Bulletin T.CXLI de l'Académie serbe des sciences et des arts − 2010
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 35

A GENERAL METHOD TO SOLVE FRACTIONAL DIFFERENTIAL EQUATIONS ON R

B. STANKOVIĆ

(Presented at the 4th Meeting, held on May 21, 2010)

A b s t r a c t. Linear differential equations with constant coefficients and Riemann-Liouville fractional derivatives defined on the real axis are analysed in a subspace of tempered distributions. This method can also give the classical solutions.

AMS Mathematics Subject Classification (2000): 26A33,34A08
Key Words: fractional differential equations, Wright's function

1. *Introduction*

The present paper is devoted to a method to obtain explicit solutions to equation

$$\sum_{i=0}^{m} A_i({}_{-\infty}D_t^{\alpha_i} y)(t) = f(t), \quad -\infty < t < \infty, \tag{1.1}$$

where ${}_{-\infty}D_t^{\alpha_i}$ are Riemann-Liouville left fractional derivatives. In the literature there are different results of special cases of equation (1.1) defined on a bounded interval, on real axis and on Half-axis. In the monograph [3], published in year 2006, such results have been quoted giving a rich literature on equation (1.1).

In the meantime, other papers, books and conference proceedings have also appeared. We mention only two books: [13] published in 2005 and [9] in 2008. We believe that such a method, as it is ours, which gives the generalized solutions and classical too, to equation (1.1) was a missing link.

2. *Preliminaries*

2.1. *Wright's functions*

The Wright's function (cf. [12]) is defined by the series

$$\phi(\beta,\rho;z)=\sum_{i=0}^{\infty}\frac{z^i}{\Gamma(i+1)\Gamma(\rho i+\beta)},\ z\in C,$$

where $-1<\rho<0$ and $\beta\in R$ are fixed.

We use functions $F_\nu(x)$,

$$F_\nu(x)=\phi(0,-\nu,-x^{-\nu}),\ x>0,\ 0<\nu<1.$$

We quote some properties of F_ν (cf. [12] and [7]):

1) $\int\limits_0^\infty \exp(-st)F_\nu(t/x^{1/\nu})\frac{dt}{\nu x}=s^{\nu-1}\exp(-xs^\nu), x>0,\ s\in C,\ Re\ s>0;$

2) $F_\nu(x)\sim\frac{1}{\pi}\sin\nu\pi\frac{\Gamma(\nu+1)}{x^\nu}, x\to\infty;$

3) $F_\nu(x)>0,\ x>0;$

4) $F_\nu(\sigma/x^{1/\nu})=y^{1/2}\exp(-y)\sum\limits_{m=0}^{M}A_m y^{-m}+O(y^{-M}),$
   where $y=(1-\nu)\nu^{\frac{\nu}{1-\nu}}x^{\frac{1}{1-\nu}}/\sigma^{\frac{\nu}{1-\nu}}$ and A_m are constants depending on ν, $m=0,...,M$.
   It follows that $|F_\nu(\sigma/x^{1/\nu})|\le C(x^{1/(1-\nu)}/\sigma^{\nu/1-\nu})^{1/2}\exp(-\gamma x^{1/(1-\nu)}/\sigma^{\nu/1-\nu})$, where C and γ are positive constants.

5) $\int\limits_0^\infty F_\nu(\sigma/x^{1/\nu})x^\alpha\frac{dx}{\nu x}=\frac{\Gamma(\alpha+1)}{\Gamma\nu\alpha+1}\sigma^{\nu\alpha},\ \alpha>-1.$

2.2. *The Fourier transform in $\mathcal{S}'$ ($\mathcal{S}'$ is the space of tempered distributions)*

1) Let $\varphi \in \mathcal{S}$, the Fourier transform $\mathcal{F}\varphi$ is

$$\mathcal{F}\varphi = \int\limits_{-\infty}^{\infty} e^{-\omega t}\varphi(t)dt,\ \mathcal{F}^{-1}(\varphi) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{i\omega t}\varphi(t)dt. \tag{2.1}$$

If $f \in \mathcal{S}'$, then $\mathcal{F}f$ is defined as

$$\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}\varphi\rangle,\ \varphi \in \mathcal{S}.$$

2) If the distribution f is defined by the function f belonging to $L^1(\mathbf{R})$, then $\mathcal{F}f$ is given by the same formula as in the case $f \in \mathcal{S}$.

3) The Fourier transform of the n-the derivative of $f \in \mathcal{S}'$ is

$$\mathcal{F}(f^{(m)}) = (i\omega)^m \mathcal{F}f. \tag{2.2}$$

4) Finally we have for $b < 0$ and $\beta \geq 1$ (cf. [6], p.207):

$$\mathcal{F}(H(-t)e^{b|t|}\frac{|t|^{\beta-1}}{\Gamma(\beta)})(i\omega) = \frac{(-1)^\beta}{(b+i\omega)^\beta},\ \beta \geq 1.$$

For the space of tempered distributions $\mathcal{S}'$ cf. [5], [8] and [10].

