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COMPOSITION PROPERTY AND AUTOMATIC EXTENSION OF LOCAL CONVOLUTED C-COSINE FUNCTIONS

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A b s t r a c t. The central theme of the paper is to clarify composition property of convoluted C-cosine functions. As an application, we obtain an extension type theorem for local convoluted C-cosine functions.

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1. *Introduction and preliminaries*

Generalizing the notion of fractionally integrated C-semigroups ([5], [8]-[9], [13]), I. Ciorănescu and G. Lumer [2]-[4] were introduced local convoluted C-semigroups and related them to ultra-distribution semigroups ([7], [9]). The present author was introduced the class of convoluted C-cosine functions in [6] with a view to study ill-posed abstract second order Cauchy problems in a Banach space setting. Convoluted C-cosine functions allow one to consider in a unified treatment the notion of α-times integrated C-semigroups (cf. [1] for $\alpha \in \mathbb{N}$ and $C = I$; [5], [10], [14] for $\alpha \in (0, \infty)$ and $C = I$; [11] for $\alpha \in \mathbb{N}$ and arbitrary C, and finally [13] for the general case). We refer the reader to [1], [5], [7], [9] and [13] for examples of differential operators generating convoluted C-cosine functions.

In this paper, we establish the composition property of (local) convoluted C-cosine functions and extend some results obtained by S. W. Wang and Z. Huang in [11] as well as J. Zhang and Q. Zheng in [14]. Although the composition property enables one to define the class of convoluted C-cosine functions in a somewhat different way (cf. [10, p. 175], [11, Definition 1.2] and [14]), we omit such an approach since the use of it is confined and becomes quite inoperative in some situations. From this point of view, the main objective of paper is to complete the structural theory of convoluted C-cosine functions developed in [6] and [7]. Finally, the composition property is essentially utilized in proving of an extension type theorem for local convoluted C-cosine functions (cf. [3], [8] and [12] for similar results).

Throughout this paper E denotes a non-trivial complex Banach space, $L(E)$ denotes the space of bounded linear operators from E into E and A denotes a closed linear operator acting on E. From now on, we assume $L(E) \ni C$ is an injective operator, $\tau \in (0, \infty]$, $K(\cdot)$ is a complex-valued locally integrable function in $[0, \tau)$ and $K(\cdot)$ is not identical to zero. Put $\Theta(t) := \int_0^t K(s)ds$ and $\Theta^{-1}(t) := \int_0^t \Theta(s)ds,\ t \in [0, \tau)$; then $\Theta(\cdot)$ is an absolutely continuous function in $[0, \tau)$ and $\Theta'(t) = K(t)$, for a.e. $t \in [0, \tau)$. Let us remind that a function $K \in L^1_{loc}([0, \tau))$ is called a kernel if for every $\phi \in C([0, \tau))$, the assumption $\int_0^t K(t-s)\phi(s)ds = 0,\ t \in [0, \tau)$, implies $\phi \equiv 0$; due to the famous Titchmarsh's theorem, the condition $0 \in \operatorname{supp} K$ implies that K is a kernel. In what follows, we employ the convolution like mapping $*_0$ which is given by $f *_0 g(t) := \int_0^t f(t-s)g(s)ds$.

We recall the definitions of convoluted C-cosine functions and convoluted C-semigroups.

Definition 1.1. ([6]-[7]) *Let A be a closed operator, $K \in L^1_{loc}([0, \tau))$ and $0 < \tau \leq \infty$. If there exists a strongly continuous operator family $(C_K(t))_{t \in [0,\tau)}$ ($C_K(t) \in L(E),\ t \in [0, \tau)$) such that:*

(i) $C_K(t)A \subset AC_K(t),\ t \in [0, \tau)$,

(ii) $C_K(t)C = CC_K(t),\ t \in [0,\tau)$ *and*

(iii) for all $x \in E$ *and* $t \in [0,\tau)$*:* $\int_0^t (t-s)C_K(s)x\,ds \in D(A)$ *and*

$$A\int_0^t (t-s)C_K(s)x\,ds = C_K(t)x - \Theta(t)Cx, \tag{1}$$

then it is said that A *is a subgenerator of a (local)* K*-convoluted* C*-cosine function* $(C_K(t))_{t\in[0,\tau)}$.

