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PERTURBATION THEOREMS FOR CONVOLUTED C-SEMIGROUPS AND COSINE FUNCTIONS

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A b s t r a c t. *In this paper, we prove several different types of additive perturbation theorems for (local) convoluted C-semigroups and cosine functions.*

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1. *Introduction and Preliminaries*

Convoluted C-semigroups and cosine functions ([6]-[8], [14], [16]-[17], [25]) allow one to consider in a unified treatment the notion of fractionally integrated C-semigroups and cosine functions ([1]-[3], [20], [22], [39], [41]). We refer the reader to [1]-[2], [5], [9]-[11], [16]-[17], [25], [36] and [42] for examples of differential operators generating various types of convoluted C-semigroups and cosine functions. In the present paper, we study additive

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perturbation theorems for such classes of operator semigroups and cosine functions and continue the researches raised in [12], [16], [20], [28]-[29], [31], [35] and [39]-[40] (cf. also [4], [9]-[10], [21], [26], [32]-[34], [37]-[38], and [23]-[24], for similar results).

Throughout this paper E denotes a non-trivial complex Banach space, $L(E)$ denotes the space of bounded linear operators from E into E, A denotes a closed linear operator acting on E and $[D(A)]$ denotes the Banach space $D(A)$ equipped with the norm $||x||_{[D(A)]} := ||x|| + ||Ax||$, $x \in D(A)$. By $R(A)$ is denoted the range of operator A. Henceforward $L(E) \ni C$ is an injective operator, $\tau \in (0, \infty]$, K is a complex-valued locally integrable function in $[0, \tau)$ and K is not identical to zero. Given $t \in \mathbf{R}$, set $\lfloor t \rfloor := \sup\{k \in \mathbf{Z} : k \leq t\}$ and $\lceil t \rceil := \inf\{k \in \mathbf{Z} : k \geq t\}$. Set $\Theta(t) := \int_0^t K(s)ds$ and $\Theta^{-1}(t) := \int_0^t \Theta(s)ds,\ t \in [0, \tau)$; then Θ is an absolutely continuous function in $[0, \tau)$ and $\Theta'(t) = K(t)$ for a.e. $t \in [0, \tau)$. Let us recall that a function $K \in L^1_{loc}([0, \tau))$ is called a kernel if for every $\phi \in C([0, \tau))$, the assumption $\int_0^t K(t-s)\phi(s)ds = 0,\ t \in [0, \tau)$, implies $\phi \equiv 0$; thanks to the famous Titchmarsh's theorem, the condition $0 \in \text{supp}K$ implies that K is a kernel. We mainly use the following condition:

(P1) K is Laplace transformable, i.e., it is locally integrable on $[0, \infty)$ and there exists $\beta \in \mathbf{R}$ so that $\tilde{K}(\lambda) = \mathcal{L}(K)(\lambda) := \lim_{b \to \infty} \int_0^b e^{-\lambda t} K(t)dt := \int_0^\infty e^{-\lambda t} K(t)dt$ exists for all $\lambda \in \mathbf{C}$ with $\text{Re}\lambda > \beta$. Put $\text{abs}(K) := \inf\{\text{Re}\lambda : \tilde{K}(\lambda) \text{ exists}\}$.

In Theorem 2.9, we use the following condition:

(P2) $\tilde{K}(\lambda) \neq 0$ for all $\lambda \in \mathbf{C}$ with $\text{Re}\lambda > \text{abs}(K)$.

Definition 1.1. ([14], [16]-[17]) Let A be a closed operator and let $0 < \tau \leq \infty$. If there exists a strongly continuous operator family $(S_K(t))_{t \in [0,\tau)}$ $(S_K(t) \in L(E),\ t \in [0, \tau))$ such that:

(i) $S_K(t)A \subseteq AS_K(t)$, $t \in [0, \tau)$,

(ii) $S_K(t)C = CS_K(t),\ t \in [0, \tau)$ and

(iii) for all $x \in E$ and $t \in [0, \tau)$: $\int_0^t S_K(s)xds \in D(A)$ and

$$A \int_0^t S_K(s)xds = S_K(t)x - \Theta(t)Cx,$$

then it is said that A is a subgenerator of a (local) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$. If $\tau = \infty$, then we say that $(S_K(t))_{t\geq 0}$ is an exponentially bounded K-convoluted C-semigroup with a subgenerator A if, additionally, there exist $M \geq 1$ and $\omega \in \mathbf{R}$ such that $||S_K(t)|| \leq Me^{\omega t},\ t \geq 0$.

Definition 1.2. ([14], [16]-[17]) Let A be a closed operator and let $0 < \tau \leq \infty$. If there exists a strongly continuous operator family $(C_K(t))_{t\in[0,\tau)}$ such that:

(i) $C_K(t)A \subseteq AC_K(t)$, $t \in [0, \tau)$,

(ii) $C_K(t)C = CC_K(t),\ t \in [0, \tau)$ and

(iii) for all $x \in E$ and $t \in [0, \tau)$: $\int_0^t (t-s)C_K(s)xds \in D(A)$ and

$$A \int_0^t (t-s)C_K(s)xds = C_K(t)x - \Theta(t)Cx,$$

then it is said that A is a subgenerator of a (local) K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$. If $\tau = \infty$, then we say that $(C_K(t))_{t\geq 0}$ is an exponentially bounded K-convoluted C-cosine function with a subgenerator A if, additionally, there exist $M \geq 1$ and $\omega \in \mathbf{R}$ such that $||C_K(t)|| \leq Me^{\omega t},\ t \geq 0$.

Plugging $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ in Definition 1.1 and Definition 1.2, where $\alpha > 0$ and $\Gamma(\cdot)$ denotes the Gamma function, we obtain the well-known classes of α-times integrated C-semigroups and cosine functions; a (local) 0-times integrated C-semigroup, resp. C-cosine function, is defined to be a (local) C-semigroup, resp. C-cosine function. The integral generator of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$, is defined by

$$\Big\{(x,y) \in E \times E : S_K(t)x - \Theta(t)Cx = \int_0^t S_K(s)yds,\ t \in [0,\tau)\Big\},\ \text{resp.},$$

$$\Big\{(x,y)\in E\times E : C_K(t)x-\Theta(t)Cx=\int\limits_0^t (t-s)C_K(s)yds,\ t\in[0,\tau)\Big\},$$

and it is a closed linear operator which is an extension of any subgenerator of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$. Suppose that A is a subgenerator of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$. By [18, Proposition 1.1], we know that the integral generator $\hat{A}$ of $(S_K(t))_{t\in[0,\tau)}$, resp. $(C_K(t))_{t\in[0,\tau)}$, satisfies $\hat{A}=C^{-1}\hat{A}C=C^{-1}AC$.

Lemma 1.3. ([17]) *Let A be a closed operator and let $0<\tau\leq\infty$. Then the following assertions are equivalent:*

(i) *A is a subgenerator of a K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$ in E.*

(ii) *The operator $\mathcal{A}\equiv\begin{pmatrix}0 & I\\ A & 0\end{pmatrix}$ is a subgenerator of a Θ-convoluted $\mathcal{C}$-semigroup $(S_\Theta(t))_{t\in[0,\tau)}$ in $E\times E$, where $\mathcal{C}\equiv\begin{pmatrix}C & 0\\ 0 & C\end{pmatrix}$.*

In this case:

$$S_\Theta(t)=\begin{pmatrix}\int\limits_0^t C_K(s)ds & \int\limits_0^t (t-s)C_K(s)ds\\ C_K(t)-\Theta(t)C & \int\limits_0^t C_K(s)ds\end{pmatrix},\ 0\leq t<\tau,$$

and the integral generators of $(C_K(t))_{t\in[0,\tau)}$ and $(S_\Theta(t))_{t\in[0,\tau)}$, denoted respectively by B and $\mathcal{B}$, satisfy $\mathcal{B}=\begin{pmatrix}0 & I\\ B & 0\end{pmatrix}$.

Definition 1.4. Let $0<\alpha\leq\frac{\pi}{2}$ and let $(S_K(t))_{t\geq 0}$ be a K-convoluted C-semigroup. Then we say that $(S_K(t))_{t\geq 0}$ is an analytic K-convoluted C-semigroup of angle α, if there exists an analytic function $\mathbf{S}_K:\Sigma_\alpha\to L(E)$ which satisfies

(i) $\mathbf{S}_K(t)=S_K(t),\ t>0$ and

(ii) $\lim\limits_{z\to 0,\ z\in\Sigma_\gamma}\mathbf{S}_K(z)x=0$ for all $\gamma\in(0,\alpha)$ and $x\in E$.