2.3. *The left Riemann-Lionville derivative on* $\mathbf{R}$

Let $Y \in L^p(\mathbf{R})(x)$ and $\alpha = k + \gamma,\ k \in \mathbf{N}_0,\ \gamma \in [0, 1)$. Then

$${}_{-\infty}I_x^{1-\gamma}Y(x) = \frac{1}{\Gamma(1-\gamma)}\int\limits_{-\infty}^{x}\frac{Y(t)dt}{(x-t)^\gamma}. \tag{2.3}$$

The left fractional integral ${}_{-\infty}I_x^{1-\gamma}Y$ is defined for $Y \in L^p(\mathbf{R}),\ 1 \leq p < \frac{1}{1-\gamma}$. The operator ${}_{-\infty}I_x^{1-\gamma}$ is bounded from $L^p(\mathbf{R})$ to $L^q(\mathbf{R})$ if and only if $1 < p < \frac{1}{\gamma-1}$ and $q = \frac{p}{1-(1-\gamma)p}$. (cf. [4],p.102-103).

The left fractional derivative is

$${}_{-\infty}D_t^\alpha Y = (\frac{d}{dt})^{k+1}{}_{-\infty}I_t^{1-\gamma}Y.$$

We extend the left fractional derivative ${}_{-\infty}D_t^\alpha$ to a subclass of tempered distributions $\mathcal{S}'$.

The function

$$h_\eta(t) = \begin{cases} t^{\eta-1}/\Gamma(\eta), & t > 0 \\ 0, & t < 0, \quad \eta > 0, \end{cases} \tag{2.4}$$

defines a regular tempered distribution. Then the fractional integral ${}_{-\infty}I_t^{1-\gamma}Y$ given by (2.3) can be written as:

$${}_{-\infty}I_t^{1-\gamma}Y = h_\gamma * Y$$

(cf. [4],p.94).

Definition 2.1. *Let $f \in \mathcal{S}'$. If $h_\gamma * f$ exists and belongs to $\mathcal{S}'$, then*

$${}_{-\infty}D_t^\alpha f = \delta^{(k+1)} * (h_\gamma * f),\ \alpha = k + \gamma,\ \alpha \in \mathbf{N}_0,\ \gamma \in [0, 1).$$

We quote some classes of tempered distributions for which there exist ${}_{-\infty}D_t^\alpha$:

1) Space of rapidly decreasing distributions (denoted by $\mathcal{O}'_C$). Since $h_\alpha \in \mathcal{S}'$, $\alpha > 0$, for every $g \in \mathcal{O}'_C$, $h_\alpha * g \in \mathcal{S}'$ (cf. [5], T.II, p.103). Then ${}_{-\infty}D_t^\alpha g$ exists for every $\alpha > 0$.

2) $f \in L^p(\mathbf{R}),\ 1 < p < \frac{1}{1-\gamma}$, has ${}_{-\infty}D_t^\alpha f,\ \alpha = k + \gamma$. If $f \in L^p(\mathbf{R}),\ 1 < p < \frac{1}{1-\gamma}$, we have seen that ${}_{-\infty}I_t^{1-\gamma}f \in L^q(\mathbf{R}),\ q = \frac{p}{1-(1-\gamma)p}$.

3) Class ρ:
$f \in L^1_{loc}(\mathbf{R}),\ supp f \subset [a, \infty),\ a > -\infty$ and for some $k \in \mathbf{N}_0,\ 0 < M < \infty$, admitting the estimation $|f(x)| \le M(x-a)^k$ for x sufficiently large.

   If $f \in \rho$, then ${}_{-\infty}D_t^\alpha f$ exists for every $\alpha > 0$.

   Namely, for $\gamma < 1$,

$$\begin{aligned} |({}_{-\infty}I_t^{1-\gamma}f)(t)| &= \left|\frac{1}{\Gamma(1-\gamma)}\int_{-\infty}^{t} \frac{f(\tau)}{(t-\tau)^\gamma}d\tau\right| \\ &\le H(t-a)\frac{M(t-a)^{k+1-\gamma}}{\Gamma(2-\gamma)}. \end{aligned}$$

   Hence, ${}_{-\infty}I_t^{1-\gamma}f \in \rho \subset \mathcal{S}'$ and ${}_{-\infty}D_t^\alpha f$ exists.

Lemma 2.1. *If* $f = D^l F = \delta^{(l)} * F,\ l \in \mathbf{N}$, *then*

$$ {}_{-\infty}D_t^{\alpha} f = \delta^{(k+1)} * (h_\gamma * D^l F) = \delta^{(k+1+l)} * (h_\gamma * F). $$

The proof follows from the property of the derivative of the convolution (cf. [10],p.65 or [5], T. II , p.16).

3. *A theorem for the Fourier transform*

Theorem 3.1. *Let* $0 < \nu < 1,\ \beta \geq 1$.