Definition 1.2. ([6]-[7]) *Let* A *be a closed operator,* $K \in L^1_{loc}([0,\tau))$ *and* $0 < \tau \le \infty$. *If there exists a strongly continuous operator family* $(S_K(t))_{t\in[0,\tau)}$ *such that:*

(i) $S_K(t)A \subset AS_K(t),\ t \in [0,\tau)$,

(ii) $S_K(t)C = CS_K(t),\ t \in [0,\tau)$ *and*

(iii) for all $x \in E$ *and* $t \in [0,\tau)$*:* $\int_0^t S_K(s)x\,ds \in D(A)$ *and*

$$A\int_0^t S_K(s)x\,ds = S_K(t)x - \Theta(t)Cx, \tag{2}$$

then it is said that A *is a subgenerator of a (local)* K*-convoluted* C*-semigroup* $(S_K(t))_{t\in[0,\tau)}$.

Plugging $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ in Definition 1.1 and Definition 1.2, where $\alpha > 0$ and $\Gamma(\cdot)$ denotes the Gamma function, we obtain the well-known classes of fractionally integrated C-cosine functions and semigroups. The integral generator of $(C_K(t))_{t\in[0,\tau)}$, resp. $(S_K(t))_{t\in[0,\tau)}$, is defined by

$$\{(x,y) \in E^2 : C_K(t)x - \Theta(t)Cx = \int_0^t (t-s)C_K(s)y\,ds,\ t \in [0,\tau)\},\ \text{resp.},$$

$$\{(x,y) \in E^2 : S_K(t)x - \Theta(t)Cx = \int_0^t S_K(s)y\,ds,\ t \in [0,\tau)\}.$$

The integral generator of $(C_K(t))_{t\in[0,\tau)}$, resp. $(S_K(t))_{t\in[0,\tau)}$, is a closed linear operator which is an extension of any subgenerator of $(C_K(t))_{t\in[0,\tau)}$, resp. $(S_K(t))_{t\in[0,\tau)}$. It is well known that the set of all subgenerators of a global, exponentially bounded C-cosine function (cf. [13] for the notion) need not be monomial; furthermore, such a set can be consisted of infinitely many elements ([7]). It is clear that the previous assertions remain true in the case of (local) convoluted C-cosine functions and semigroups.

2. *Composition property of convoluted C-cosine functions*

In order to establish the composition property of a convoluted C-cosine function, we pass to the corresponding theory of convoluted C-semigroups. First of all, we need some auxiliary results.

Lemma 2.1. ([7]) *Let A be a closed operator, $K \in L^1_{loc}([0,\tau))$ and $0 < \tau \le \infty$. Then the following assertions are equivalent:*

(a) A is a subgenerator of a K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$ in E.

(b) The operator $\mathcal{A} \equiv \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ is a subgenerator of a Θ-convoluted $\mathcal{C}$-semigroup $(S_\Theta(t))_{t\in[0,\tau)}$ in E^2, where $\mathcal{C} \equiv \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$. In this case:

$$S_\Theta(t) = \begin{pmatrix} \int_0^t C_K(s)ds & \int_0^t (t-s)C_K(s)ds \\ C_K(t) - \Theta(t)C & \int_0^t C_K(s)ds \end{pmatrix}, \ 0 \le t < \tau.$$

Lemma 2.2. ([3], [6]) *Suppose A is a subgenerator of a (local) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$, $x \in E$ and $0 \le t,\ s,\ t+s < \tau$. Then the following holds:*

$$S_K(t)S_K(s)x = \Big[\int_0^{t+s} - \int_0^{t} - \int_0^{s}\Big]K(t+s-r)S_K(r)Cx\,dr. \tag{3}$$

The following simple equality is left to the reader as an easy exercise.

Lemma 2.3. *Suppose* $0 < \tau \leq \infty$ *and* $K \in C([0,\tau))$. *Then*

$$\left[\int_0^{t+s} - \int_0^t - \int_0^s\right] K(t+s-r)K(r)dr = 0, \quad 0 \leq t,\ s,\ t+s < \tau. \tag{4}$$

Now we are in a position to prove the composition property of convoluted C-cosine functions.

Theorem 2.4. *Let* A *be a subgenerator of a (local)* K*-convoluted* C*-cosine function* $(C_K(t))_{t\in[0,\tau)}$, $x \in E$, $t,\ s \in [0,\tau)$ *and* $t+s < \tau$. *Then we have the following:*

$$2C_K(t)C_K(s)x = \begin{cases} (\int_t^{t+s} - \int_0^s)K(t+s-r)C_K(r)Cxdr \\ + \int_{t-s}^{t} K(r-t+s)C_K(r)Cxdr \\ + \int_0^s K(r+t-s)C_K(r)Cxdr,\ t \geq s; \\ (\int_s^{t+s} - \int_0^t)K(t+s-r)C_K(r)Cxdr \\ + \int_{s-t}^{s} K(r+t-s)C_K(r)Cxdr \\ + \int_0^t K(r-t+s)C_K(r)Cxdr,\ t < s. \end{cases}$$