It is said that $(S_K(t))_{t\geq 0}$ is an exponentially bounded, analytic K-convoluted C-semigroup of angle α, if for every $\gamma\in(0,\alpha)$, there exist $M_\gamma\geq 0$ and $\omega_\gamma\geq 0$ such that $||\mathbf{S}_K(z)||\leq M_\gamma e^{\omega_\gamma Rez},\ z\in\Sigma_\gamma$.

Since there is no risk for confusion, we shall also write S_K for $\mathbf{S}_K$.

2. *Perturbations*

The following rescaling result for subgenerators of (local) convoluted C-semigroups extends [16, Proposition 3.2].

Theorem 2.1. *Suppose $z \in \mathbf{C}$, K and F satisfy* (P1), *there exists $a \geq 0$ such that*

$$\frac{\tilde{K}(\lambda) - \tilde{K}(\lambda + z)}{\tilde{K}(\lambda + z)} = \int_0^\infty e^{-\lambda t} F(t)dt, \ Re\lambda > a, \ \tilde{K}(\lambda + z) \neq 0, \tag{1}$$

and A is a subgenerator (the integral generator) of a (local) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$. Then $A - z$ is a subgenerator (the integral generator) of a (local) K-convoluted C-semigroup $(S_{K,z}(t))_{t\in[0,\tau)}$, where

$$S_{K,z}(t)x := e^{-tz} S_K(t)x + \int_0^t F(t-s)e^{-zs} S_K(s)xds, \ x \in E, \ t \in [0, \tau). \tag{2}$$

Furthermore, in the case $\tau = \infty$, $(S_{K,z}(t))_{t\geq 0}$ is exponentially bounded provided that F and $(S_K(t))_{t\geq 0}$ are exponentially bounded.

P r o o f. It is clear that $(S_{K,z}(t))_{t\in[0,\tau)}$ is a strongly continuous operator family that commutes with C and $A - z$. Let $x \in E$ be fixed. Then we obtain

$$(A-z)\int_0^t S_{K,z}(s)xds = (A-z)\int_0^t [e^{-zs}S_K(s)x + \int_0^s F(s-r)e^{-zr}S_K(r)xdr]ds$$

$$= (A-z)[e^{-zt}\int_0^t S_K(s)xds + z\int_0^t e^{-sz}\int_0^s S_K(r)xdrds]$$

$$+(A-z)\int_0^t\int_0^s F(s-r)e^{-zr}S_K(r)xdrds$$

$$= e^{-zt}[S_K(t)x - \Theta(t)Cx] - ze^{-zt}\int_0^t S_K(s)xds + z\int_0^t e^{-sz}[S_K(s)x - \Theta(s)Cx]ds$$

$$-z^2\int_0^t e^{-sz}\int_0^s S_K(r)xdrds + (A-z)\int_0^t F(t-s)\int_0^s e^{-zr}S_K(r)xdrds$$

$$= e^{-zt}[S_K(t)x - \Theta(t)Cx] - ze^{-zt}\int_0^t S_K(s)xds$$

$$+z\int_0^t e^{-sz}[S_K(s)x-\Theta(s)Cx]ds - z^2\int_0^t e^{-sz}\int_0^s S_K(r)xdrds$$
$$+\int_0^t F(t-s)(A-z)[e^{-zs}\int_0^s S_K(r)xdr + z\int_0^s e^{-zr}\int_0^r S_K(v)xdvdr]ds$$
$$= S_{K,z}(t)x - f_1(t) - f_2(t)Cx,\ x\in E,\ \text{where:}$$

$$f_1(t) = ze^{-zt}\int_0^t S_K(s)xds - z\int_0^t e^{-sz}S_K(s)xds + z^2\int_0^t e^{-sz}\int_0^s S_K(r)xdrds$$
$$+z\int_0^t e^{-zs}F(t-s)\int_0^s S_K(r)xdrds - z\int_0^t F(t-s)\int_0^s e^{-zr}[S_K(r)x-\Theta(r)Cx]drds$$
$$+z^2\int_0^t F(t-s)\int_0^s e^{-zr}\int_0^r S_K(v)xdvdrds,\ t\in[0,\tau) \text{ and}$$

$$f_2(t) = \Theta(t)e^{-zt} + z\int_0^t e^{-zs}\Theta(s)ds$$
$$+\int_0^t F(t-s)e^{-zs}\Theta(s)ds - \int_0^t F(t-s)\int_0^s e^{-zr}\Theta(r)drds,\ t\in[0,\tau).$$

Fix a number $t\in(0,\tau)$ and define afterwards a function $\widetilde{S}_K : [0,\infty)\to L(E)$ by setting:

$$\widetilde{S}_K(s) =: \begin{cases} S_K(s),\ s\in[0,t], \\ S_K(t),\ s>t. \end{cases}$$

Clearly, $(\widetilde{S}_K(t))_{t\geq 0}$ is a strongly continuous operator family and there exist $M>0$ and $\omega\in\mathbf{R}$ such that $||\widetilde{S}_K(t)||\leq Me^{\omega t},\ t\geq 0$. Define $\tilde{f}_1 : [0,\infty)\to L(E)$ by replacing S_K in the representation formula for f_1 with $\widetilde{S}_K$. Then $\tilde{f}_1$ extends continuously the function f_1 to the whole non-negative real axis, and moreover, $\tilde{f}_1$ is Laplace transformable. Using the elementary operational properties of Laplace transform and (1), one obtains $\mathcal{L}(f_1(t))(\lambda) = \mathcal{L}(f_2(t))(\lambda) = 0$ for all sufficiently large real numbers λ. An application of the uniqueness theorem for the Laplace transform gives that $A-z$ is a subgenerator of a (local) K-convoluted C-semigroup $(S_{K,z}(t))_{t\in[0,\tau)}$. Suppose now that A is the integral generator of $(S_K(t))_{t\geq 0}$. Then one has $C^{-1}AC = A$ and this implies that $C^{-1}(A-z)C = A-z$ is the integral generator of $(S_{K,z}(t))_{t\in[0,\tau)}$. Finally, the exponential boundedness of $(S_{K,z}(t))_{t\geq 0}$ simply follows from (2) and the exponential boundedness of F and $(S_K(t))_{t\geq 0}$.

Suppose $K = \mathcal{L}^{-1}(\frac{p_m(\lambda)}{p_k(\lambda)})$, where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform, p_k and p_m are polynomials of degree k and m, respectively, and $k > m$. Then the condition (1) holds for a suitable exponentially bounded function F. Suppose now $\alpha > 0$ and $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$. Then there exists a sufficiently large positive real number a such that $\frac{\tilde{K}(\lambda)-\tilde{K}(\lambda+z)}{\tilde{K}(\lambda+z)} = (1+\frac{z}{\lambda})^{\alpha} - 1 = \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{z^n}{\lambda^n} = \mathcal{L}(\sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{z^n t^{n-1}}{(n-1)!})(\lambda),\ \lambda > a$, where $1^{\alpha} = 1$. Since $\sup_{n\in\mathbf{N}} |\binom{\alpha}{n}| =: L_0 < \infty$, we obtain $|\sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{z^n t^{n-1}}{(n-1)!}| \leq L_0 |z| e^{|z|t},\ t \geq 0$. With Theorem 2.1 and this observation in view, one obtains the following extension of [20, Proposition 2.4(b)] and [28, Proposition 3.3]; for the global case, see [21].

Corollary 2.2. *Suppose $z \in \mathbf{C}$, $\alpha > 0$ and A is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) α-times integrated C-semigroup $(S_{\alpha}(t))_{t\in[0,\tau)}$. Then $A - z$ is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) α-times integrated C-semigroup $(S_{\alpha,z}(t))_{t\in[0,\tau)}$, which is given by:*

$$S_{\alpha,z}(t)x = e^{-zt}S_{\alpha}(t)x + \int_0^t \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{z^n t^{n-1}}{(n-1)!} e^{-zs} S_{\alpha}(s)x\,ds,\ t \in [0,\tau),\ x \in E.$$

The following perturbation theorem generalizes [16, Theorem 4.1].

Theorem 2.3. *Suppose $B \in L(E)$, K is a kernel and satisfies* (P1), *A is a subgenerator (the integral generator) of a (local) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$, $BA \subseteq AB$, $BC = CB$ and there exists $a > 0$ such that the following conditions hold:*

(i) *For every $n \in \mathbf{N}$, there is a function K_n satisfying* (P1) *and*

$$\widetilde{K_n}(\lambda) = \tilde{K}(\lambda)\frac{d^n}{d\lambda^n}\Big(\frac{1}{\tilde{K}}\Big)(\lambda),\ \lambda > a,\ \tilde{K}(\lambda) \neq 0.$$

Put $\Theta_n(t) := \int_0^t |K_n(s)|ds,\ t \geq 0,\ n \in \mathbf{N}$.