1) If $f(t) = H(t)e^{-at}\frac{t^{\beta-1}}{\Gamma(\beta)}$, *Re* $a > 0$, *then*

$$ \begin{aligned} &(\mathcal{F}\, {}_{-\infty}D_t^{1-\nu} H(t) \int_{-\infty}^{\infty} F_\nu(t/\tau^{1/\nu}) H(\tau) e^{-a\tau} \frac{\tau^{\beta-1}}{\Gamma(\beta)} \frac{d\tau}{\nu\tau})(i\omega) \\ = &\ \frac{1}{(a+(i\omega)^\nu)^\beta}. \end{aligned} \tag{3.1} $$

2) If $f(t) = H(-t)e^{b|t|}\frac{|t|^{\beta-1}}{\Gamma(\beta)}$, *Re* $b < 0$, *then*

$$ \begin{aligned} &(\mathcal{F}\, {}_{-\infty}D_t^{1-\nu} H(t) \int_{-\infty}^{\infty} F_\nu(t/|\tau|^{1/\nu}) H(-\tau) e^{b|\tau|} \frac{\tau^{\beta-1}}{\Gamma(\beta)} \frac{d\tau}{\nu\tau}) \\ = &\ \frac{(-1)^\beta}{(b+(i\omega)^\nu)^\beta}. \end{aligned} \tag{3.2} $$

P r o o f. First we prove some properties of the functions

$$ \begin{aligned} \psi_{1,\beta,1/q_0}(a,t) &\equiv H(t) \int_{-\infty}^{\infty} F_{1/q_0}(t/\tau^{q_0}) H(\tau) e^{-a\tau} \frac{\tau^{\beta-1}}{\Gamma(\beta)} \frac{q_0 d\tau}{\tau}, \qquad (3.3) \\ \psi_{2,\beta,1/q_0}(b,t) &\equiv H(t) \int_{-\infty}^{\infty} F_{1/q_0}(t/|\tau|^{q_0}) H(-\tau) e^{b|\tau|} \frac{|\tau|^{\beta-1}}{\Gamma(\beta)} \frac{q_0 d\tau}{\tau}, \end{aligned} $$

where $a > 0,\ b < 0$.

We consider first the function $\psi_{1,\beta,1/q_0}(a,t)$.

Since $F_{1/q_0}(x) > 0,\ x > 0$ and $F_{1/q_0}(x) \to 0,\ x \to 0$, (cf. 2.1 3) and 4)), we have

$$|\psi_{1,\beta,1/q_0}(a,t)| \leq \int_0^\infty F_{1/q_0}(t/\tau^{q_0}) \frac{\tau^{\beta-1}}{\Gamma(\beta)} \frac{q_0 d\tau}{\tau}.$$

By (2.1 5)) it follows that

$$|\psi_{1,\beta,1/q_0}(a,t)| \leq H(t)Ct^{\frac{1}{q_0}(\beta-1)},\ t \in (-\infty,\infty),\ \beta \geq 1.$$

Also, if we take care that the function

$$e^{-a\tau}\frac{\tau^{\beta-1}}{\Gamma(\beta)},\ a > 0,\ \beta \geq 1,$$

is bounded on $[0,\infty)$, we have by 2.1 5)

$$|\psi_{1,\beta,1/q_0}(a,t)| \leq H(t)K \int_0^\infty F_{1/q_0}(t/\tau^{1/\nu}) \frac{q_0 d\tau}{\tau} = K.$$

By the last two inequalities it follows that $\psi_{1,\beta,1/q_0}(a,t)$ is a bounded function on $(-\infty,\infty)$, $supp\psi_{1,\beta,1/q_0}(a,t) \subset [0,\infty)$ and $\psi_{1,\beta,1/q_0}(a,t) \sim Ct^{\frac{1}{q_0}(\beta-1)}$, $t \to 0$.

For the function $\psi_{2,\beta,1/q_0}(b,t)$ the procedure is just the same because

$$\begin{aligned} |\psi_{2,\beta,1/q_0}(b,t)| &\leq H(t)|\int_{-\infty}^0 F_\nu(t/|\tau|^{1/\nu})H(-\tau)e^{b|\tau|}\frac{|\tau|^{\beta-1}}{\Gamma(\beta)}\frac{d\tau}{\nu\tau}| \\ &\leq H(t)\int_0^\infty F_\nu(t/u^{1/\nu})e^{bu}\frac{u^{\beta-1}}{\Gamma(\beta)}\frac{du}{\nu u}. \end{aligned}$$

The function $\psi_{2,\beta,1/q_0}(b,t)$ is also a bounded function on $(-\infty,\infty)$, $supp\psi_{2,\beta,1/q_0}(b,t) \subset [0,\infty)$, $\psi_{2,\beta,1/q_0}(b,t) \sim t^{\frac{1}{q_0}(\beta-1)}$, $t \to 0$.

