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On $I_{\pi g\beta^\star}$-closed sets in ideal topological spaces

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Abstract. In this paper, a new class of sets called $I_{\pi g\beta^\star}$-closed sets is introduced and its properties are studied in ideal topological space. Moreover $I_{\pi g\beta^\star}$-continuity and the notion of quasi-$\beta^\star$-I-normal spaces are introduced.

1. Introduction and preliminaries

An ideal topological space is a topological space (X, τ) with an ideal I on X, and is denoted by (X, τ, I). $A^*(I) = \{x \in X \mid U \cap A \notin I$ for each open neighborhood U of x$\}$ is called the local function of A with respect to I and τ [**11**]. When there is no chance for confusion $A^*(I)$ is denoted by A^*. For every ideal topological space (X, τ, I), there exists a topology τ^* finer than τ, generated by the base $\beta(I, \tau) = \{U\backslash I \mid U\in \tau$ and $I \in I\}$. In general $\beta(I, \tau)$ is not always a topology [**10**]. Observe additionally that $Cl^*(A) = A^* \cup A$ [**17**] defines a Kuratowski closure operator for τ^*. $Int^\star(A)$ will denote the interior of A in (X, $\tau^\star$).

In this paper, we define and study a new notion $I_{\pi g\beta^\star}$-closed set by using the notion of $\beta_I^\star$-open set. Some new notions depending on $I_{\pi g\beta^\star}$-closed sets such as $I_{\pi g\beta^\star}$-open sets, $I_{\pi g\beta^\star}$-continuity and $I_{\pi g\beta^\star}$-irresoluteness are also introduced and a decomposition of $\beta^\star$-I-continuity is given. Also by using $I_{\pi g\beta^\star}$-closed sets characterizations of quasi-$\beta^\star$-I-normal spaces are obtained. Several preservation theorems for quasi-$\beta^\star$-I-normal spaces are given.

Throughout this paper, space (X, τ) (or simply X) always means topological space on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively.

A subset A of a topological space (X, τ) is said to be regular open [**15**](resp. regular closed [**15**]) if A = Int(Cl(A)) (resp. A = Cl(Int(A))).

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The finite union of regular open sets is said to be π-open [**18**] in (X, τ). The complement of a π-open set is π-closed [**18**].

A subset A of a topological space (X, τ) is said to be β-open [**1**] if A $\subseteq$ Cl(Int(Cl(A))) and the complement of a β-open set is called β-closed [**1**].

The intersection of all β-closed sets containing A is called the β-closure [**2**] of A and is denoted by βCl(A).

Note that βCl(A) = A $\cup$ Int(Cl(Int(A))) [**3**].

A subset A of a space (X, τ) is said to be πg-closed [**4**] (resp. $\pi g\beta$-closed [**16**]) if Cl(A) $\subseteq$ U (resp. βCl(A) $\subseteq$ U) whenever A $\subseteq$ U and U is π-open in X.

A function f : (X, τ) $\rightarrow$ (Y, σ) is said to be m-π-closed [**7**] if f(V) is π-closed in (Y, σ) for every π-closed in (X, τ).

A function f : (X, τ) $\rightarrow$ (Y, σ) is said to be πg-continuous [**4**] (resp. $\pi g\beta$-continuous [**16**]) if f^{-1}(V) is πg-closed (resp. $\pi g\beta$-closed) in (X, τ) for every closed set V of (Y, σ).

A space (X, τ) is said to be quasi-β-normal [**13**] if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint β-open sets U, V of X such that A $\subseteq$ U and B $\subseteq$ V.

A space (X, τ) is said to be quasi-normal [**18**] if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint open sets U, V of X such that A $\subseteq$ U and B $\subseteq$ V.

An ideal I is said to be codense [**5**] if $\tau \cap I = \emptyset$.