(ii) $\sum_{n=1}^{\infty} \Theta_n(t) < \infty,\ t \geq 0$.

Then $A+B$ is a subgenerator (the integral generator) of a (local) K-convoluted C-semigroup $(S_K^B(t))_{t\in[0,\tau)}$, which satisfies for every $x \in E$ and $t \in [0,\tau)$:

$$S_K^B(t)x = e^{tB}S_K(t)x+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)s^{i-n}S_K(s)xds. \quad (3)$$

Furthermore, the following holds:

(a) $||S_K^B(t) - e^{tB}S_K(t)|| \leq e^{||B||}\max\limits_{s\in[0,t]}||S_K(s)||\sum\limits_{n=1}^{\infty}\Theta_n(t)e^{t||B||},\ t \in [0,\tau).$

(b) *Suppose $\tau = \infty$, $(S_K(t))_{t\in[0,\tau)}$ is exponentially bounded and there exist constants $M > 0$ and $\omega \geq 0$ such that*

$$\sum_{n=1}^{\infty}\Theta_n(t) \leq Me^{\omega t},\ t \geq 0. \quad (4)$$

Then $(S_K^B(t))_{t\in[0,\tau)}$ is also exponentially bounded.

P r o o f. First of all, notice that the commutation of B with C and A implies that the function u_1, resp. u_2, given by $u_1(t) := \int_0^t S_K(s)Bxds,\ t \in [0,\tau)$, resp. $u_2(t) := \int_0^t BS_K(s)xds,\ t \in [0,\tau)$, is a solution of the initial value problem

$$\begin{cases} u \in C([0,\tau) : [D(A)]) \cap C^1([0,\tau) : E), \\ u'(t) = Au(t) + \Theta(t)CBx,\ t \in [0,\tau), \\ u(0) = 0. \end{cases}$$

Since K is a kernel, we have the uniqueness of solutions of the preceding problem ([14], [17]) and this implies $BS_K(t)x = S_K(t)Bx,\ t \in [0,\tau),\ x \in E$. Then we obtain:

$$\begin{aligned} ||S_K^B(t) \quad -e^{tB}S_K(t)|| &\leq \max\nolimits_{s\in[0,t]}||S_K(s)||\sum_{n=1}^{\infty}\Theta_n(t)\sum_{i=1}^{\infty}\sum_{n=1}^{i}\tfrac{||B||^i}{i!}\tbinom{i}{n}t^{i-n} \\ &= \max\nolimits_{s\in[0,t]}||S_K(s)||\sum_{n=1}^{\infty}\Theta_n(t)\sum_{i\geq 1}\tfrac{||B||^i}{i!}t^i\sum_{n=1}^{i}\tbinom{i}{n}t^{-n} \\ &\leq \max\nolimits_{s\in[0,t]}||S_K(s)||\sum_{n=1}^{\infty}\Theta_n(t)\tfrac{||B||^i}{i!}t^i\tfrac{(t+1)^i}{t^i} \\ &= e^{||B||}\max\nolimits_{s\in[0,t]}||S_K(s)||\sum_{n=1}^{\infty}\Theta_n(t)e^{t||B||},\ t \in (0,\tau). \end{aligned}$$

Hence, the assertion (a) holds. The previous computation also shows that $(S_K^B(t))_{t\in[0,\tau)}$ is a strongly continuous operator family that commutes with $A+B$ and C. Let $x \in E$ be fixed. Then the dominated convergence theorem, the closedness of A and integration by parts, as well as the argumentation used in the estimation of term $||S_K^B(t) - e^{tB}S_K(t)||$, imply:

$$
\begin{aligned}
(A+B)\int_0^t S_K^B(s)x\,ds &= (A+B)\int_0^t e^{sB}S_K(s)x\,ds \\
&+ \sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(A+B)\int_0^t\int_0^s K_n(s-r)r^{i-n}S_K(r)x\,dr\,ds \\
&= e^{tB}[S_K(t)x - \Theta(t)Cx] + B\int_0^t e^{sB}\Theta(s)Cx\,ds \\
&+ \sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(A+B)\int_0^t K_n(t-s)\int_0^s r^{i-n}S_K(r)x\,dr\,ds \\
&= e^{tB}\Big[S_K(t)x - \Theta(t)Cx\Big] + B\int_0^t e^{sB}\Theta(s)Cx\,ds \\
&+ \sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(A+B)\times \\
&\times\int_0^t K_n(t-s)\Big[s^{i-n}\int_0^s S_K(r)x\,dr - (i-n)\int_0^s r^{i-n-1}\int_0^r S_K(v)x\,dv\,dr\Big]ds \\
&= e^{tB}[S_K(t)x - \Theta(t)Cx] + B\int_0^t e^{sB}\Theta(s)Cx\,ds \\
&+ \sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^{i+1}}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)\times \\
&\times\Big[s^{i-n}\int_0^s S_K(r)x\,dr - (i-n)\int_0^s r^{i-n-1}\int_0^r S_K(v)x\,dv\,dr\Big]ds \\
&+ \sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)s^{i-n}[S_K(s)x - \Theta(s)Cx]ds \\
&+ \sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(n-i)\int_0^t K_n(t-s)\int_0^s r^{i-n-1}[S_K(r)x - \Theta(r)Cx\,dr]ds \\
&= S_K^B(t)x - f_1(t) - f_2(t)Cx,\ \ t\in[0,\tau),\ \text{ where:}
\end{aligned}
$$

$$
\begin{aligned}
f_1(t) &= \sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^{i+1}}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)\times \\
&\times[s^{i-n}\int_0^s S_K(r)x\,dr - (i-n)\int_0^s r^{i-n-1}\int_0^r S_K(v)x\,dv\,dr]ds
\end{aligned}
$$

$$+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(n-i)\int_0^t K_n(t-s)\int_0^s r^{i-n-1}S_K(r)x\,dr\,ds,\ t\in[0,\tau)$$

and

$$f_2(t)=e^{tB}\Theta(t)-B\int_0^t e^{sB}\Theta(s)\,ds+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)s^{i-n}\Theta(s)\,ds$$
$$+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(n-i)\int_0^t K_n(t-s)\int_0^s r^{i-n-1}\Theta(r)\,dr\,ds,\ t\in[0,\tau).$$

Then the partial integration implies:

$$f_1(t)=\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^{i+1}}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)\int_0^s r^{i-n}\int_0^r S_K(r)x\,dr\,ds$$
$$+\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{B^i}{i!}(-1)^n\binom{i}{n}(n-i)\int_0^t K_n(t-s)\int_0^s r^{i-n-1}\int_0^r S_K(r)x\,dr\,ds,\ t\in[0,\tau).$$

The coefficient of $B^i,\ i\geq 2$ in the expression of $f_1(t)$ equals

$$\sum_{n=1}^{i-1}(-1)^n\Big(\frac{n-i}{i!}\binom{i}{n}+\frac{1}{(i-1)!}\binom{i-1}{n}\Big)$$

$$\cdot\int_0^t K_n(t-s)\int_0^s r^{i-n-1}\int_0^r S_K(r)x\,dr\,ds=0,$$

because $\frac{n-i}{i!}\binom{i}{n}+\frac{1}{(i-1)!}\binom{i-1}{n}=0$. Thereby, $f_1(t)=0,\ t\in[0,\tau)$. On the other hand, the usual series arguments imply that the coefficient of B^i in the expression of $f_2(t)$ equals $\Theta(t),\ t\geq 0$ if $i=0$, and

$$f_{2,i}(t):=\frac{t^i}{i!}\Theta(t)-\int_0^t\frac{s^{i-1}}{(i-1)!}\Theta(s)\,ds$$
$$+\sum_{n=1}^{i}\frac{1}{i!}(-1)^n\binom{i}{n}\int_0^t K_n(t-s)s^{i-n}\Theta(s)\,ds$$
$$+\sum_{n=1}^{i}\frac{1}{i!}(-1)^n\binom{i}{n}(n-i)\int_0^t K_n(t-s)\int_0^s r^{i-n-1}\Theta(r)\,dr\,ds,\ t\geq 0,$$

if $i\geq 1$. Proceeding as before, one obtains, as a consequence of the condition (iii), that the function $t\mapsto f_{2,i}(t),\ t\geq 0$ satisfies (P1) and that there exists