P r o o f. First of all, we will prove the composition property in the case when $K(\cdot)$ is an absolutely continuous function in $[0,\tau)$. In order to do that, suppose $\tau_0 \in (0,\tau)$ and put $D_{\tau_0} := \{(t,s) \in \mathbb{R}^2 : 0 \leq t,\ s,\ t+s \leq \tau_0,\ s \leq t\}$. Fix an $x \in E$ and define

$$u(t,s) := \int_0^t C_K(r)(C_K(s)x - \Theta(s)Cx)dr,\ (t,s) \in D_{\tau_0} \text{ and} \tag{5}$$

$$F(t,s) := \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] K(t+s-r)C_K(r)Cxdr - \Theta(s)C_K(t)Cx, \tag{6}$$

for any $(t,s) \in D_{\tau_0}$. Designate by $C^1(D_{\tau_0} : E)$ the vector space of all functions from D_{τ_0} into E which are continuously differentiable in $\mathrm{int} D_{\tau_0}$ and whose partial derivatives can be extended continuously throughout D_{τ_0}. Further on, let us consider the problem (P):

$$(P) : \begin{cases} u \in C^1(D_{\tau_0} : E), \\ u_t(t,s) + u_s(t,s) = F(t,s),\ (t,s) \in D_{\tau_0}, \\ u(t,0) = 0. \end{cases} \tag{7}$$

The uniqueness of solutions of the problem (P) can be proved by means of the elementary theory of quasi-linear partial differential equations of first order. On the other hand, an application of Lemma 2.1 shows that $\mathcal{A}$ is a subgenerator of Θ-convoluted $\mathcal{C}$-semigroup $(S_\Theta(t))_{t\in[0,\tau)}$ in E^2. Thanks to Lemma 2.2, one obtains:

$$S_\Theta(t)S_\Theta(s)(0\ ,\ x)^T$$

$$= (\int_0^t C_K(v) \int_0^s (s-r)C_K(r)x\,dr\,dv + \int_0^t (t-v)C_K(v) \int_0^s C_K(r)x\,dr\,dv\ ,$$

$$C_K(t)\int_0^s (s-r)C_K(r)x\,dr - \Theta(t)\int_0^s (s-r)C_K(r)Cx\,dr + \int_0^t\int_0^s C_K(v)C_K(r)x\,dr\,dv)^T$$

$$= [\int_0^{t+s} - \int_0^t - \int_0^s]\Theta(t+s-r)(\int_0^r (r-v)C_K(v)Cx\,dv\ ,\ \int_0^r C_K(v)Cx\,dv)^T dr,$$

for any $(t,s) \in D_{\tau_0}$. Hence,

$$A\left[\int_0^t C_K(v)\int_0^s (s-r)C_K(r)x\,dr\,dv + \int_0^t (t-v)C_K(v)\int_0^s C_K(r)x\,dr\,dv\right]$$

$$= A\left\{\left[\int_0^{t+s} - \int_0^t - \int_0^s\right]\Theta(t+s-r)\int_0^r (r-v)C_K(v)Cx\,dv\,dr\right\},\ (t,s) \in D_{\tau_0}.$$

The last equality and Lemma 2.3 imply

$$\int_0^t C_K(v)(C_K(s)x - \Theta(s)Cx)dv + C_K(t)\int_0^s C_K(r)x\,dr - \Theta(t)\int_0^s C_K(r)Cx\,dr$$

$$= \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] \Theta(t+s-r)C_K(r)Cxdr,\ (t,s) \in D_{\tau_0}. \tag{8}$$

Fix, for the time being, a number $t \in [0, \tau)$. The standard arguments enable one to conclude that

$$\frac{d}{ds}\left[\int_0^{t+s} - \int_0^t - \int_0^s\right] \Theta(t+s-r)C_K(r)Cxdr$$

$$= \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] K(t+s-r)C_K(r)Cxdr - \Theta(t)C_K(s)Cx,\ s \in [0, \tau - t).$$

Differentiate (8) with respect to s in order to see that the function $u(t,s)$, given by (5), is a solution of (P). Further on, put

$$v_1(t,s) := \frac{1}{2}\int_0^s (\int_s^{s+v} - \int_0^v) K(v+s-r)C_K(r)Cxdrdv, \tag{9}$$

$$v_2(t,s) := \frac{1}{2}\int_0^s \int_{s-v}^s K(r-s+v)C_K(r)Cxdrdv, \tag{10}$$

$$v_3(t,s) := \frac{1}{2}\int_0^s \int_0^v K(r+s-v)C_K(r)Cxdrdv, \tag{11}$$

$$v_4(t,s) := \frac{1}{2}\int_s^t (\int_v^{s+v} - \int_0^s) K(v+s-r)C_K(r)Cxdrdv \tag{12}$$