A subset A of an ideal topological space X is said to be $\star$-dense-in-itself [**9**](resp. $\beta_I^\star$-open [**6**], $\alpha^\star$-I-open [**8**], β-I-open [**8**]) if A $\subseteq$ A* (resp. A $\subseteq$ Cl(Int*(Cl(A))), Int(Cl*(Int(A))) = Int(A), A $\subseteq$ Cl(Int(Cl*(A)))).

The complement of $\beta_I^\star$-open set is $\beta_I^\star$-closed [**6**].

A subset A of an ideal topological space X is said to be $I_{\pi g}$-closed [**12**] if A$^\star\subseteq$ U whenever A $\subseteq$ U and U is π-open in X.

A function f : (X, τ, I) $\rightarrow$ (Y, σ) is said to be $I_{\pi g}$-continuous [**12**] if f^{-1}(V) is $I_{\pi g}$-closed in (X, τ, I) for every closed set V of (Y, σ).

Lemma 1.1. [**14**] *Let (X, τ, I) be an ideal topological space and A $\subseteq$ X. If A $\subseteq$ A*, then A* = Cl(A*) = Cl(A) = Cl*(A).*

Theorem 1.2. [**12**] *Every πg-closed set is $I_{\pi g}$-closed but not conversely.*

Theorem 1.3. [**12**] *For a function f : (X, τ, I) $\rightarrow$ (Y, σ), the following holds: Every πg-continuous function is $I_{\pi g}$-continuous but not conversely.*

Proposition 1.4. [**8**] *Every β-I-open set is β-open but not conversely.*

2. $I_{\pi g\beta^\star}$-closed sets

Definition 2.1. *Let (X, τ, I) be an ideal topological space and let A be a subset of X. The union of all $\beta_I^\star$-open sets contained in A is called the $\beta_I^\star$-interior of A and is denoted by $\beta_I^\star$Int(A).*

Definition 2.2. *Let (X, τ, I) be an ideal topological space and let A be a subset of X. The intersection of all $\beta_I^\star$-closed sets containing A is called the $\beta_I^\star$-closure of A and is denoted by $\beta_I^\star$Cl(A).*

LEMMA 2.3. *Let (X, τ, I) be an ideal topological space. For a subset A of X, the followings hold:*
(1) $\beta_I^\star Cl(A) = A \cup Int(Cl^\star(Int(A)))$,
(2) $\beta_I^\star Int(A) = A \cap Cl(Int^\star(Cl(A)))$.

DEFINITION 2.4. *A subset A of an ideal topological space (X, τ, I) is called $I_{\pi g\beta^\star}$-closed if $\beta_I^\star Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is π-open in X.*
The complement of $I_{\pi g\beta^\star}$-closed set is said to be $I_{\pi g\beta^\star}$-open.

PROPOSITION 2.5. *Every β-open set is $\beta_I^\star$-open but not conversely.*

PROOF. Let A be β-open set. Then A $\subseteq$ Cl(Int(Cl(A))) which implies A $\subseteq$ Cl(Int*(Cl(A))). Hence A is $\beta_I^\star$-open set.

EXAMPLE 2.6. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}\}$ and $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $\{b\}$ is $\beta_I^\star$-open set but not β-open.*

THEOREM 2.7. *Every $\star$-dense-in-itself and $I_{\pi g\beta^\star}$-closed set is a $\pi g\beta$-closed set.*

PROOF. Let A$\subseteq$ U, and U is π-open in X. Since A is $I_{\pi g\beta^\star}$-closed, $\beta_I^\star$Cl(A) $\subseteq$ U. By Lemmas 1.1 and 2.3, $\beta_I^\star$Cl(A) = A $\cup$ Int(Cl*(Int(A))) = A $\cup$ Int(Cl(Int(A))) = βCl(A). Then, βCl(A) $\subseteq$ U. So A is $\pi g\beta$-closed.