Hence, $\psi_{1,\beta,1/q_0}$ and $\psi_{2,\beta,1/q_0}$ belong to the class ρ and there exist ${}_{-\infty}D_t^\alpha\psi_{1,\beta,1/q_0}(a,t)$ and ${}_{-\infty}D_t^\alpha\psi_{2,\beta,1/q_0}(b,t)$, $\alpha > 0$. Now we can prove the first part of the Theorem.

Since $f \in L^1(\mathbf{R})$, we have

$$\int_{-\infty}^\infty e^{-i\omega t}f(t)dt = \int_0^\infty e^{-(a+i\omega)t}\frac{t^{\beta-1}}{\Gamma(\beta)}dt = \frac{1}{(a+i\omega)^\beta}. \tag{3.4}$$

We have to prove that for $\omega \in (-\infty, \infty)$:

$$
\begin{aligned}
& (i\omega)^{1-\nu} \int_{-\infty}^{\infty} e^{-i\omega t} dt H(t) \int_{-\infty}^{\infty} F_\nu(t/\tau^{1/\nu}) H(\tau) e^{-a\tau} \frac{\tau^{\beta-1}}{\Gamma(\beta)} \frac{d\tau}{\nu\tau} \\
= \; & (i\omega)^{1-\nu} \int_{-\infty}^{\infty} H(\tau) e^{-a\tau} \frac{\tau^{\beta-1}}{\Gamma(\beta+1)} d\tau \int_{-\infty}^{\infty} H(t) e^{-i\omega t} F_\nu(t/\tau^{1/\nu}) \frac{dt}{\nu\tau}.
\end{aligned} \tag{3.5}
$$

By Fubini's theorem for $0 < \tau_1 \leq \tau \leq \tau_2 < \infty$, and by the properties 2) and 4) of the function F_ν (cf. 2.1) we have

$$
\begin{aligned}
& (i\omega)^{1-\nu} \int_{\tau_1}^{\tau_2} f(\tau) d\tau \int_0^{\infty} e^{-i\omega t} F_\nu(t/\tau^{1/\nu}) \frac{d\tau}{\nu\tau} \\
= \; & (i\omega)^{1-\nu} \int_0^{\infty} e^{-i\omega t} dt \int_{\tau_1}^{\tau_2} F_\nu(t/\tau^{1/\nu}) f(\tau) \frac{d\tau}{\nu\tau}.
\end{aligned} \tag{3.6}
$$

Let us prove that

$$
\begin{aligned}
& \lim_{\tau_1 \to 0} \lim_{\tau_2 \to \infty} (i\omega)^{1-\nu} \int_0^{\infty} e^{i\omega t} dt \int_{\tau_1}^{\tau_2} F_\nu(t/\tau^{1/\nu}) f(\tau) \frac{d\tau}{\nu\tau} = \\
= \; & (-i\omega)^{1-\nu} \int_0^{\infty} e^{-i\omega t} \int_0^{\infty} F_\nu(t/\tau^{1/\nu}) f(\tau) \frac{d\tau}{\nu\tau}.
\end{aligned}
$$

First we show that

$$
\lim_{\tau_1 \to 0} I = \lim_{\tau_1 \to 0} (i\omega)^{1-\nu} \int_0^{\infty} e^{-i\omega t} dt \int_0^{\tau_1} F_\nu(t/\tau^{1/\nu}) f(\tau) \frac{d\tau}{\nu\tau} = 0.
$$

By (3.6) and by property 1) of F_ν, for any $\epsilon > 0$ there exists τ_1^ϵ such that $0 < \tau_1'' < \tau_1' < \tau^\epsilon$ and

$$
\begin{aligned}
|I(\tau_1') - I(\tau_1'')| \; & = \; |(i\omega)^{1-\nu} \int_0^{\infty} e^{-i\omega t} dt \int_{\tau_1''}^{\tau_1'} F_\nu(t/\tau^{1/\nu}) f(\tau) \frac{d\tau}{\nu\tau}| \\
& \leq \; |e^{-\tau''(\omega i)^\nu}| \int_{\tau_1''}^{\tau_1'} |f(\tau)| d\tau < \epsilon.
\end{aligned}
$$

Analogously we can proceed in the case $\tau_2 \to \infty$. Thus (3.5) is proved. If we use once more property 1) of F_ν and (3.5) we have by (3.4):

$$\begin{aligned} &(\mathcal{F}_{-\infty}D_x^{1-\nu}H(t)\int_{-\infty}^{\infty} F_\nu(t/\tau^{1/\nu})H(\tau)e^{-a\tau}\frac{\tau^{\beta-1}}{\Gamma(\beta)}\frac{d\tau}{\nu\tau})(i\omega) \\ = \ &(i\omega)^{1-\nu}(\mathcal{F}H(t)\int_{-\infty}^{\infty} F_\nu(t/\tau^{1/\nu})H(\tau)e^{-a\tau}\frac{\tau^{\beta-1}}{\Gamma(\beta)}\frac{d\tau}{\nu\tau})(i\omega) \\ = \ &(i\omega)^{1-\nu}\int_{-\infty}^{\infty} H(\tau)e^{-a\tau}\frac{\tau^{\beta-1}}{\Gamma(\beta)}(i\omega)^{\nu-1}e^{-\tau(i\omega)^\nu}d\tau \\ = \ &\Big(\frac{1}{a+(i\omega)^\nu}\Big)^\beta, \end{aligned}$$

where $Re\ \tau(i\omega)^\nu = \tau|\omega|\cos\nu\frac{\pi}{2} > 0,\ \tau > 0.$

The first part of Theorem 3.1 is proved.