$a'' > 0$ such that

$$\begin{aligned}\mathcal{L}(f_{2,i}(t))(\lambda) &= \tfrac{1}{i!}(-1)^i\Big(\tfrac{\tilde{K}(\cdot)}{\cdot}\Big)^{(i)}(\lambda) - \tfrac{1}{\lambda}\tfrac{1}{(i-1)!}(-1)^{i-1}\Big(\tfrac{\tilde{K}(\cdot)}{\cdot}\Big)^{(i-1)}(\lambda)\\ &\quad + \sum_{n=1}^{i} \tfrac{1}{i!}(-1)^i\tbinom{i}{n}\tilde{K}(\lambda)\Big(\tfrac{1}{\tilde{K}(\cdot)}\Big)^{(n)}(\lambda)\tfrac{1}{\lambda}\tilde{K}^{(i-n)}(\lambda)\\ &= \tfrac{1}{i!}(-1)^i\Big(\tfrac{\tilde{K}(\cdot)}{\cdot}\Big)^{(i)}(\lambda) - \tfrac{1}{\lambda}\tfrac{1}{(i-1)!}(-1)^{i-1}\Big(\tfrac{\tilde{K}(\cdot)}{\cdot}\Big)^{(i-1)}(\lambda)\\ &\quad + \tfrac{\tilde{K}(\lambda)}{\lambda}\tfrac{(-1)^i}{i!}\Big(-\tfrac{1}{\tilde{K}(\lambda)}\Big)\tilde{K}^{(i)}(\lambda)\\ &= \tfrac{1}{i!}(-1)^i\Big(\tfrac{\tilde{K}(\cdot)}{\cdot}\Big)^{(i)}(\lambda) - \tfrac{1}{\lambda}\tfrac{1}{(i-1)!}(-1)^{i-1}\Big(\tfrac{\tilde{K}(\cdot)}{\cdot}\Big)^{(i-1)}(\lambda)\\ &\quad + \tfrac{(-1)^{i+1}}{i!}\tfrac{\tilde{K}^{(i)}(\lambda)}{\lambda} = 0,\end{aligned}$$

for all $\lambda > a''$ with $\tilde{K}(\lambda) \neq 0$. This enables one to deduce that $f_2(t) = \Theta(t),\ t \in [0,\tau)$ and that $(S_K^B(t))_{t\in[0,\tau)}$ is a (local) K-convoluted C-semigroup with a subgenerator $A + B$. The proof of (b) follows from a simple computation; furthermore, the supposition that A is the integral generator of $(S_K(t))_{t\in[0,\tau)}$ implies that $C^{-1}AC = A$ and that $C^{-1}(A + B)C = A + B$ is the integral generator of $(S_K^B(t))_{t\in[0,\tau)}$. This completes the proof of theorem.

Remark 2.4.

(i) The assumption (i) of Theorem 2.3 is satisfied for the function $K = \mathcal{L}^{-1}(\frac{a}{p_k(\lambda)})$, where p_k is a polynomial of degree $k \in \mathbf{N}$ and $a \in \mathbf{C} \setminus \{0\}$. Then $n_0 = k$ and $K_n \equiv 0,\ n \geq k+1$. In this case, we have the existence of positive real numbers M and ω such that (4) holds.

(ii) ([16]) Let $n > 1$ and let P be an analytic function in the right half plane $\{\lambda \in \mathbf{C} :\ \mathrm{Re}\lambda > \lambda_0\}$ for some $\lambda_0 \geq 1$. Suppose that $P(\lambda) \neq 0,\ \mathrm{Re}\lambda > \lambda_0$, and that there exist $C > 0$ and $r \in (0, 1]$ with:

$$|P(\lambda)| \geq C|\lambda|^n,\ \mathrm{Re}\lambda > \lambda_0,$$

$$|\frac{d^i}{d\lambda^i}P(\lambda)| \leq C|\lambda|^{-ir}|P(\lambda)|,\ \mathrm{Re}\lambda > \lambda_0,\ i \in \mathbf{N},$$

$$\frac{P^{(j)}}{P} \in LT(\mathbf{C}),\ j \leq 1/r,\ j \in \mathbf{N},$$

where $LT(\mathbf{C})$ denotes the set of all Laplace transforms of exponentially bounded functions. Then the condition (i) of Theorem 2.3 holds for the function $K = \mathcal{L}^{-1}(1/P)$ and there exist $M > 0$ and $\omega \geq 0$ such that (4) holds.

(iii) The conditions (ii) and (iii) quoted in the formulation of Theorem 2.3 can be replaced with:

(ii)' there exist $M_1 \geq 1$ and $\omega_1 \geq 0$ such that

$$\sum_{i=1}^{\infty}\sum_{n=1}^{i}\frac{||B||^i}{i!}\binom{i}{n}\int_0^t |K_n(t-s)|s^{i-n}\,ds \leq M_1 e^{\omega_1 t},\ t \geq 0$$

and

(iii)' to every $i \in \mathbf{N}$, there exists $a_i > 0$ such that the function $t \mapsto \max_{s\in[0,t]} |\Theta(s)|e^{-a_i t}\sum_{n=1}^{i}\frac{(2t+2)^i}{i!}\Theta_n(t)$, $t \geq 0$ belongs to the space $L^1([0,\infty) : \mathbf{R})$.

Notice only that one can prove that $f_1 \equiv 0$ by direct computation of coefficient of B^i, $i \in \mathbf{N}$ and that the condition (iii)' is necessary in our striving to show that, for every $i \in \mathbf{N}$, the function $t \mapsto f_{2,i}(t)$, $t \geq 0$ satisfies (P1); it is also clear that (iii)' holds provided that Θ is exponentially bounded and that, for every $n \in \mathbf{N}$, Θ_n is exponentially bounded. Let us prove now that (ii)' and (iii)' hold for the function $K = \mathcal{L}^{-1}(e^{-\lambda^\sigma})$, where $\sigma \in (0,1)$. First of all, we know that K is an exponentially bounded, continuous kernel. Let $f(\lambda) = e^{\lambda^\sigma}$, $\lambda \in \mathbf{C}\setminus(-\infty,0]$. Then the mapping $\lambda \mapsto f(\lambda)$, $\lambda \in \mathbf{C}\setminus(-\infty,0]$ is analytic, $f'(\lambda) = \sigma\lambda^{\sigma-1}f(\lambda)$ and

$$f^{(n)}(\lambda) = \sum_{i=0}^{n-1}\binom{n-1}{i}\left(\cdot^{\sigma-1}\right)^{(n-i-1)}(\lambda)f^{(i)}(\lambda),\ \lambda \in \mathbf{C}\setminus(-\infty,0]. \quad (5)$$

Using (5), one concludes inductively that, for every $n \in \mathbf{N}$, there exist real numbers $p_{i,n}(\sigma)$, $1 \leq i \leq n$ such that, for every $t \geq 0$:

$$\widetilde{K_n}(\lambda) = \sum_{i=1}^{n} p_{i,n}(\sigma)\lambda^{i\sigma-n},\ Re\lambda > 0 \text{ and } \Theta_n(t) \leq \sum_{i=1}^{n}\frac{|p_{i,n}(\sigma)t^{n-i\sigma}|}{\Gamma(n+1-i\sigma)}.$$

Put $p_{0,n}(\sigma) := 0$, $n \in \mathbf{N}$. By the foregoing, we have

$$\left(e^{\cdot^\sigma}\right)^{(n)}(\lambda) = e^{\lambda^\sigma}\sum_{i=1}^{n} p_{i,n}(\sigma)\lambda^{i\sigma-n}$$

and

$$\left(e^{\cdot^\sigma}\right)^{(n+1)}(\lambda) = e^{\lambda^\sigma} \sum_{i=1}^{n+1} \Big(p_{i,n}(\sigma)(i\sigma - n) + \sigma p_{i-1,n}(\sigma)\Big)\lambda^{i\sigma-(n+1)},$$

for all $n \in \mathbf{N}$ and $\lambda \in \mathbf{C}$ with $Re\lambda > 0$. Hence, $p_{1,n}(\sigma) = \sigma(\sigma-1)\cdots(\sigma-(n-1))$, $n \in \mathbf{N} \setminus \{1\}$, $p_{n,n}(\sigma) = \sigma^n$, $n \in \mathbf{N}$ and

$$p_{i,n+1}(\sigma) = p_{i,n}(\sigma)(i\sigma - n) + \sigma p_{i-1,n}(\sigma),\ n \in \mathbf{N},\ 2 \le i \le n. \tag{6}$$