THEOREM 2.8. *Let A be $I_{\pi g\beta^\star}$-closed in (X, τ, I). Then $\beta_I^\star Cl(A) \setminus A$ does not contain any non-empty π-closed set.*

PROOF. Let F be a π-closed set such that F $\subseteq$ $\beta_I^\star$Cl(A) $\setminus$ A. Then F $\subseteq$ X $\setminus$ A implies A $\subseteq$ X $\setminus$ F. Therefore $\beta_I^\star$Cl(A) $\subseteq$ X $\setminus$ F. That is F $\subseteq$ X $\setminus$ $\beta_I^\star$Cl(A). Hence F $\subseteq$ $\beta_I^\star$Cl(A) $\cap$ (X $\setminus$ $\beta_I^\star$Cl(A))= $\emptyset$. This shows F = $\emptyset$.

PROPOSITION 2.9. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then the following properties hold:*
(1) *If A is $\pi g\beta$-closed, then A is $I_{\pi g\beta^\star}$-closed,*
(2) *If A is $I_{\pi g}$-closed, then A is $I_{\pi g\beta^\star}$-closed.*

PROOF. The proof is obvious.

REMARK 2.10. *From Theorem 1.2, Theorem 1.4 and Proposition 2.9, we have the following diagram.*

$$\begin{array}{ccc} \pi g\textbf{-closed} & \longrightarrow & \pi g\beta\textbf{-closed} \\ \downarrow & & \downarrow \\ I_{\pi g}\textbf{-closed} & \longrightarrow & I_{\pi g\beta^\star}\textbf{-closed} \end{array}$$

where none of these implications is reversible as shown in the following examples.

EXAMPLE 2.11. *(1) Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}, \{b, c, d\}\}$ and $I = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$. Then $A = \{b\}$ is $\pi g\beta$-closed set but not πg-closed set. Also $C = \{b\}$ is $I_{\pi g\beta^\star}$-closed set but not $I_{\pi g}$-closed.*
(2) Let $X = \{a, b, c, d, e\}$, $\tau = \{X, \emptyset, \{a\}, \{e\}, \{a, e\}, \{a, b, e\}, \{a, b, d, e\}\}$ and $I = \{\emptyset, \{a\}, \{e\}, \{a, e\}\}$. Then B= $\{a, e\}$ is $I_{\pi g\beta^\star}$-closed set but it is not $\pi g\beta$-closed.

THEOREM 2.12. *Every π-open and $I_{\pi g\beta^\star}$-closed set is $\alpha^\star$-I-open.*

PROOF. $\beta_I^\star$Cl(A) $\subseteq$ A, since A is π-open and $I_{\pi g\beta^\star}$-closed. We have Int(Cl*(Int(A))) $\subseteq$ A and Int(Cl*(Int(A))) $\subseteq$ Int(A). Always Int(A) $\subseteq$ Int(Cl*(Int(A))). Therefore Int(A) = Int(Cl*(Int(A))), which shows that A is $\alpha^\star$-I-open.

REMARK 2.13. *The union of two $I_{\pi g\beta^\star}$-closed sets need not be $I_{\pi g\beta^\star}$-closed.*

EXAMPLE 2.14. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $I = \{\emptyset, \{a\}\}$. Then $A = \{a\}$ and $B = \{b\}$ are $I_{\pi g\beta^\star}$-closed sets but their union $\{a, b\}$ is not $I_{\pi g\beta^\star}$-closed.*

REMARK 2.15. *The intersection of two $I_{\pi g\beta^\star}$-closed sets need not be $I_{\pi g\beta^\star}$-closed.*

EXAMPLE 2.16. *Consider the Example 2.14. Let $A = \{a, b, c\}$ and $B = \{a, b, d\}$ are $I_{\pi g\beta^\star}$-closed sets but their intersection $\{a, b\}$ is not $I_{\pi g\beta^\star}$-closed.*

THEOREM 2.17. *If A is $I_{\pi g\beta^\star}$-closed and $A \subseteq B \subseteq \beta_I^\star Cl(A)$, then B is $I_{\pi g\beta^\star}$-closed.*

PROOF. Let A be $I_{\pi g\beta^\star}$-closed and B $\subseteq$ U, where U is π-open. Then A $\subseteq$ B implies A $\subseteq$ U. Since A is $I_{\pi g\beta^\star}$-closed, $\beta_I^\star$Cl(A) $\subseteq$ U. B $\subseteq$ $\beta_I^\star$Cl(A) implies $\beta_I^\star$Cl(B) $\subseteq$ $\beta_I^\star$Cl(A). Therefore $\beta_I^\star$Cl(B) $\subseteq$ U and hence B is $I_{\pi g\beta^\star}$-closed.