$$v_5(t,s) := \frac{1}{2}\int_s^t \int_{v-s}^v K(r-v+s)C_K(r)Cxdrdv, \tag{13}$$

$$v_6(t,s) := \frac{1}{2}\int_s^t \int_0^s K(r+v-s)C_K(r)Cxdrdv - \Theta(s)\int_0^t C_K(r)Cxdr \tag{14}$$

and

$$v(t,s) := \sum_{i=1}^{6} v_i(t,s),\ (t,s) \in D_{\tau_0}. \tag{15}$$

To prove that $v(t,s)$ is also a solution of the problem (P), notice that the usual limit procedure implies

$$2\frac{\partial v_1}{\partial s}(t,s)=\Big(\int_s^{2s}-\int_0^{s}\Big)K(2s-r)C_K(r)Cx\,dr-\int_0^{s}\int_0^{v}K'(v+s-r)C_K(r)Cx\,dr\,dv$$

$$+\int_0^{s}\int_s^{s+v}K'(v+s-r)C_K(r)Cx\,dr\,dv-\Theta(s)C_K(s)Cx+K(0)\int_0^{s}C_K(s+v)Cx\,dv, \tag{16}$$

$$2\frac{\partial v_2}{\partial s}(t,s)=\int_0^{s}K(r)C_K(r)Cx\,dr-\int_0^{s}\int_{s-v}^{s}K'(r-s+v)C_K(r)Cx\,dr\,dv$$

$$-K(0)\int_0^{s}C_K(s-v)Cx\,dv+\Theta(s)C_K(s)Cx, \tag{17}$$

$$2\frac{\partial v_3}{\partial s}(t,s)=\int_0^{s}K(r)C_K(r)Cx\,dr+\int_0^{s}\int_0^{v}K'(r+s-v)C_K(r)Cx\,dr\,dv, \tag{18}$$

$$2\frac{\partial v_4}{\partial s}(t,s)=\int_s^{t}\Big(\int_v^{s+v}-\int_0^{s}\Big)K'(v+s-r)C_K(r)Cx\,dr\,dv$$

$$+K(0)\int_s^{t}C_K(s+v)Cx\,dv-\int_s^{t}K(v)\,dv\,C_K(s)Cx-\Big(\int_s^{2s}-\int_0^{s}\Big)K(2s-r)C_K(r)Cx\,dr, \tag{19}$$

$$2\frac{\partial v_5}{\partial s}(t,s)=\int_s^{t}\int_{v-s}^{v}K'(r-v+s)C_K(r)Cx\,dr$$

$$+K(0)\int_s^{t}C_K(v-s)Cx\,dv-\int_0^{s}K(r)C_K(r)Cx\,dr, \tag{20}$$

$$2\frac{\partial v_6}{\partial s}(t,s)=-\int_s^{t}\int_0^{s}K'(r+v-s)C_K(r)Cx\,dr\,dv$$

$$+\int_s^{t}K(r)\,dr\,C_K(s)Cx-\int_0^{s}K(r)C_K(r)Cx\,dr-2K(s)\int_0^{t}C_K(r)Cx\,dr, \tag{21}$$

for any $(t,s) \in D_{\tau_0}$. Adding these six summands, one gets

$$2\frac{\partial v}{\partial s}(t,s) = K(0)\Big(\int_0^{t-s} + \int_s^{2s} - \int_0^{s} + \int_{2s}^{t+s}\Big)C_K(r)Cxdr$$

$$-2K(s)\int_0^t C_K(r)Cxdr + I_1 + I_2 + I_3,\ (t,s) \in D_{\tau_0}, \tag{22}$$

where we put

$$I_1 := \Big(\int_0^s\int_s^{s+v} - \int_0^s\int_0^v + \int_s^t\int_v^{s+v} - \int_s^t\int_0^s\Big)K'(v+s-r)C_K(r)Cxdrdv, \tag{23}$$

$$I_2 := \Big(\int_0^s\int_0^v + \int_s^t\int_{v-s}^v\Big)K'(r+s-v)C_K(r)Cxdrdv \text{ and} \tag{24}$$

$$I_3 := \Big(-\int_0^s\int_{s-v}^s - \int_s^t\int_0^s\Big)K'(r-s+v)C_K(r)Cxdrdv,\ (t,s) \in D_{\tau_0}. \tag{25}$$