Clearly, $L_\sigma := \sup_{n\in\mathbf{N}_0} |\binom{\sigma}{n}| < \infty$. Applying (6) we infer that for every $n \ge 2:$

$$\sum_{i=1}^{n+1} i!|p_{i,n+1}(\sigma)|$$

$$\le \Big|\sigma(\sigma-1)\cdots(\sigma-n)\Big| + \sum_{i=2}^{n}\Big[\sigma i!|p_{i-1,n}(\sigma)| + n\Big(\sigma+1\Big)i!|p_{i,n}(\sigma)|\Big] + (n+1)!$$

$$\le L_\sigma\Big(\sigma+n\Big)n! + n\sigma\sum_{i=1}^{n-1} i!|p_{i,n}(\sigma)| + n\Big(\sigma+1\Big)\sum_{i=2}^{n} i!|p_{i,n}(\sigma)| + (n+1)!.$$

The preceding inequality implies that, for every $\zeta \ge 2 + 4\sigma + 2L_\sigma$, the following holds:

$$\sum_{i=1}^{n} i!|p_{i,n}(\sigma)| \le \zeta^n n! \text{ for all } n \in \mathbf{N}. \tag{7}$$

Denote by ζ_σ the minimum of all numbers satisfying (7). Then a simple computation shows that, for every $x \in E:$

$$\sum_{i=1}^{\infty}\sum_{n=1}^{i} \frac{||B||^i}{i!}\binom{i}{n}\int_0^t ||K_n(t-s)s^{i-n}S_K(s)x||\,ds$$

$$\le \max_{s\in[0,t]} ||S_K(s)x|| \sum_{i=1}^{\infty} \frac{||B||^i\zeta_\sigma^i}{i!} \sum_{n=1}^{i}\sum_{l=1}^{n} \frac{t^{i+1-l\sigma}i!}{\Gamma(i+2-l\sigma)l!},\ t \ge 0. \tag{8}$$

On the other hand, it is easily verified that:

$$\sum_{n=1}^{i}\sum_{l=1}^{n} \frac{i!}{\Gamma(i+2-l\sigma)l!} \le i2^{(2-\sigma)i},\ i \in \mathbf{N}. \tag{9}$$

Combining (8)-(9), it follows that, for every $t \in [0, \min(1, \tau))$,

$$\left\| S_K^B(t) - e^{tB} S_K(t) \right\| \leq t||B||\zeta_\sigma 2^{2-\sigma} e^{||B||\zeta_\sigma 2^{2-\sigma}} \max_{s\in[0,t]} ||S_K(s)||$$

and, in case $\tau > 1$,

$$\left\| S_K^B(t) - e^{tB} S_K(t) \right\| \leq t^2||B||\zeta_\sigma 2^{2-\sigma} e^{||B||\zeta_\sigma 2^{2-\sigma} t} \max_{s\in[0,t]} ||S_K(s)||,\ t \in [1, \tau),$$

proving the condition (ii)'; furthermore, in case that $\tau = \infty$ and that $(S_K(t))_{t\geq 0}$ is exponentially bounded, then $(S_K^B(t))_{t\geq 0}$ is also exponentially bounded. It is clear that (iii)' holds and that the above conclusions remain true in case $K = \mathcal{L}^{-1}(e^{-a\lambda^\sigma})$, where $\sigma \in (0,1)$ and $a > 0$.

(iv) Suppose $\alpha > 0$, $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$, $L_0 := \sup_{n\in\mathbf{N}} |\binom{\alpha}{n}|$ and A is a subgenerator of a (local, global exponentially bounded) α-times integrated C-semigroup $(S_\alpha(t))_{t\in[0,\tau)}$. Then $K_n(t) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} t^{n-1}$, $L_0 < \infty$, $\Theta_n(t) = |\binom{\alpha}{n}| t^n$, $t \geq 0$, $n \in \mathbf{N}$ and this implies that the condition (iii) of Theorem 2.3 does not hold if $\alpha \notin \mathbf{N}$. Nevertheless, the series appearing in (3) still converges, the estimate $||S_K^B(t) - e^{tB} S_K(t)|| \leq L_0 \max_{s\in[0,t]} ||S_K(s)|| e^{2t||B||}$, $t \in [0, \tau)$ follows analogically and the proof of Theorem 2.3 can be repeated verbatim. Having in mind these observations, we obtain the next important generalization of [16, Corollary 4.5] and [35, Theorem 2.3] (cf. also [21, Theorem 3.5]):

Theorem 2.5. *Suppose $\alpha > 0$, A is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) α-times integrated C-semigroup $(S_\alpha(t))_{t\in[0,\tau)}$, $B \in L(E)$, $BA \subseteq AB$ and $BC = CB$. Then $A+B$ is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) α-times integrated C-semigroup $(S_\alpha^B(t))_{t\in[0,\tau)}$, which satisfies, for every $x \in E$ and $t \in [0, \tau)$,*

$$S_\alpha^B(t)x = e^{tB} S_\alpha(t)x + \sum_{i=1}^{\infty} \sum_{n=1}^{i} \frac{B^i}{i!} (-1)^n n \binom{i}{n} \binom{\alpha}{n} \int_0^t (t-s)^{n-1} s^{i-n} S_\alpha(s)x\,ds.$$

Notice ([36]) that the previous formula can be rewritten in the following form:

$$S_{\alpha}^{B}(t)x = e^{tB}S_{\alpha}(t)x+\sum_{i=1}^{\infty}\binom{\alpha}{i}(-B)^{i}\int_{0}^{t}\frac{(t-s)^{i-1}}{(i-1)!}e^{Bs}S(s)xds,\ x\in E,\ t\in[0,\tau). \tag{10}$$

The following perturbation theorem for generators of exponentially bounded, analytic integrated C-semigroups is applicable on a class of (differential) operators analyzed by R. deLaubenfels in [10, Section XXI, Section XXIV].

Theorem 2.6. *Suppose $r > 0$, $\alpha \in (0, \frac{\pi}{2}]$, A is a subgenerator, resp. the integral generator, of an exponentially bounded, analytic r-times integrated C-semigroup $(S_r(t))_{t\geq 0}$ of angle α, $B \in L(E)$, $BA \subseteq AB$ and $BC = CB$. Then $A + B$ is a subgenerator, resp. the integral generator, of an exponentially bounded, analytic r-times integrated C-semigroup $(S_r^B(t))_{t\geq 0}$ of angle α, where*

$$S_{r}^{B}(z)x := e^{zB}S_{r}(z)x+\sum_{i=1}^{\infty}\binom{\alpha}{i}(-B)^{i}\int_{0}^{z}\frac{(z-s)^{i-1}}{(i-1)!}e^{Bs}S_{r}(s)xds,\ x\in E,\ z\in\Sigma_{\alpha}. \tag{11}$$

P r o o f. Clearly, $L_0 = \sup\limits_{n\in\mathbf{N}} |\binom{r}{n}| < \infty$. Notice that, for every $z \in \Sigma_\alpha$, the series appearing in (11) is absolutely convergent and that, for every $\gamma \in (-\alpha, \alpha)$ such that $|\gamma| > \arg(z)$, we have the following:

$$\begin{aligned} ||S_r^B(z) - e^{zB}S_r(z)|| &\leq \sum_{i\geq 1} L_0||B||^i \int_0^{\mathrm{Re}z} \tfrac{|z|^{i-1}}{(i-1)!} e^{||B|||z|} M_\gamma e^{\omega_\gamma \mathrm{Re}z} ds \\ &\leq \mathrm{Re}z M_\gamma L_0 ||B|| e^{(2||B||+\omega_\gamma)\mathrm{Re}z}. \end{aligned} \tag{12}$$

This implies that $(S_r(z))_{z\in\Sigma_\alpha}$ is a strongly continuous operator family satisfying the conditions (i) and (ii) stated in the formulation of Definition 1.4. It remains to be shown that the mapping

$$z \mapsto \sum_{i=1}^{\infty}\binom{\alpha}{i}(-B)^{i}\int_{0}^{z}\frac{(z-s)^{i-1}}{(i-1)!}e^{Bs}S_{r}(s)ds,\ z\in\Sigma_{\alpha} \tag{13}$$

is analytic. By standard arguments, the mapping $f_0(z)=\int_0^z e^{-Bs}S_r(s)ds,\ z\in \Sigma_\alpha$ is analytic and $f_0'(z)=e^{-Bz}S_r(z),\ z\in\Sigma_\alpha$. This simply yields that, for every $i\in\mathbf{N}$, the mapping $f_i(t)=\int_0^z \frac{(z-s)^{i-1}}{(i-1)!}e^{Bs}S_r(s)ds,\ z\in\Sigma_\alpha$ is analytic and that $f_i'(z)=f_{i-1}(z),\ z\in\Sigma_\alpha$. Furthermore, the series in (11) is locally uniformly convergent; this follows from the next obvious estimate:

$$\Big\|\binom{\alpha}{i}(-B)^i\int_0^z \frac{(z-s)^{i-1}}{(i-1)!}e^{Bs}S_r(s)ds\Big\|$$
$$\leq M_\gamma\Big(\sup_{z\in K}|z|||B||\Big)^i\frac{e^{(||B||+\omega)\sup_{z\in K}|z|}}{(i-1)!},$$

where K is an arbitrary compact subset of Σ_α and γ is chosen so that $K\subseteq\Sigma_\gamma$. An application of the Weierstrass theorem completes the proof of theorem.

