \sum_{j,k\in J} c_{j,k}(\varepsilon)\phi^\varepsilon_{j,k}, \tag*{(7.5)$''$}$$

for every $\varepsilon \in (0,1]$, where

$$c_{j,k}(\varepsilon) = (f, \psi^{\varepsilon}_{j,k})_{L^2(\mathbf{R}^\mathbf{d})}. \tag{7.6$''$}$$

We remark that if the pair in (7.9) is admissible Gabor pair, then it follows from the investigations in [11] that the lattice pair (Λ_1, Λ_2) in Definition 1 is strongly admissible.

In the following proposition we prove that if Λ_1 and Λ_2 are the same as in Definition 1, $\{\phi_{j,k}\}_{j,k\in J}$ and $\{\psi_{j,k}\}_{j,k\in J}$ are dual Gabor frames which satisfy

$$\sum_{x_j\in\Lambda_1} \phi(\,\cdot\, - x_j)\psi(\,\cdot\, - x_j) = \|\Lambda_1\|^{-1}, \tag{7.10}$$

and $\phi^{\varepsilon}_{j,k}$ and $\psi^{\varepsilon}_{j,k}$ are given by (7.8), then $(\{\phi^{\varepsilon}_{j,k}\}_{j,k\in J}, \{\psi^{\varepsilon}_{j,k}\}_{j,k\in J})$ is an admissible Gabor pair.

Proposition 1. *Let $\phi, \psi \in C_0^\infty(\mathbf{R}^\mathbf{d})$ be non-negative functions and let $\phi_{j,k}$, $\psi_{j,k}$, $\phi^{\varepsilon}_{j,k}$ and $\psi^{\varepsilon}_{j,k}$, $\varepsilon \in (0,1]$, be given by (7.4) and (7.8). If $\{\phi_{j,k}\}_{j,k\in J}$ and $\{\psi_{j,k}\}_{j,k\in J}$ are dual Gabor frames such that (7.10) holds, then the pair in (7.9) is an admissible Gabor pair.*

8. *Discrete wave-front sets*

Definition 2. *Let $f \in \mathcal{D}'(X)$, X is open in $\mathbf{R}^\mathbf{d}$, $x_0 \in X$, and let (Λ_1, Λ_2) be a strongly admissible lattice pair in $\mathbf{R}^\mathbf{d}$ and $\{\xi_k\}_{j\in J} = \Lambda_2$. Moreover, let $D \in \mathcal{A}(*_\infty)$ contain x_0. Then the discrete wave-front set $DF_\mathcal{B}(f)$ consists of all $(x_0, \xi_0) \in \mathbf{R}^\mathbf{d} \times (\mathbf{R}^\mathbf{d} \setminus 0)$ such that for each $\varphi \in C_0^\infty(D \cap X)$ with $\varphi(x_0) \neq 0$ and each open conical neighbourhood Γ of ξ_0, it holds*

$$|\varphi f|^D_{\mathcal{B}(\Gamma)} = \infty.$$

For the definition of discrete wave-front sets of modulation space type, we consider admissible Gabor pairs $(\{\phi^{\varepsilon}_{j,k}\}_{j,k\in J}, \{\psi^{\varepsilon}_{j,k}\}_{j,k\in J})$, $\varepsilon \in (0,1]$, and let

$$J_{x_0}(\varepsilon) = J_{x_0}(\varepsilon, \phi, \psi) = J_{x_0}(\varepsilon, \phi, \psi, \Lambda_1)$$

be the set of all $j \in J$ such that

$$x_0 \in \ \mathrm{supp}\ \phi^{\varepsilon}_{j,k} \quad \text{or} \quad x_0 \in \ \mathrm{supp}\ \psi^{\varepsilon}_{j,k}$$

Definition 3. *Let $f \in \mathcal{D}'(X)$, X is open in $\mathbf{R^d}$, $x_0 \in X$, and let $(\{\phi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$ be an admissible Gabor pair with respect to the lattices Λ_1 and Λ_2 in $\mathbf{R^d}$. Then the discrete wave-front set $DF_{\mathcal{C}}(f)$ (recall, $\mathcal{C} = M^{p,q}_{(\omega)}(\mathbf{R^d})$) consists of all $(x_0, \xi_0) \in \mathbf{R^d} \times (\mathbf{R^d} \setminus 0)$ such that for each $\varepsilon \in (0,1]$ and each open conical neighbourhood Γ of ξ_0, it holds*

$$\Big(\sum_{\{\xi_k\}\in\Gamma\cap\Lambda_2} \Big(\sum_{j\in J_{x_0}(\varepsilon)} |c_{j,k}(\varepsilon)\omega(\xi_k)|^p \Big)^{q/p} \Big)^{1/q} = \infty,$$

where $f = \sum_{j,k\in J} c_{j,k(\varepsilon)}\phi^\varepsilon_{j,k}$, and $c_{j,k}(\varepsilon) = C_{\phi,\psi}(f, \psi^\varepsilon_{j,k})_{L^2(\mathbf{R^d})}$ and the constant $C_{\phi,\psi}$ depends on ϕ and ψ only.

Roughly speaking, $(x_0, \xi_0) \in \mathrm{D}F_{\mathcal{C}}(f)$ means that f is not locally in $\mathcal{C}$ i n the direction ξ_0. The following result shows that a lot of our wave-front sets coincide.

Theorem7. *Let $X \subseteq \mathbf{R^d}$ be open and let $f \in \mathcal{D}'(X)$. Then Theorem 6 holds.*

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