The elementary calculus shows that

$$I_3 = -\int_0^s\int_{s-r}^s K'(r-s+v)C_K(r)Cxdvdr - \int_0^s\int_s^t K'(r-s+v)C_K(r)Cxdvdr$$

$$= -\int_0^s (K(r)-K(0))C_K(r)Cxdr$$

$$-\int_0^s (K(t-s+r)-K(r))C_K(r)Cxdr. \tag{26}$$

Applying the same arguments, one yields:

$$I_1 = K(s)\int_0^t C_K(r)Cxdr - K(0)\int_s^{t+s} C_K(r)Cxdr$$

$$+\Big(\int_t^{t+s} - \int_0^s\Big)K(t+s-r)C_K(r)Cxdr \text{ and} \tag{27}$$

$$I_2 = -\int_{t-s}^{t} K(r+s-t)C_K(r)Cxdr$$

$$+K(s)\int_0^t C_K(r)Cxdr - K(0)\int_0^{t-s} C_K(r)Cxdr,\ (t,s) \in D_{\tau_0}. \tag{28}$$

Furthermore, $v(t,0) = 0,\ t \in [0, \tau_0]$,

$$2\frac{\partial v}{\partial t}(t,s) = (\int_t^{t+s} - \int_0^s)K(t+s-r)C_K(r)Cxdr + \int_{t-s}^{t} K(r-t+s)C_K(r)Cxdr$$

$$+\int_0^s K(r+t-s)C_K(r)Cxdr - 2\Theta(s)C_K(t)Cx, \tag{29}$$

$v(\cdot,\cdot) \in C^1(D_{\tau_0} : E)$ and a simple computation involving (22)-(29) implies that the function $v(\cdot,\cdot)$ solves (P). Owing to the uniqueness of solutions of the problem (P), one immediately yields:

$$C_K(t)C_K(s)x = v_t(t,s) + \Theta(s)C_K(t)Cx,\ (t,s) \in D_{\tau_0}. \tag{30}$$

The previous equality and arbitrariness of τ_0 enable one to deduce that the composition property holds whenever $K(\cdot)$ is an absolutely continuous function in $[0,\tau)$, $x \in E$, $0 \le t,\ s,\ t+s < \tau$ and $s \le t$. Put $C_\Theta(t)x := \int_0^t C_K(r)xdr,\ t \in [0,\tau),\ x \in E$; then $(C_\Theta(t))_{t\in[0,\tau)}$ is a Θ-convoluted C-cosine function with a subgenerator A and the first part of the proof implies that, for every $x \in E$ and $(t,s) \in [0,\tau) \times [0,\tau)$ with $t+s < \tau$ and $s \le t$:

$$2C_\Theta(t)C_\Theta(s)x = (\int_t^{t+s} - \int_0^s)\Theta(t+s-r)C_\Theta(r)Cxdr + \int_{t-s}^{t} \Theta(r-t+s)C_\Theta(r)Cxdr$$

$$+\int_0^s \Theta(r+t-s)C_\Theta(r)Cxdr. \tag{31}$$

Notice also that the partial integration implies that, for every $x \in E$ and $(t,s) \in [0,\tau) \times [0,\tau)$ with $t+s < \tau$ and $s \le t$:

$$(\int_t^{t+s} - \int_0^s)\Theta(t+s-r)C_\Theta(r)Cxdr$$

$$= \Theta^{-1}(s)C_\Theta(t)Cx + \Theta^{-1}(t)C_\Theta(s)Cx + \int\limits_t^{t+s} \Theta^{-1}(t+s-r)]C_K(r)Cxdr, \quad (32)$$

$$\int\limits_{t-s}^{t} \Theta(r-t+s)C_\Theta(r)Cxdr = \Theta^{-1}(s)C_\Theta(t)Cx$$

$$- \int\limits_{t-s}^{t} \Theta^{-1}(r-t+s)C_K(r)Cxdr \text{ and} \quad (33)$$

$$\int\limits_{0}^{s} \Theta(r+t-s)C_\Theta(r)Cxdr = \Theta^{-1}(t)C_\Theta(s)Cx$$

$$- \int\limits_{0}^{s} \Theta^{-1}(r+t-s)C_K(r)Cxdr. \quad (34)$$

Now one can rewrite (31) by means of (32)-(34):

$$2C_\Theta(t)C_\Theta(s)x = 2\Theta^{-1}(s)C_\Theta(t)Cx + 2\Theta^{-1}(t)C_\Theta(s)Cx$$

$$+(\int\limits_t^{t+s} - \int\limits_0^{s})\Theta^{-1}(t+s-r)C_K(r)Cxdr$$

$$- \int\limits_{t-s}^{t} \Theta^{-1}(r-t+s)C_K(r)Cxdr - \int\limits_{0}^{s} \Theta^{-1}(r+t-s)C_K(r)Cxdr. \quad (35)$$