The following theorem extends [30, Theorem 3.8] and [35, Theorem 2.4, Theorem 2.5, Corollary 2.6] (cf. also [40, Theorem 2.3]). The proof is omitted since it follows by the use of the argumentation given in [35], [29, Section 10] and [40].

Theorem 2.7. *Suppose $n\in\mathbf{N}$, A is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) n-times integrated C-semigroup $(S(t))_{t\in[0,\tau)}$, $B\in L(E)$, $R(B)\subseteq C(D(A^n))$ and $BCx=CBx,\ x\in D(A)$. Then $A+B$ is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) n-times integrated C-semigroup $(S_B(t))_{t\in[0,\tau)}$ which satisfies the following integral equation:*

$$S_B(t)x=S(t)x+\int_0^t \frac{d^n}{dt^n}S(t-s)C^{-1}BS_B(s)xds,\ t\in[0,\tau),\ x\in E.$$

With Theorem 2.7 in view, one can prove the following extension of [29, Theorem 10.1] that is comparable with [35, Theorem 4.6] and [39, Theorem 3.1]; notice only that the assertions related to the study of unbounded perturbations of generators of integrated C-semigroups (cf. [35, Theorem 3.1, Theorem 3.2] and [23, Theorem 3.1]) and cosine functions can be proved similarly. It seems possible to prove the assertions of Theorem 2.7 and Theorem 2.8 in the case of (local) fractionally integrated C-semigroups and cosine functions.

Theorem 2.8. *Suppose $n \in \mathbf{N}$, A is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) n-times integrated C-cosine function $(C(t))_{t\in[0,\tau)}$, $B \in L(E)$, $R(B) \subseteq C(D(A^{\lfloor \frac{n+1}{2} \rfloor}))$ and $BCx = CBx$, $x \in D(A)$. Then $A+B$ is a subgenerator, resp. the integral generator, of a (local, global exponentially bounded) n-times integrated C-cosine function $(C_B(t))_{t\in[0,\tau)}$.*

The following theorem mimics an interesting perturbation result of C. Kaiser and L. Weis ([12]-[13]) which can be additionally refined if the Fourier type of the space E ([1], [12]) is also taken into consideration.

Theorem 2.9. *Suppose K satisfies* (P1), (P2) *and there exists $\beta \in (abs(K), \infty)$ such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ with*

$$\frac{1}{|\tilde{K}(\lambda)|} \leq C_\varepsilon e^{\varepsilon|\lambda|},\ \lambda \in \mathbf{C},\ \mathrm{Re}\lambda > \beta. \tag{14}$$

(i) *Suppose A generates an exponentially bounded K-convoluted semigroup $(S_K(t))_{t\geq 0}$ such that $||S_K(t)|| \leq M_1 e^{\omega t}$, $t \geq 0$ for some $M_1 > 0$ and $\omega \geq 0$. Let B be a linear operator such that $D(A) \subseteq D(B)$ and that there exist $M \in (0,1)$ and $\lambda_0 \in (\max(\beta,\omega), \infty)$ satisfying $||BR(\lambda : A)|| \leq M$, $\lambda \in \mathbf{C}, Re\lambda = \lambda_0$. Then, for every $\alpha > 1$, the operator $A+B$ generates an exponentially bounded, $(K *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)})$-convoluted semigroup.*

(ii) *Suppose A generates an exponentially bounded K-convoluted semigroup $(S_K(t))_{t\geq 0}$ such that $||S_K(t)|| \leq M_1 e^{\omega t}$, $t \geq 0$ for some $M_1 > 0$ and $\omega \geq 0$. Let B be a densely defined linear operator such that there exist $M \in (0,1)$ and $\lambda_0 \in (\max(\beta,\omega), \infty)$ satisfying $||R(\lambda : A)Bx|| \leq M||x||$, $x \in D(B)$, $\lambda \in \mathbf{C}, Re\lambda = \lambda_0$. Then there exists a closed extension D of the operator $A+B$ such that, for every $\alpha > 1$, the operator D generates an exponentially bounded, $(K *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)})$-convoluted semigroup. Furthermore, if A and A^* are densely defined, then D is the part of the operator $(A^* + B^*)^*$ in E.*

(iii) *Suppose A generates an exponentially bounded K-convoluted cosine function $(C_K(t))_{t\geq 0}$ such that $||C_K(t)|| \leq M_1 e^{\omega t}$, $t \geq 0$ for some $M_1 > 0$ and $\omega \geq 0$. Let B be a linear operator such that $D(A) \subseteq D(B)$ and that there exist $M > 0$ and $\lambda_0 \in (\max(\beta,\omega), \infty)$ satisfying $||BR(\lambda^2 : A)|| \leq \frac{M}{|\lambda|}$, $\lambda \in \mathbf{C}, Re\lambda = \lambda_0$. Then, for every $\alpha > 1$, the operator $A+B$ generates an exponentially bounded, $(K *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)})$-convoluted cosine function.*

(iv) *Suppose A generates an exponentially bounded K-convoluted cosine function $(C_K(t))_{t\geq 0}$ such that $||C_K(t)|| \leq M_1 e^{\omega t}$, $t \geq 0$ for some $M_1 > 0$ and $\omega \geq 0$. Let B be a densely defined linear operator such that there exist $M > 0$ and $\lambda_0 \in (\max(\beta, \omega), \infty)$ satisfying $||R(\lambda^2 : A)Bx|| \leq \frac{M}{|\lambda|}||x||$, $x \in D(B)$, $\lambda \in \mathbf{C}, Re\lambda = \lambda_0$. Then there exists a closed extension D of the operator $A+B$ such that, for every $\alpha > 1$, the operator D generates an exponentially bounded, $(K *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)})$-convoluted cosine function. Furthermore, if A and A^* are densely defined, then D is the part of the operator $(A^* + B^*)^*$ in E.*

P r o o f. We will prove only (iii) and (iv). By [17, Theorem 3.1], we have that $\{\lambda^2 : \lambda \in \mathbf{C},\ \mathrm{Re}\lambda > \max(\beta, \omega)\} \subseteq \rho(A)$ and that $||R(\lambda^2 : A)|| \leq \frac{M_1}{|\lambda||\tilde{K}(\lambda)|(\mathrm{Re}\lambda - \omega)}$, $\lambda \in \mathbf{C}$, $\mathrm{Re}\lambda > \max(\beta, \omega)$. Suppose $z \in \mathbf{C}$ and $\mathrm{Re}z > \lambda_0$. Put $\lambda = \lambda_0 + i\mathrm{Im}z$ and notice that

$$\begin{aligned}
||BR(z^2 : A)|| &= \Big\|BR(\lambda^2 : A)\Big(I + (\lambda^2 - z^2)R(z^2 : A)\Big)\Big\| \\
&\leq \Big\|BR(\lambda^2 : A)\Big\|\Big(1 + |\lambda - z||\lambda + z|||R(z^2 : A)||\Big) \\
&\leq \frac{M}{|\lambda|}\Big(1 + |\lambda - z||\lambda + z|\frac{M_1}{|z||\tilde{K}(z)|(\mathrm{Re}z - \omega)}\Big) \\
&\leq \frac{M}{|\lambda|}\Big(1 + |\lambda + z|\frac{M_1}{|z||\tilde{K}(z)|}\Big) \\
&\leq M\Big(\frac{1}{|\lambda|} + (1 + \frac{|z|}{|\lambda|})\frac{M_1}{|z||\tilde{K}(z)|}\Big) \\
&\leq M\Big(\frac{1}{\lambda_0} + \frac{M_1}{|z||\tilde{K}(z)|} + \frac{M_1}{\lambda_0|\tilde{K}(z)|}\Big). \qquad (15)
\end{aligned}$$

Consider now the function $h : \{z \in \mathbf{C} : \mathrm{Re}z \geq 0\} \to L(E)$ defined by $h(z) := zBR((z + \lambda_0)^2 : A)$, $z \in \mathbf{C}$, $\mathrm{Re}z \geq 0$. Then $||h(it)|| \leq M$, $t \in \mathbf{R}$ and, owing to (14) and (15), we have that, for every $\varepsilon > 0$, there exists $\overline{C_\varepsilon} > 0$ such that $||h(z)|| \leq \overline{C_\varepsilon} e^{\varepsilon|z|}$ for all $z \in \mathbf{C}$ with $\mathrm{Re}z \geq 0$. An application of the Phragmén-Lindelöf type theorems (cf. for instance [1, Theorem 3.9.8, p. 179]) gives that $||h(z)|| \leq M$ for all $z \in \mathbf{C}$ with $\mathrm{Re}z \geq 0$. This, in turn, implies that there exists $a > \lambda_0$ such that $||BR(\lambda^2 : A)|| < \frac{1}{2}$, $\lambda^2 \in \rho(A+B)$ and that, for every $\lambda \in \mathbf{C}$ with $\mathrm{Re}\lambda > a$:

$$\Big\|\lambda R(\lambda^2 : A + B)\Big\| = \Big\|\lambda R(\lambda^2 : A)(I - BR(\lambda^2 : A))^{-1}\Big\| \leq \frac{1}{|\tilde{K}(\lambda)|}.$$

The proof of (iii) follows by making use of [17, Theorem 3.1] and [36, Theorem 1.12] while the proof of (iv) is a consequence of [12, Lemma 3.2] and a similar reasoning.