THEOREM 2.18. *Let (X, τ, I) be an ideal topological space. Then every subset of X is $I_{\pi g\beta^\star}$-closed if and only if every π-open set is $\alpha^\star$-I-open.*

PROOF. Necessity: It is obvious from Theorem 2.12.
Sufficiency: Suppose that every π-open set is $\alpha^\star$-I-open. Let A be a subset of X and U be π-open such that A $\subseteq$ U. By hypothesis Int(Cl*(Int(A))) $\subseteq$ Int(Cl*(Int(U))) = Int(U) $\subseteq$ U. Then $\beta_I^\star$Cl(A)$\subseteq$ U. So A is $I_{\pi g\beta^\star}$-closed.

THEOREM 2.19. *Let (X, τ, I) be an ideal topological space. $A \subseteq X$ is $I_{\pi g\beta^\star}$-open if and only if $F \subseteq \beta_I^\star Int(A)$ whenever F is π-closed and $F \subseteq A$.*

PROOF. Necessity: Let A be $I_{\pi g\beta^\star}$-open and F be π-closed such that F $\subseteq$ A. Then X\A $\subseteq$ X\F where X\F is π-open. $I_{\pi g\beta}$-closedness of X\A implies $\beta_I^\star$Cl(X\A) $\subseteq$ X\F. Then F $\subseteq$ $\beta_I^\star$Int(A).
Sufficiency: Suppose F is π-closed and F $\subseteq$ A implies F $\subseteq$ $\beta_I^\star$Int(A). Let X\A $\subseteq$ U where U is π-open. Then X\U $\subseteq$ A where X\U is π-closed. By hypothesis X\U $\subseteq$ $\beta_I^\star$Int(A). That is $\beta_I^\star$Cl(X\A) $\subseteq$ U. So, A is $I_{\pi g\beta^\star}$-open.

DEFINITION 2.20. *A subset A of an ideal topological space (X, τ, I) is called M_I-set if $A = U \cup V$ where U is π-closed and V is $\beta_I^\star$-open.*

PROPOSITION 2.21. *Every π-closed set is M_I-set but not conversely.*

EXAMPLE 2.22. *Consider the Example 2.14. Let $A = \{a, b\}$. Then A is M_I-set but not π-closed.*

PROPOSITION 2.23. *Every $\beta_I^\star$-open set is M_I-set but not conversely.*

EXAMPLE 2.24. *Consider the Example 2.14. Let A= $\{c,\ d\}$. Then A is M_I-set but not $\beta^\star_I$-open.*

PROPOSITION 2.25. *Every $\beta^\star_I$-open set is $I_{\pi g\beta^\star}$-open but not conversely.*

PROOF. Let A be $\beta^\star_I$-open set. Then A ⊆ Cl(Int*(Cl(A))). Assume that F is π-closed and F ⊆ A. Then F ⊆ Cl(Int*(Cl(A))) which implies F ⊆ A ∩ Cl(Int*(Cl(A))) = $\beta^\star_I$Int(A) by Lemma 2.3. Hence, by Theorem 2.19, A is $I_{\pi g\beta^\star}$-open.