Taking into account (35), it can be straightforwardly proved that, for every $x \in E$ and $(t,s) \in [0,\tau) \times [0,\tau)$ with $t+s < \tau$ and $s \le t$:

$$2C_K(t)C_\Theta(s)x = 2\frac{d}{dt}C_\Theta(t)C_\Theta(s)x = 2\Theta(t)C_\Theta(s)Cx$$

$$+(\int\limits_t^{t+s} - \int\limits_0^{s})\Theta(t+s-r)C_K(r)Cxdr$$

$$+ \int\limits_{t-s}^{t} \Theta(r-t+s)C_K(r)Cxdr - \int\limits_{0}^{s} \Theta(r+t-s)C_K(r)Cxdr. \quad (36)$$

Differentiation of the last equality with respect to s immediately implies the validity of composition property for all $x \in E$ and $(t,s) \in [0,\tau) \times [0,\tau)$ with $t+s<\tau$ and $s \leq t$; the proof in the case $s>t$ can be obtained analogously. This completes the proof of theorem.

3. *Automatic extension of local convoluted C-cosine functions*

The following extension type theorem for local convoluted C-cosine functions essentially follows from an application of the composition property.

Theorem 3.1. *Let A be a subgenerator of a local K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$, $\tau_0 \in (\frac{\tau}{2},\tau)$ and let $K = K_{1_{|[0,\tau)}}$ for an appropriate complex-valued function $K_1 \in L^1_{loc}([0,2\tau))$. (Put $\Theta_1(t) = \int\limits_0^t K_1(s)ds$ and $\Theta_1^{-1}(t) = \int\limits_0^t \Theta_1(s)ds$, $t \in [0,2\tau)$; since it makes no misunderstanding, we will also write $K(\cdot)$, $\Theta(\cdot)$ and $\Theta^{-1}(\cdot)$ for $K_1(\cdot)$, $\Theta_1(\cdot)$ and $\Theta_1^{-1}(\cdot)$, respectively, and denote by $(K*_0K)(\cdot)$ the restriction of this function to any subinterval of $[0,2\tau)$.) Then A is a subgenerator of a local $(K*_0K)$-convoluted C^2-cosine function $(C_{K*_0K}(t))_{t\in[0,2\tau_0)}$, which is given by:*

$$C_{K*_0K}(t)x = \begin{cases} \int\limits_0^t K(t-s)C_K(s)Cxds, \ t \in [0,\tau_0], \\ 2C_K(\tau_0)C_K(t-\tau_0)x + (\int\limits_0^{t-\tau_0} + \int\limits_0^{\tau_0})K(t-r)C_K(r)Cxdr \\ - \int\limits_{2\tau_0-t}^{\tau_0} K(r+t-2\tau_0)C_K(r)Cxdr \\ - \int\limits_0^{t-\tau_0} K(r+2\tau_0-t)C_K(r)Cxdr, \ t \in (\tau_0,2\tau_0), \ x \in E. \end{cases}$$

Furthermore, the condition $0 \in \operatorname{supp}K$ implies that A is a subgenerator of a local $(K*_0K)$-convoluted C^2-cosine function on $[0,2\tau)$.

P r o o f. It can be simply verified $K*_0K \in L^1_{loc}([0,2\tau))$, $K*_0K$ is not identical to zero and $(C_{K*_0K}(t))_{t\in[0,2\tau_0)}$ is a strongly continuous operator family which commutes with A and C. Proceeding as in the proof of [6, Lemma 4.4], one gets that $((K*_0C_KC)(t))_{t\in[0,\tau)}$ is a local $(K*_0K)$-convoluted C^2-cosine function having A as a subgenerator, and consequently, the condition (iii) quoted in the formulation of Definition 1.1 holds for every

$t \in [0, \tau_0]$ and $x \in E$. It remains to be shown that this condition holds for every $t \in (\tau_0, 2\tau_0)$ and $x \in E$; to this end, denote $\sum = \int_0^t (t-s) C_{K*_0K}(s)x ds$ and notice that

$$\sum = \int_0^{\tau_0} (\tau_0 - s) \int_0^s K(s-r) C_K(r) Cx dr ds + \int_0^{\tau_0} (t-\tau_0) \int_0^s K(s-r) C_K(r) Cx dr ds$$

$$+2C_K(\tau_0) \int_0^{t-\tau_0} (t-\tau_0-s) C_K(s) x ds + I_1 + I_2 - I_3 - I_4, \quad \text{where:} \tag{37}$$

$$I_1 := \int_{\tau_0}^t (t-s) \int_0^{s-\tau_0} K(s-r) C_K(r) Cx dr ds, \tag{38}$$

$$I_2 := \int_{\tau_0}^t (t-s) \int_0^{\tau_0} K(s-r) C_K(r) Cx dr ds, \tag{39}$$

$$I_3 := \int_{\tau_0}^t (t-s) \int_{2\tau_0 - s}^{\tau_0} K(r+s-2\tau_0) C_K(r) Cx dr ds \text{ and} \tag{40}$$

$$I_4 := \int_{\tau_0}^t (t-s) \int_0^{s-\tau_0} K(r+2\tau_0-s) C_K(r) Cx dr ds. \tag{41}$$