By the proof of Theorem 2.9, we immediately obtain the following corollary.

Corollary 2.10.

(i) *Suppose A generates a cosine function $(C(t))_{t\geq 0}$ satifying $||C(t)|| \leq Me^{\omega t}$, $t \geq 0$ for appropriate $M > 0$ and $\omega \geq 0$. If B is a linear operator such that $D(A) \subseteq D(B)$ and that there exist $M' > 0$ and $\lambda_0 \in (\omega, \infty)$ satisfying $||BR(\lambda^2 : A)|| \leq \frac{M}{|\lambda|}$, $\lambda \in \mathbf{C}, Re\lambda = \lambda_0$, then, for every $\alpha > 1$, the operator $A+B$ generates an exponentially bounded, α-times integrated cosine function.*

(ii) *Suppose A generates a cosine function $(C(t))_{t\geq 0}$ satifying $||C(t)|| \leq Me^{\omega t}$, $t \geq 0$ for appropriate $M > 0$ and $\omega \geq 0$. Let B be a densely defined linear operator such that there exist $M' > 0$ and $\lambda_0 \in (\omega, \infty)$ satisfying $||R(\lambda^2 : A)Bx|| \leq \frac{M}{|\lambda|}||x||$, $x \in D(B)$, $\lambda \in \mathbf{C}, Re\lambda = \lambda_0$. Then there exists a closed extension D of the operator $A+B$ such that, for every $\alpha > 1$, the operator D generates an exponentially bounded, α-times integrated cosine function. Furthermore, if A and A^* are densely defined, then D is the part of the operator $(A^* + B^*)^*$ in E.*

We close the paper with the following illustrative example.

Example 2.11.

(i) ([22]) Let $E := C_0(\mathbf{R}) \oplus C_0(\mathbf{R}) \oplus C_0(\mathbf{R})$, $C(f, g, h) := (f, g, \sin(\cdot)h(\cdot))$, $f,\ g,\ h \in C_0(\mathbf{R})$ and $A(f, g, h) := (f' + g', g', (\chi_{[0,\infty)} - \chi_{(-\infty,0]})h)$, $(f, g, h) \in D(A) = \{(f, g, h) \in E : f' \in C_0(\mathbf{R}),\ g' \in C_0(\mathbf{R}),\ h(0) = 0\}$. Arguing as in [22, Example 8.1, Example 8.2], one gets that A is the integral generator of an exponentially bounded once integrated C-semigroup and that A is not a subgenerator of any local C-semigroup. Suppose now $m_i \in C^1(\mathbf{R})$, $i = 1, 2$, the mappings $t \mapsto |t|m_i(t)$, $t \in \mathbf{R}$ and $t \mapsto |t|m_i'(t)$, $t \in \mathbf{R}$ are bounded for $i = 1, 2$; $C(\mathbf{R}) \ni m_3$ is bounded and satisfies $m_3(0) = 0$. Put $B(f, g, h) := (m_1(\cdot)\int_0^{\cdot} f(s)ds, m_2(\cdot)\int_0^{\cdot} g(s)ds, \sin(\cdot)m_3(\cdot)h(\cdot))$, $f,\ g,\ h \in C_0(\mathbf{R})$. Then one can simply verify $B \in L(E)$, $R(B) \subseteq C(D(A))$ and $BC(f, g, h) = CB(f, g, h)$, $(f, g, h) \in E$. By Theorem 2.7, one obtains that $A + B$ is the integral generator of an exponentially bounded once integrated C-semigroup.

(ii) Let $E := L^1(\mathbf{R})$ and let $D := d/dx$ with maximal distributional domain. Then it is well known (cf. also [12, Corollary 3.4, Example

7.1]) that E has the Fourier type 1, and in particular, that E is not a $B-$convex Banach space. Furthermore, $A := D^2 = d^2/dx^2$ generates a bounded cosine function $(C(t))_{t\geq 0}$ given by

$$\Big(C(t)f\Big)(x) := \frac{1}{2}\Big(f(x+t) + f(x-t)\Big),\ t \geq 0,\ x \in \mathbf{R},\ f \in L^1(\mathbf{R}),$$

and Sobolev imbedding theorem implies $D(A) = W^{1,2}(\mathbf{R}) \subseteq C(\mathbf{R}) \cap L^\infty(\mathbf{R})$. Suppose $g \in L^1(\mathbf{R}) \setminus L^\infty(\mathbf{R})$ and define a linear operator $B : L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}) \to L^1(\mathbf{R})$ by $Bf(x) := f(x)g(x),\ f \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$. In general, B cannot be extended to a bounded linear operator from $L^1(\mathbf{R})$ into $L^1(\mathbf{R})$ and $R(B) \nsubseteq D(A)$. Clearly,

$$\begin{aligned} ||B(2\lambda R(\lambda^2 : A)f)|| &= \int_{-\infty}^{\infty} |g(x)|| \int_0^{\infty} e^{-\lambda t}(f(x+t) + f(x-t))dt|dx \\ &\leq \int_{-\infty}^{\infty} |g(x)| \int_0^{\infty} (|f(x+t)| + |f(x-t)|)dtdx \\ &\leq 2||g||||f||,\ \lambda \in \mathbf{C},\ \mathrm{Re}\lambda > 0,\ f \in L^1(\mathbf{R}), \end{aligned}$$

and this implies that all assumptions quoted in the formulation of Corollary 2.10(i) holds with $\lambda_0 = 1$. Hence, $A + B$ generates an exponentially bounded α-times integrated cosine function for every $\alpha > 1$; let us also point out that it is not clear whether there exists $\beta \in [0, 1)$ such that $A + B$ generates a (local) β-times integrated cosine function although one can simply prove that there exist $a > 0$ and $M > 0$ such that $||\lambda R(\lambda^2 : A + B)|| \leq \frac{M}{\mathrm{Re}\lambda},\ \lambda \in \mathbf{C},\ \mathrm{Re}\lambda > a$.

(iii) Suppose A generates a (local) α-times integrated cosine function for some $\alpha > 0$, $B \in L(E)$ and $BA \subseteq AB$. Then the proof of [15, Theorem 4.3] and the analysis given in [17, Example 7.3] imply that, for every $s \in (1, 2)$, $\pm iA$ generate global $K_{1/s}$-semigroups and that $\pm iA$ generate local $K_{1/2}$-semigroups, where $K_\sigma(t) = \mathcal{L}^{-1}(e^{-\lambda^\sigma})(t),\ t \geq 0,\ \sigma \in (0, 1)$. By Theorem 2.3 and Remark 2.4(iii), we have that $\pm i(A+B)$ generate global $K_{1/s}$-semigroups for every $s \in (1, 2)$ and that $\pm i(A+B)$ generate local $K_{1/2}$-semigroups. Therefore, a large class of differential operators (cf. [2], [11] and [42]) generating integrated cosine functions can be used to provide applications of Theorem 2.3.