As regards the second part of the Theorem, we start with (cf. 2.2 4))

$$\mathcal{F}(H(-\tau)e^{b|\tau|}\frac{|\tau|^{\beta-1}}{\Gamma(\beta)})(i\omega) = \frac{1}{(-b-(i\omega))^\beta} = \frac{(-1)^\beta}{(b+(i\omega))^\beta}. \tag{3.7}$$

Then for $f(t) = e^{b|\tau|}\frac{|\tau|^{\beta-1}}{\Gamma(\beta)},\ t \geq 0$, we have, similar to (3.6),

$$\begin{aligned} I \ = \ &(i\omega)^{1-\nu}\int_{-\infty}^{\infty} e^{-i\omega t}H(t)dt\int_{-\tau_1}^{-\tau_2} F_\nu(t/|\tau|^{1/\nu})H(-\tau)f(\tau)\frac{d\tau}{\nu\tau} \\ = \ &(i\omega)^{1-\nu}\int_{-\tau_1}^{-\tau_2} H(-\tau)f(\tau)d\tau\int_0^{\infty} e^{-i\omega t}F_\nu(t/|\tau|^{1/\nu})\frac{1}{\nu\tau}dt \\ = \ &(i\omega)^{1-\nu}\int_{-\tau_1}^{-\tau_2} H(-\tau)f(\tau)(i\omega)^{\nu-1}e^{-|\tau|(i\omega)^\nu}\frac{d\tau}{\nu\tau}. \end{aligned} \tag{3.8}$$

The proof that

$$\begin{aligned} &\lim_{\tau_1\to\infty}\lim_{\tau_2\to 0} I = \\ = \ &(i\omega)^{1-\nu}\int_{-\infty}^{\infty} e^{-\omega t}H(t)dt\int_{-\infty}^{\infty} F_\nu(t/|\tau|^{1/\nu})H(-\tau)f(t)\frac{d\tau}{\nu\tau}) \end{aligned} \tag{3.9}$$

is the same as the proof of (3.6). It remains only to use (3.7) in (3.8) taking care of (3.9), which gives:

$$\lim_{\tau_1\to\infty}\lim_{\tau_2\to 0} I = \frac{(-1)^\beta}{(b+(i\omega)^\nu)^\beta}.$$

This proves Theorem 3.1.

4. *Equation with left fractional derivatives on* **R**

4.1. *A method to find solutions*

Assume for equation

$$\sum_{i=0}^{m} A_i({}_{-\infty}D_t^{\alpha_i}Y)(t) = f(t),\ t\in\mathbf{R},\ f\in\mathcal{S}', \tag{4.1}$$

the following conditions: $\alpha_0 = 0,\ \alpha_i = \frac{p_i}{q_i} = \frac{\beta_i}{q_0} = k_i+\gamma_i;\ p_i,\beta_i,k_i\in\mathbf{N}_0$ and $q_1, q_0\in\mathbf{N};\ \gamma_i\in[0,1)$ for $i=1,...,m;\ \alpha_1<\alpha_2<...<\alpha_m$.

Let us suppose that $Y\subset\mathcal{S}'$ and such that ${}_{-\infty}D_t^{\alpha_i}Y,\ i=1,...,m$, exist (cf. Definition 2.1).

We apply the Fourier transform to (4.1) which gives

$$\sum_{i=0}^{m} A_i(i\omega)^{\alpha_i}(\mathcal{F}Y)(i\omega) = (\mathcal{F}f)(i\omega),\ \omega\in\mathbf{R}. \tag{4.2}$$

Hence,

$$\begin{aligned}(\mathcal{F}Y)(i\omega) &= \frac{1}{\sum\limits_{i=0}^{m} A_i(i\omega)^{\alpha_i}}(\mathcal{F}f)(i\omega)\\ &= \frac{1}{\sum\limits_{i=0}^{m} A_i((i\omega)^{1/q_0})^{\beta_i}}(\mathcal{F}f)(i\omega).\end{aligned} \tag{4.3}$$

Let $P(z)$ and $Q(z)$ denote the following functions:

$$P(z) = \sum_{i=0}^{m} A_i z^{\beta_i},\quad Q(z) = \frac{1}{P(z)}. \tag{4.4}$$

Then $Q(z)$ can be written as the sum of elements of the form $C_r(z+r)^{-k_r}$ and $zC'_r(z^2+r^2)^{-k_r}$, where C_r and C'_r are constants, $r \in \mathbf{R}$ and $k \in \mathbf{N}$. If $z = (i\omega)^{1/q_0}$, we have by (4.3)

$$Y = (\mathcal{F}^{-1}Q((i\omega)^{1/q_0})) * f. \tag{4.5}$$

This gives only formally a solution to (4.1). Therefore we have to analyse (4.5).