We compute I_1 as follows:

$$I_1 = \int_{\tau_0}^t (t-s) \int_0^{s-\tau_0} K(s-r) C_K(r) Cx dr ds = \int_0^{t-\tau_0} \int_{r+\tau_0}^t (t-s) K(s-r) C_K(r) Cx ds dr$$

$$= \int_0^{t-\tau_0} [-\Theta(\tau_0)(t-\tau_0-r) + \int_{r+\tau_0}^t \Theta(s-r) ds] C_K(r) Cx dr$$

$$= -\Theta(\tau_0) \int_0^{t-\tau_0} (t-\tau_0-r) C_K(r) Cx dr + \int_0^{t-\tau_0} [\Theta^{-1}(t-r) - \Theta^{-1}(\tau_0)] C_K(r) Cx dr$$

$$= -\Theta(\tau_0) \int_0^{t-\tau_0} (t-\tau_0-r) C_K(r) Cx dr + \int_0^{t-\tau_0} \Theta(t-r) \int_0^r C_K(v) Cx dv dr$$

$$= -\Theta(\tau_0)\int_0^{t-\tau_0}(t-\tau_0-r)C_K(r)Cxdr + \Theta(\tau_0)\int_0^{t-\tau_0}(t-\tau_0-r)C_K(r)Cxdr$$

$$+\int_0^{t-\tau_0}K(t-r)\int_0^r(r-v)C_K(v)Cxdvdr = \int_0^{t-\tau_0}K(t-r)\int_0^r(r-v)C_K(v)Cxdvdr. \tag{42}$$

Applying the same argumentation, we easily infer that:

$$I_2 = -\int_0^{\tau_0}(t-\tau_0)\Theta(\tau_0-r)C_K(r)Cxdr + \Theta(t-\tau_0)\int_0^{\tau_0}(\tau_0-r)C_K(r)Cxdr$$

$$+\Theta^{-1}(t-\tau_0)\int_0^{\tau_0}C_K(r)Cxdr+\int_0^{\tau_0}[K(t-r)-K(\tau_0-r)]\int_0^r(r-v)C_K(v)Cxdvdr, \tag{43}$$

$$I_3 = -\Theta(t-\tau_0)\int_0^{\tau_0}(\tau_0-r)C_K(r)Cxdr + \Theta^{-1}(t-\tau_0)\int_0^{\tau_0}C_K(r)Cxdr$$

$$+\int_{2\tau_0-t}^{\tau_0}K(r+t-2\tau_0)\int_0^r(r-v)C_K(v)Cxdvdr \text{ and} \tag{44}$$

$$I_4 = \int_0^{t-\tau_0}K(r+2\tau_0-t)\int_0^r(r-v)C_K(v)Cxdvdr. \tag{45}$$

Exploiting (37)-(45) and the following simple equality:

$$\int_0^{\tau_0}(t-\tau_0)\int_0^s K(s-r)C_K(r)Cxdrds = \int_0^{\tau_0}(t-\tau_0)\Theta(\tau_0-r)C_K(r)Cxdr,$$

one obtains:

$$\sum = \int_0^{\tau_0}(\tau_0-s)\int_0^s K(s-r)C_K(r)Cxdrds + 2C_K(\tau_0)\int_0^{t-\tau_0}(t-\tau_0-s)C_K(s)xds$$

$$+\int_0^{t-\tau_0}K(t-r)\int_0^r(r-v)C_K(v)Cxdvdr$$

$$+\int_0^{\tau_0}[K(t-r)-K(\tau_0-r)]\int_0^r(r-v)C_K(v)Cxdvdr$$

$$-\int_{2\tau_0-t}^{\tau_0}K(r+t-2\tau_0)\int_0^r(r-v)C_K(v)Cxdvdr$$

$$-\int_0^{t-\tau_0}K(r+2\tau_0-t)\int_0^r(r-v)C_K(v)Cxdvdr+2\Theta(t-\tau_0)\int_0^{\tau_0}(\tau_0-r)C_K(r)Cxdr. \tag{46}$$