EXAMPLE 2.26. *Consider the Example 2.14. Let A = $\{c\}$. Then A is $I_{\pi g\beta^\star}$-open set but not $\beta^\star_I$-open.*

THEOREM 2.27. *For a subset A of (X, τ, I) the following conditions are equivalent:*

(1) *A is $\beta^\star_I$-open,*
(2) *A is $I_{\pi g\beta^\star}$-open and a M_I-set.*

PROOF. (1) ⇒ (2) It is obvious.
(2) ⇒ (1) Let A be $I_{\pi g\beta^\star}$-open and a M$_I$-set. Then there exist a π-closed set U and $\beta^\star_I$-open set V such that A = U ∪ V. Since U ⊆ A and A is $I_{\pi g\beta^\star}$-open, by Theorem 2.19, U ⊆ $\beta^\star_I$Int(A) and U ⊆ Cl(Int*(Cl(A))). Also, V ⊆ Cl(Int*(Cl(V))) ⊆ Cl(Int*(Cl(A))). Then A ⊆ Cl(Int*(Cl(A))). So A is $\beta^\star_I$-open.

The following examples show that concepts of $I_{\pi g\beta^\star}$-open set and M$_I$-set are independent.

EXAMPLE 2.28. *Let (X, τ, I) be the same ideal topological space as in Example 2.14. Then $\{c,\ d\}$ is a M_I-set but not $I_{\pi g\beta^\star}$-open.*

EXAMPLE 2.29. *Let (X, τ, I) be the same ideal topological space as in Example 2.14. Then $\{d\}$ is $I_{\pi g\beta^\star}$-open set but not a M_I-set.*

3. $I_{\pi g\beta^\star}$-continuity and $I_{\pi g\beta^\star}$-irresoluteness

DEFINITION 3.1. *A function f : (X, τ, I) → (Y, σ) is said to be $I_{\pi g\beta^\star}$-continuous (resp. $\beta^\star$-I-continuous) if f^{-1}(V) is $I_{\pi g\beta^\star}$-closed (resp. $\beta^\star_I$-closed) in X for every closed set V of Y.*

DEFINITION 3.2. *A function f : (X, τ, I) → (Y, σ, J) is said to be $I_{\pi g\beta^\star}$-irresolute if f^{-1}(V) is $I_{\pi g\beta^\star}$-closed in X for every $J_{\pi g\beta^\star}$-closed set V of Y.*

DEFINITION 3.3. *A function f : (X, τ, I) → (Y, σ) is said to be M_I-continuous if f^{-1}(V) is M_I-set in (X, τ, I) for every closed set V of Y.*

THEOREM 3.4. *A function f : (X, τ, I) → (Y, σ) is $\beta^\star$-I-continuous if and only if it is M_I-continuous and $I_{\pi g\beta^\star}$-continuous.*

PROOF. This is an immediate consequence of Theorem 2.27.

REMARK 3.5. *The following Examples show that:*

(1) *every $I_{\pi g\beta^\star}$-continuous function is not $\pi g\beta$-continuous,*

(2) *every $I_{\pi g\beta^\star}$-continuous function is not $I_{\pi g}$-continuous.*

EXAMPLE 3.6. *Let (X, τ, I) be the same ideal topological space as in Example 2.11(2). Let Y = {x, y, z} and σ = {Y, $\emptyset$, {y, z}}. Define a function f : (X, τ, I) $\rightarrow$ (Y, σ) as follows: f(a) = f(e) = x, f(c) = f(d) = y and f(b) = z. Then f is a $I_{\pi g\beta^\star}$-continuous function but it is not $\pi g\beta$-continuous.*

EXAMPLE 3.7. *Let (X, τ, I) be the same ideal topological space as in Example 2.11(1). Let Y = {x, y, z} and σ = {Y, $\emptyset$, {x, y}}. Define a function f : (X, τ, I) $\rightarrow$ (Y, σ) as follows: f(a) = f(d) = f(e)= x, f(b) = z and f(c) = y. Then f is a $I_{\pi g\beta^\star}$-continuous function but it is not $I_{\pi g}$-continuous.*

THEOREM 3.8. *For a function f : (X, τ, I) $\rightarrow$ (Y, σ), the following properties hold:*

$$\begin{array}{ccc} \pi g\textbf{-continuous} & \longrightarrow & \pi g\beta\textbf{-continuous} \\ \downarrow & & \downarrow \\ \boldsymbol{I}_{\pi g}\textbf{-continuous} & \longrightarrow & \boldsymbol{I}_{\pi g\beta^\star}\textbf{-continuous} \end{array}$$

PROOF. The proof is obvious by Remark 2.10.