4.2. *The function* $\mathcal{F}^{-1}Q((i\omega)^{1/q_0})$

Let in the polynomial $P(z) = \sum_{i=0}^{m} A_i z^{\beta_i}$ the coefficient $A_0 \neq 0$. With our supposition $\alpha_0 = 0$, we have $P(0) \neq 0$ and

$$Q(z) = \frac{1}{P(z)} = \sum_{p=1}^{p_0}\sum_{i=1}^{k_p} \frac{M_{i,p}}{(z-r_p)^i}, \tag{4.6}$$

where r_p are the zeros of $P(z)$ and k_p are theirs multiplicities, $p = 1, ..., p_0,\ k_1 + ... + k_{p_0} = \beta_m,\ k_1 \leq ... \leq k_{p_0}$. In this case we have $r_p \neq 0,\ p = 1, ..., p_0$. First we consider this situation. But if $A_0 = 0$, then $P(z) = z^l P_1(z),\ l \in \mathbf{N}$ where $P_1(0) \neq 0$. This will be considered separately.
Case $A_0 \neq 0$**.** To realize $\mathcal{F}^{-1}Q((i\omega)^{1/q_0})$, we have to find

$$(\mathcal{F}^{-1}((i\omega)^{1/q_0} + r)^{-k})(t),\ Re\ r \neq 0 \tag{4.7}$$

and

$$(i\omega)^{1/q_0}((i\omega)^{2/q_0} + (Im\ r)^2)^{-k}(t),\ Re\ r = 0,\ Im\ r \neq 0. \tag{4.8}$$

By Theorem 3.1 we have to separate three cases:
$Re\ r_p > 0$, $Re\ r_p < 0$ and $Re\ r_p = 0$. In cases $Re\ r_p > 0$ and $Re\ r_p < 0$, (4.7) is realized by Theorem 3.1, i.e., by (3.1) and by (3.2), respectively. We have for $r_p > 0$

$$\begin{aligned}((i\omega)^{1/q_0} + r_p)^{-k_p} &= \mathcal{F}\Big({}_{-\infty}D_t^{1-1/q_0} H(t) \times \\ &\times \int_{-\infty}^{\infty} F_{1/q_0}(t/\tau^{q_0}) H(\tau) e^{-r_p\tau} \frac{\tau^{k_p-1}}{\Gamma(k_p)} \frac{q_0 d\tau}{\tau}\Big)(i\omega),\end{aligned} \tag{4.9}$$

and for $r_p < 0$:

$$\frac{(-1)^{k_p}}{((i\omega)^{1/q_0}+r_p)} = \mathcal{F}({}_{-\infty}D_t^{1-1/q_0}H(t)\int\limits_{-\infty}^{\infty} F_{1/q_0}(t/|\tau|^{q_0}) \times \times H(-\tau)e^{-r_p|\tau|}\frac{|\tau|^{k_p-1}}{\Gamma(k_p)}\frac{q_0 d\tau}{\tau}\Big)(i\omega), \qquad (4.10)$$

It remains the case $Re\ r = 0$. If $Re\ r = 0$ and $Im\ r \neq 0$, we can take $q_0 > 2$ (otherwise equation (4.1) reduces to ordinary differential equation). We can apply (3.1) to (4.8) to obtain

$$\begin{aligned} &(i\omega)^{1/q_0}((i\omega)^{2/q_0}+(Im\ r_p)^2)^{-k_p} = \\ &= \mathcal{F}[{}_{-\infty}D_t^{1-1/q_0}H(t)\int\limits_{-\infty}^{\infty} F_{2/q_0}(t/\tau^{q_0/2})H(\tau)e^{-(Im\ r_p)^2\tau}\frac{\tau^{\beta-1}}{\Gamma(\beta)}\frac{q_0 d\tau}{2\tau}](i\omega) \\ &= \mathcal{F}[{}_{-\infty}D_t^{1-1/q_0}\psi_{1,i,2/q_0}((Im\ r_p)^2,t)](i\omega). \qquad (4.11)\end{aligned}$$

Case $A_0 = 0$**.** Then $P(z) = z^l P_1(z),\ l \in \mathbf{N}$ and $P_1(0) \neq 0$. Hence,

$$Q(z) = \frac{1}{P(z)} = \frac{1}{z^l P_1(z)},$$

and

$$Q((i\omega)^{1/q_0}) = (i\omega)^{-l/q_0}\frac{1}{P_1((i\omega)^{1/q_0})} = (i\omega)^{-l/q_0}Q_1((i\omega)^{1/q_0}). \qquad (4.12)$$

4.3. *Existence and the analytical form of the solutions to (4.1)*

Supposing that $Y(t)$ in (4.1) has all the left fractional derivatives ${}_{-\infty}D_t^{\alpha_i},\ i = 1,...,m,$ (Definition 2.1) and that they belong to $\mathcal{S}'$. We find its possible analytical form given by (4.5). Selecting an f in (4.1) we have to prove that Y given by (4.5) has all presumed properties.