The last equality implies $\sum \in D(A)$ and

$$A(\sum)=C_{K*_0K}(t)-f(t)C^2x,\ \text{where} \tag{47}$$

$$f(t)=\int_0^{\tau_0}K(\tau_0-r)\Theta(r)dr+\int_0^{t-\tau_0}K(t-r)\Theta(r)dr$$

$$+\int_0^{\tau_0}[K(t-r)-K(\tau_0-r)]\Theta(r)dr-\int_{2\tau_0-t}^{\tau_0}K(r+t-2\tau_0)\Theta(r)dr$$

$$-\int_0^{t-\tau_0}K(r+2\tau_0-t)\Theta(r)dr+2\Theta(\tau_0)\Theta(t-\tau_0). \tag{48}$$

Notice also that $(\Theta(t)I)_{t\in[0,2\tau)}$ is a local K-convoluted cosine function generated by $\mathbf{0}$ and that the following identity follows immediately from an application of Theorem 2.4:

$$2\Theta(\tau_0)\Theta(t-\tau_0)=(\int_{\tau_0}^{t}-\int_0^{t-\tau_0})K(t-r)\Theta(r)dr$$

$$+\int_{2\tau_0-t}^{\tau_0}K(r+t-2\tau_0)\Theta(r)dr+\int_0^{t-\tau_0}K(r+2\tau_0-t)\Theta(r)dr. \tag{49}$$

The use of (47)-(49) enables one to see that $f(t)=(K*_0\Theta)(t)$ and that A is a subgenerator of a local $(K *_0 K)$-convoluted C^2-cosine function $(C_{K*_0K}(t))_{t\in[0,2\tau_0)}$. The supposition $0 \in \operatorname{supp}K$ implies that the function $(K *_0 K)_{|[0,\tau')}$ is a kernel for all $\tau' \in (0,2\tau]$; in this case, $(C_{K*_0K}(t))_{t\in[0,2\tau_0)}$

is a unique local $(K*_0K)$-convoluted C^2-cosine function with a subgenerator A ([7]) and the proof of Theorem 3.1 ends a routine argument.

Corollary 3.2. *Suppose $\alpha > 0$ and A is a subgenerator of a local α-times integrated C-cosine function $(C_\alpha(t))_{t\in[0,\tau)}$. Then A is a subgenerator of a local (2α)-times integrated C^2-cosine function $(C_{2\alpha}(t))_{t\in[0,2\tau)}$.*

Corollary 3.3. *Suppose A is a subgenerator of a (local) K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$ and $0 \in \operatorname{supp}K$. Then:*

$$C_K(t)C_K(s) = C_K(s)C_K(t),\ 0 \le t,\ s < \tau. \tag{50}$$

P r o o f. The assertion in global case follows immediately from Theorem 2.4 and herein it is worthwhile to point out that, in this case, we can neglect the supposition $0 \in \operatorname{supp}K$. Suppose now $\tau < \infty$, $0 \le t,\ s < \tau$, $x \in E$ and $\tau_0 \in (\max(t,s),\tau)$. Since no confusion seems likely, we will not distinguish the function $K(\cdot)$ and its restriction to the interval $[0,\tau_0)$. Then it is clear that there exists a function $K_1 \in L^1_{loc}([0,2\tau_0))$ such that $K = K_{1_{|[0,\tau_0)}}$ and that A is a subgenerator of a local K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau_0)}$. Owing to Theorem 3.1, we have that A is a subgenerator of a local $(K *_0 K)$-convoluted C^2-cosine function $(C_{K*_0K}(t))_{t\in[0,2\tau_0)}$ and now one can apply Theorem 2.4 in order to conclude that $C_{K*_0K}(t)C_{K*_0K}(s) = C_{K*_0K}(s)C_{K*_0K}(t)$. Using the explicit formula given in the formulation of Theorem 3.1 as well as the previous equality and the Fubini theorem, we obtain:

$$\int_0^t K(t-r)C_K(r)\int_0^s K(s-v)C_K(v)C^2x\,dv\,dr$$

$$= \int_0^s K(s-v)C_K(v)\int_0^t K(t-r)C_K(r)C^2x\,dr\,dv$$

$$= \int_0^s\int_0^t K(t-r)K(s-v)C_K(v)C_K(r)C^2x\,dv\,dr$$

$$= \int_0^t(\int_0^s K(t-r)K(s-v)C_K(v)C_K(r)C^2x\,dv)dr$$

$$= \int_0^t K(t-r)[\int_0^s K(s-v)C_K(v)C_K(r)C^2xdv]dr. \qquad (51)$$

The injectiveness of C and (51) imply:

$$\int_0^t K(t-r)[C_K(r)\int_0^s K(s-v)C_K(v)xdv]dr$$

$$= \int_0^t K(t-r)[\int_0^s K(s-v)C_K(v)C_K(r)xdv]dr. \qquad (52)$$

Now the required property follows simply from the strong continuity of $(C_K(t))_{t\in[0,\tau)}$ and the fact that $K(\cdot)$ is a kernel.

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