The composition of two $I_{\pi g\beta^\star}$-continuous functions need not be $I_{\pi g\beta^\star}$-continuous. Consider the following Example:

EXAMPLE 3.9. *Let X = {a, b, c, d}, τ = {X, $\emptyset$, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}}, I = {$\emptyset$, {a}}, Y = {x, y, z}, σ = {Y, $\emptyset$, {y, z}}, J = {$\emptyset$, {x}}, Z = {1, 2} and η = {Z, $\emptyset$, {1}}. Define f : (X, τ, I) $\rightarrow$ (Y, σ, J) by f(a)= f(b)= x, f(c)= y and f(d)= z and g : (Y, σ, J) $\rightarrow$ (Z, η) by g(x)= 1 and g(y)= g(z)= 2. Then f and g are $I_{\pi g\beta^\star}$-continuous. {2} is closed in (Z, η), $(g \circ f)^{-1}(\{2\}) = f^{-1}(g^{-1}(\{2\})) = f^{-1}(\{y, z\})$ = {c, d} which is not $I_{\pi g\beta^\star}$-closed in (X, τ, I). Hence g $\circ$ f is not $I_{\pi g\beta^\star}$-continuous.*

THEOREM 3.10. *Let f : (X, τ, I) $\rightarrow$ (Y, σ, J) and g : (Y, σ, J) $\rightarrow$ (Z, η, K) be any two functions. Then*

(1) *g $\circ$ f is $I_{\pi g\beta^\star}$-continuous, if g is continuous and f is $I_{\pi g\beta^\star}$-continuous,*
(2) *g $\circ$ f is $I_{\pi g\beta^\star}$-continuous, if g is $J_{\pi g\beta^\star}$-continuous and f is $I_{\pi g\beta^\star}$-irresolute,*
(3) *g $\circ$ f is $I_{\pi g\beta^\star}$-irresolute, if g is $J_{\pi g\beta^\star}$-irresolute and f is $I_{\pi g\beta^\star}$-irresolute.*

PROOF. (1) Let V be closed in Z. Then g^{-1}(V) is closed in Y, since g is continuous. $I_{\pi g\beta^\star}$-continuity of f implies that $f^{-1}(g^{-1}(V))$ is $I_{\pi g\beta^\star}$-closed in X. Hence g $\circ$ f is $I_{\pi g\beta^\star}$-continuous.
(2) Let V be closed in Z. Since g is $J_{\pi g\beta^\star}$-continuous, g^{-1}(V) is $J_{\pi g\beta^\star}$-closed in Y. As f is $I_{\pi g\beta^\star}$-irresolute, $f^{-1}(g^{-1}(V))$ is $I_{\pi g\beta^\star}$-closed in X. Hence g $\circ$ f is $I_{\pi g\beta^\star}$-continuous. (3) Let V be $K_{\pi g\beta^\star}$-closed in Z. Then g^{-1}(V) is $J_{\pi g\beta^\star}$-closed in Y, since g is $J_{\pi g\beta^\star}$-irresolute. Because f is $I_{\pi g\beta^\star}$-irresolute, $f^{-1}(g^{-1}(V))$ is $I_{\pi g\beta^\star}$-closed in X. Hence g $\circ$ f is $I_{\pi g\beta^\star}$-irresolute.

4. Quasi-$\beta^\star$-I-normal spaces

DEFINITION 4.1. *An ideal topological space (X, τ, I) is said to be quasi-$\beta^\star$-I-normal if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint $\beta^\star_I$-open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$.*

PROPOSITION 4.2. *If X is a quasi-β-normal space, then X is quasi-$\beta^\star$-I-normal.*

PROOF. It is obtained from Proposition 2.5.