If we find f "sufficiently good" such that Y given by (4.5) is a numerical function which has all classical fractional derivatives ${}_{-\infty}D_t^{\alpha_i},\ i = 1,...,m,$ then this function Y is a classical solution. This follows from Definition 2.1.

As an application of the proposed solving method we prove the following theorem.

Theorem 4.1. *Let* ${}_{-\infty}D_t^{1-1/q_0}f = F$*, where* F *belongs to the class* ρ*. The equation (4.1) has a generalized solution* Y *belonging to the class* ρ*, as*

F does. The solution Y is given by (4.5) which can be realized by (4.9), (4.10), (4.11) and (4.12) in case $A_0 \neq 0$ and for $A_0 = 0$ we have to use (4.13) in addition.

P r o o f. From (4.3) it follows that

$$(\mathcal{F}Y)(i\omega) = Q((i\omega)^{1/q_0})(\mathcal{F}f)(i\omega).$$

If $A_0 \neq 0$, the zeros r_p of $P(z)$ are different from zero. Hence

$$Q((i\omega)^{1/q_0})(\mathcal{F}f)(i\omega) = \sum_{p=1}^{p_0}\sum_{i=1}^{k_p} \frac{M_{i,p}}{((i\omega)^{1/q_0} + r_p)^i}(\mathcal{F}f)(i\omega),$$

where $r_p \neq 0$.

If $r_p > 0$, then by (4.9) and by using notation in (3.3) we have:

$$\begin{aligned}
&\frac{1}{((i\omega)^{1/q_0} + r_p)^i}(\mathcal{F}f)(i\omega) = \\
= \ & (i\omega)^{1-1/q_0}(\mathcal{F}\psi_{1,i,1/q_0}(r_p,\cdot))(i\omega)(\mathcal{F}f)(i\omega) \\
= \ & (\mathcal{F}\psi_{1,i,1/q_0}(r_p,\cdot))(i\omega)(\mathcal{F}_{-\infty}D^{1-1/q_0}f)(i\omega) \\
= \ & (\mathcal{F}\psi_{1,i,1/q_0}(r_p,\cdot))(i\omega)(\mathcal{F}F)(i\omega) \\
= \ & \mathcal{F}(\psi_{1,i,1/q_0}(r_p,\cdot) * F)(i\omega).
\end{aligned}$$

The function $\psi_{1,i,1/q_0}$ is a continuous and bounded function on $[0,\infty)$ and the function F belongs to the class ρ. Then

$$Y = \psi_{1,i,1/q_0}(r_p,\cdot) * F$$

belongs also to the class ρ.

If we use (4.10) and notation (3.3) we obtain for $r_p < 0$:

$$\begin{aligned}
&\frac{1}{((i\omega)^{1/q_0} + r_p)^i}(\mathcal{F}f)(i\omega) = \\
= \ & (i\omega)^{1-1/q_0}(\mathcal{F}\psi_{2,i,1/q_0}(r_p,\cdot))(i\omega)(\mathcal{F}f)(i\omega) \\
= \ & (\mathcal{F}\psi_{2,i,1/q_0}(r_p,\cdot))(i\omega)(\mathcal{F}_{-\infty}D^{1-1/q_0}f)(i\omega) \\
= \ & (\mathcal{F}\psi_{2,i,1/q_0}(r_p,\cdot) * F)(i\omega).
\end{aligned}$$

So we have the same situation as for $r_p > 0$.

Finally if $Re\ r_p = 0,\ Im\ r \neq 0$, by (4.11) is

$$\begin{aligned} & (i\omega)^{1/q_0}((i\omega)^{2/q_0} + (Im\ r_p)^2)^{-i}(\mathcal{F}f)(i\omega) = \\ = \ & (i\omega)^{1/q_0}(\mathcal{F}_{-\infty}D_t^{1-2/q_0}\psi_{1,\beta,2/q_0}((Im\ r_p)^2, t))(i\omega)(\mathcal{F}f)(i\omega) \\ = \ & (\mathcal{F}\psi_{1,\beta,2/q_0}((Im\ r_p)^2, t))(i\omega)(\mathcal{F}_{-\infty}D^{1-1/q_0}f)(i\omega) \\ = \ & (\mathcal{F}\psi_{1,\beta,2/q_0}((Im\ r_p)^2, \cdot) * F)(i\omega). \end{aligned}$$