(iv) ([19], [16]) Let $s > 1$,

$$E := \left\{ f \in C^\infty[0, 1] \mid \|f\| := \sup_{p\geq 0} \frac{\|f^{(p)}\|_\infty}{p!^s} < \infty \right\},$$

and

$$A := -d/dx, \ \ D(A) := \{f \in E : f' \in E, \ f(0) = 0\}.$$

It is well known that there exist positive real numbers m and M such that $\{\lambda \in \mathbf{C} : \mathrm{Re}\lambda \geq 0\} \subseteq \rho(A)$ and $\|R(\lambda : A)\| \leq Me^{m|\lambda|^{\frac{1}{s}}}$, $\mathrm{Re}\lambda \geq 0$ ([19]). Since $|e^{-\xi\lambda^{\frac{1}{s}}}| \leq e^{-\xi|\lambda|^{\frac{1}{s}}\cos(\frac{\pi}{2s})}$, $\xi > 0$, $\lambda \in \mathbf{C}$, $\mathrm{Re}\lambda > 0$, we have that A generates a global exponentially bounded $K_{a,\frac{1}{s}}$-convoluted semigroup for every $a > \frac{m}{\cos(\frac{\pi}{2s})}$, where $K_{a,\frac{1}{s}}(t) = \mathcal{L}^{-1}(e^{-a\lambda^{\frac{1}{s}}})(t)$, $t \geq 0$. Let $n \in \mathbf{N}$ and let $Bf(x) := \sum_{i=1}^{n} \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} f(s)ds$, $x \in [0,1]$, $f \in E$. Then it is checked at once that $B \in L(E)$ and that $BA \subseteq AB$. Owing to Theorem 2.3 and Remark 2.4(iii), we easily infer that $A+B$ generates a global exponentially bounded $K_{a,\frac{1}{s}}$-convoluted semigroup.

REFERENCES

[1] W. A r e n d t, C. J. K. B a t t y, M. H i e b e r, F. N e u b r a n d e r, *Vector-valued Laplace Transforms and Cauchy Problems*, Birkhäuser Verlag, 2001.

[2] W. A r e n d t, H. K e l l e r m a n n, *Integrated solutions of Volterra integrodifferential equations and applications,* Volterra integrodifferential equations in Banach spaces and applications, Proc. Conf., Trento/Italy 1987, Pitman Res. Notes Math. Ser. **190** (1989), 21–51.

[3] W. A r e n d t, O. E l - M e n n a o u i, V. K e y a n t u o, *Local integrated semigroups: evolution with jumps of regularity,* J. Math. Anal. Appl. **186** (1994), 572–595.

[4] W. A r e n d t, C. J. K. B a t t y, *Rank-1 perturbations of cosine functions and semigroups,* J. Funct. Anal. **238** (2006), 340–352.

[5] R. B e a l s, *On the abstract Cauchy problem,* J. Funct. Anal. **10** (1972), 281–299.

[6] I. C i o r ă n e s c u, *Local convoluted semigroups,* in: Evolution Equations (Baton Rauge, LA, 1992), 107–122, Dekker New York, 1995.

[7] I. C i o r ă n e s c u, G. L u m e r, *Problèmes d'évolution régularisés par un noyan général $K(t)$. Formule de Duhamel, prolongements, théorèmes de génération,* C. R. Acad. Sci. Paris Sér. I Math. **319** (1995), 1273–1278.

[8] I. C i o r ă n e s c u, G. L u m e r, *On $K(t)$-convoluted semigroups,* in: Recent Developments in Evolution Equations (Glasgow, 1994), 86–93. Longman Sci. Tech., Harlow, 1995.

[9] J. C h a z a r a i n, *Problémes de Cauchy abstraites et applications á quelques problémes mixtes,* J. Funct. Anal. **7** (1971), 386-446.

[10] R. d e L a u b e n f e l s, *Existence Families, Functional Calculi and Evolution Equations,* Lecture Notes in Mathematics **1570**, Springer 1994.

[11] M. H i e b e r, *Integrated semigroups and differential operators on L^p spaces,* Math. Ann. **291** (1995), 1-16.

[12] C. K a i s e r, L. W e i s, *Perturbation theorems for α-times integrated semigroups,* Arch. Math. **81** (2003), 215–228.

[13] C. K a i s e r, *Integrated semigroups and linear partial differential equations with delay,* J. Math. Anal. Appl. **292** (2004), 328–339.

[14] M. K o s t i ć, *Convoluted C-cosine functions and convoluted C-semigroups,* Bull. Cl. Sci. Math. Nat. Sci. Math. **28** (2003), 75–92.

[15] M. K o s t ić, P. J. M i a n a, *Relations between distribution cosine functions and almost-distribution cosine functions,* Taiwanese J. Math. **11** (2007), 531–543.

[16] M. K o s t i ć, S. P i l i p o v i ć, *Global convoluted semigroups,* Math. Nachr. **280** (2007), 1727–1743.

[17] M. K o s t ić, S. P i l i p o v i ć, *Convoluted C-cosine functions and semigroups. Relations with ultradistribution and hyperfunction sines,* J. Math. Anal. Appl. **338** (2008), 1224–1242.

[18] M. K o s t ić, *Convoluted C-groups,* Publ. Inst. Math., Nouv. Sér **84(98)** (2008), 73–95.

[19] P. C. K u n s t m a n n, *Stationary dense operators and generation of non-dense distribution semigroups,* J. Operator Theory **37** (1997), 111–120.

[20] M. L i, Q. Z h e n g, *α-times integrated semigroups: local and global,* Studia Math. **154** (2003), 243–252.

[21] Y. - C. L i, S. - Y. S h a w, *Perturbation of non-exponentially-bounded α-times integrated C-semigroups,* J. Math. Soc. Japan **55** (2003), 1115–1136.

[22] Y. - C. L i, S. - Y. S h a w, *N-times integrated C-semigroups and the abstract Cauchy problem,* Taiwanese J. Math. **1** (1997), 75–102.

[23] C. L i z a m a, J. S á n c h e z, *On perturbation of $K-$regularized resolvent families,* Taiwanese J. Math. **7** (2003), 217–227.

[24] C. L i z a m a, V. P o b l e t e, *On multiplicative perturbation of integral resolvent families,* J. Math. Anal. Appl. **327** (2007), 1335–1359.

[25] I. V. M e l n i k o v a, A. I. F i l i n k o v, *Abstract Cauchy Problems: Three Approaches*, Chapman Hall /CRC, 2001.

[26] T. M a t s u m o t o, S. O h a r u, H. R. T h i e m e, *Nonlinear perturbations of a class of integrated semigroups,* Hiroshima Math. J. **26** (1996), 433–473.

[27] I. M i y a d e r a, M. O k u b o, N. T a n a k a, *On integrated semigroups which are not exponentially bounded,* Proc. Japan Acad. **69** (1993), 199-204.

[28] J. M. A. M. v a n N e e r v e n, B. S t r a u b, *On the existence and growth of mild solutions of the abstract Cauchy problem for operators with polynomially bounded resolvent,* Houston J. Math. **24** (1998), 137-171.

[29] S. - Y. S h a w, *Cosine operator functions and Cauchy problems,* Conferenze del Seminario Matematica dell'Universitá di Bari. Dipartimento Interuniversitario Di Matematica Vol. **287**, ARACNE, Roma, 2002, 1–75.

[30] S. - Y. S h a w, C. - C. K u o, *Generation of local C-semigroups and solvability of the abstract Cauchy problems,* Taiwanese J. Math. **9** (2005), 291–311.

[31] S. - Y. S h a w, C. - C. K u o, Y. - C. L i, *Perturbation of local C-semigroups,* Nonlinear Analysis, Nonlinear Anal. **63** (2005), 2569-2574.

[32] N. T a n a k a, *On perturbation theory for exponentially bounded C-semigroups,* Semigroup Forum **41** (2003), 215–236.

[33] N. T a n a k a, *Perturbation theorems of Miyadera type for locally Lipschitz continuous integrated semigroups,* Studia Math. **41** (1990), 215–236.

[34] H. R. T h i e m e, *Positive perturbations of dual and integrated semigroups,* Ado. Math. Sci. Appl. **6** (1996), 445–507.

[35] S. W. W a n g, M. Y. W a n g, Y. S h e n, *Perturbation theorems for local integrated semigroups and their applications,* Studia Math. **170** (2005), 121–146.

[36] T. - J. X i a o, J. L i a n g, *The Cauchy Problem for Higher-Order Abstract Differential Equations*, Springer, 1998.

[37] T. - J. X i a o, J. L i a n g, *Perturbations of existence families for abstract Cauchy problems,* Proc. Amer. Math. Soc. **130** (2002), 2275–2285.

[38] T. - J. X i a o, J. L i a n g, F. L i, *A perturbation theorem of Miyadera type for local C-regularized semigroups,* Taiwanese J. Math. **10** (2006), 153–162.

[39] J. Z h a n g, Q. Z h e n g, *On α-times integrated cosine functions,* Math. Japon. **50** (1999), 401–408.

[40] Q. Z h e n g, *Perturbations and approximations of integrated semigroups,* Acta Math. Sinica **9** (1993), 252–260.

[41] Q. Z h e n g, *Integrated cosine functions,* Internat. J. Math. Sci. **19** (1996), 575–580.

[42] Q. Z h e n g, *Coercive differential operators and fractionally integrated cosine functions,* Taiwanese J. Math. **6** (2002), 59-65.

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