THEOREM 4.3. *The following properties are equivalent for a space X:*

1. *X is quasi-$\beta^\star$-I-normal,*
2. *for any disjoint π-closed sets A and B, there exist disjoint $I_{\pi g\beta^\star}$-open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$,*
3. *for any π-closed set A and any π-open set B containing A, there exists an $I_{\pi g\beta^\star}$-open set U such that $A \subseteq U \subseteq \beta^\star_I Cl(U) \subseteq B$.*

PROOF. (1) $\Rightarrow$ (2) The proof is obvious.
(2) $\Rightarrow$ (3) Let A be any π-closed set of X and B any π-open set of X such that A $\subseteq$ B. Then A and X\B are disjoint π-closed subsets of X. Therefore, there exist disjoint $I_{\pi g\beta^\star}$-open sets U and V such that A $\subseteq$ U and X\B $\subseteq$ V. By the definition of $I_{\pi g\beta^\star}$-open set, We have that X\B $\subseteq \beta^\star_I$Int(V) and U $\cap\, \beta^\star_I$Int(V) $= \emptyset$. Therefore, we obtain $\beta^\star_I$Cl(U) $\subseteq \beta^\star_I$Cl(X\V) and hence A $\subseteq$ U $\subset \beta^\star_I$Cl(U) $\subseteq$ B.
(3) $\Rightarrow$ (1) Let A and B be any disjoint π-closed sets of X. Then A $\subseteq$ X\B and X\B is π-open and hence there exists an $I_{\pi g\beta^\star}$-open set G of X such that A $\subseteq$ G $\subseteq \beta^\star_I$Cl(G) $\subseteq$ X\B. Put U $= \beta^\star_I$Int(G) and V = X\ $\beta^\star_I$Cl(G). Then U and V are disjoint $\beta^\star_I$-open sets of X such that A $\subseteq$ U and B $\subseteq$ V. Therefore, X is quasi-$\beta^\star$-I-normal.

THEOREM 4.4. *Let $f : X \to Y$ be an $I_{\pi g\beta^\star}$-continuous m-π-closed injection. If Y is quasi-normal, then X is quasi-$\beta^\star$-I-normal.*

PROOF. Let A and B be disjoint π-closed sets of Y. Since f is m-π-closed injection, f(A) and f(B) are disjoint π-closed sets of Y. By the quasi-normality of X, there exist disjoint open sets U and V such that f(A) $\subseteq$ U and f(B) $\subseteq$ V. Since f is $I_{\pi g\beta^\star}$-continuous, then f^{-1}(U) and f^{-1}(V) are disjoint $I_{\pi g\beta^\star}$-open sets such that A $\subseteq f^{-1}$(U) and B $\subseteq f^{-1}$(V). Therefore X is quasi-$\beta^\star$-I-normal by Theorem 4.3.

THEOREM 4.5. *Let $f : X \to Y$ be an $I_{\pi g\beta^\star}$-irresolute m-π-closed injection. If Y is quasi-$\beta^\star$-I-normal, then X is quasi-$\beta^\star$-I-normal.*

PROOF. Let A and B be disjoint π-closed sets of Y. Since f is m-π-closed injection, f(A) and f(B) are disjoint π-closed sets of Y. By quasi-$\beta^\star$-I-normality of Y, there exist disjoint $I_{\pi g\beta^\star}$-open sets U and V such that f(A) $\subseteq$ U and f(B) $\subseteq$ V. Since f is $I_{\pi g\beta^\star}$-irresolute, then f^{-1}(U) and f^{-1}(V) are disjoint $I_{\pi g\beta^\star}$-open sets such that A $\subseteq f^{-1}$(U) and B $\subseteq f^{-1}$(V). Therefore X is quasi-$\beta^\star$-I-normal.

THEOREM 4.6. *Let (X, τ, I) be an ideal topological space where I is codense. Then X is quasi-$\beta^\star$-I-normal if and only if it is quasi-β-normal.*

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