Hence, in case $Re\ r_p = 0\ \ Im\ r_p \neq 0$, we have

$$Y = \Big(\psi_{1,\beta,2/q_0}((Im\ r_p)^2, \cdot) * F\Big) \in \rho.$$

If $A_0 = 0$, then by (4.3) $\mathcal{F}Y$ is

$$(\mathcal{F}Y)(i\omega) = \frac{1}{(i\omega)^{l/q_0}} \frac{1}{P_1((i\omega)^{1/q_0})}(\mathcal{F}f)(i\omega).$$

We have seen that

$$Q((i\omega)^{1/q_0})(\mathcal{F}f)(i\omega) \equiv \mathcal{F}\mathcal{U},\ \mathcal{U} \in \rho.$$

Now we can write the solution Y to (4.1) as

$$Y = {}_{-\infty}I_t^{l/q_0}\mathcal{U} \in \rho. \tag{4.13}$$

This completes the proof.

Remark.

1) Since ${}_{-\infty}D_t^{1-1/q_0}H(t-a)(t-a)^{1/q_0-1} = 0$, and since $H(t-a)(t-a)^{1/q_0-1} \in \rho$ for any $a > -\infty$, we have the same solution Y to (4.1) even though we add to f the function $CH(t-a)(t-a)^{1/q_0-1}$, where C is a constant.

2) With the exposed method we can obtain the classical solutions too. We have only to suppose that f has "enough good" properties. So, if in Theorem 4.2 we have for F additional property: $F \in AC^{k_m+1}([a,b])$ for every $b < \infty$, $F^{(k)}(a) = 0,\ k = 0, ..., k_m$, then the solution to equation (4.1), with $A_0 \neq 0$, is given as in 4.2. Let us remark that $AC^n([a,b])$ is the space of absolute continuous functions. To prove this assertion it is enough to show that there exists

$${}_{-\infty}D_t^{\alpha_i}(\psi_{j,i,1/q_0}(r_p, \cdot) * F)(t)$$

for $j = 1, 2,\ i = 1, ..., m$ and for all zeros r_p of $P(z)$.

We know (cf. [4], p.39) that ${}_{-\infty}D_t^{\alpha}G,\ \alpha > 0$, exists on (a, b) if $G \in AC^n([a, b])$. Then by [2], p.119 and properties of F, the derivative of the convolution is

$$E = (\frac{d}{dt})^{k_m}(\psi_{j,i,1/q_0}(r_p, \cdot) * F) = \psi_{j,i,1/q_0}(r_p, \cdot) * (\frac{d}{dt})^{k_m} F$$
$$= \int_0^t (\psi_{j,i,1/q_0}(r_p, \cdot) * F^{(k_m+1)})(\tau) d\tau.$$

This proves that $E \in AC[a, b]$ and that

$$\left(\psi_{j,i,1/q_0}(r_p, \cdot) * F\right) \in AC^{k_m+1},\ j = 1, 2.$$

REFERENCES

[1] Y u. A. B r i c h k o v, A. P. P r u d n i k o v, *Integral transforms of generalized functions*, Nauka, Moscow, 1957 (in Russian).

[2] G. D o e t s c h, *Handbuch der Lalace-Transformation I*, Birkhäuser-Verlag, Basel, 1950.

[3] A. A. K i l b a s, H. M. S r i v a s t a v a, J. J. T r u j i l l o, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[4] S. G. S a m k o, A. A. K i l b a s, O. I. M a r i c h e v, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach Science Publishers, Amsterdam, 1993.

[5] L. S c h w a r t z, *Théorie des Distributions*, Hermann, Paris, 1966.

[6] L. S c h w a r t z, *Méthodes mathématiques pour les sciences physiques*, Hermann, Paris, 1961.

[7] B. S t a n k o v i ć, *Sur une classe d' équations intégrales singuliéres*, Recueil des travaux de l'Académie Serbe des Sciences XLIII, 4:81-130, 1955.

[8] Z. S z m y d t, *Fourier transformation and linear differential equations*, Polish Scientific Publisher, Warszawa, 1977.

[9] V. V. U c h a y k i n, *Method of Fractional derivatives*, Artishok, Uljanovsk, 2008 (in Russian).

[10] V. S. V l a d i m i r o v, *Generalized Functions in Mathematical Phisics*, Mir, Moscow, 1979.

[11] E. M. W r i g h t, *On the coefficients of power series having exponential singularities*, J. Lond. Math. Soc, 8:71-79,1953.

[12] E. M. W r i g h t, *The Generalized Bessel Function of order greater then one*, Quert. J. Math. Oxford series, pages 36-48, 1940.

[13] G. M. Z a s l a v s k y, *Hamiltonian Chaos and Fractional Dynamics*, University Press, Oxford, 2005.

Department of mathematics and informatics
University of Novi Sad
Trg Dositeja Obradovića 4
21000 Novi Sad
Serbia
E-mail: borasta@eunet.rs