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A Measure of Graphs Vulnerability: Edge Scattering Number

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Abstract

In a communication network, several vulnerability measures are used to determine the resistance of the network to disruption of operation after the failure of certain stations or communication links. This study introduces a new vulnerability parameter, edge scattering number. The edge scattering number of a noncomplete connected graph G is defined to be $es(G) = max\{\omega(G-S)-|S| : S \subseteq E(G),\ \omega(G-S) > 1\}$ where $\omega(G-S)$ denote the number of components in $G-S$. A set $S \subseteq E(G)$, is said to be the es-set of G, if $es(G) = \omega(G-S) - |S|$. In this paper contains results on bounds for the edge scattering number. Moreover we obtain edge scattering number of some graphs.

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Key words and phrases: Vulnerability, network design and communication, connectivity, edge connectivity, integrity, edge integrity, scattering number.

1 Introduction

The stability of a (computer, communication, or transportation) network composed of (processing) nodes and (communication or transportation) links is of prime importance to network designers. As the network begins losing links or nodes, eventually it loses effectiveness. Communication networks are designed such that they are not easily disrupted under external attack and, moreover, such that they can easily be reconstructed if they are disrupted [12]. These desirable properties of networks can be measured by various parameters such as connectivity and edge-connectivity. However, these parameters do not take

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into account what remains after the graph is disconnected. Consequently, a number of other parameters have recently been introduced in an attempt to cope with this. These include connectivity and edge-connectivity [15], integrity and edge-integrity [2, 13], toughness and edge-toughness [5, 11], tenacity and edge-tenacity [6, 12], scattering number [7].

Terminology and notation not defined in this paper can be found in [4]. Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. In the graph G, n and m denote the number of vertices and the number of edges, respectively. Let $\omega(G-S)$ denote the number of components and $m(G-S)$ denote the order of a largest component in $G-S$. Then the following parameters are defined.

- Connectivity [15]

$$\kappa(G) = min\{|S| : S \subset V(G),\ \omega(G-S) > 1\}.$$

- Edge-connectivity [15],

$$\lambda(G) = min\{|S| : S \subseteq E(G),\ \omega(G-S) > 1\}.$$

- Integrity [2],

$$I(G) = min\{|S| + m(G-S) : S \subset V(G)\}.$$

- Edge-integrity [13],

$$I'(G) = min\{|S| + m(G-S) : S \subseteq E(G)\}.$$

- Tenacity [6],

$$T(G) = min\{[|S| + m(G-S)]/\omega(G-S) : S \subset V(G),\ \omega(G-S) > 1\}.$$

- Edge-tenacity [12],

$$T'(G) = min\{[|S| + m(G-S)]/\omega(G-S) : S \subseteq E(G),\ \omega(G-S) > 1\}.$$

The concept of scattering number was first introduced by Jung in [7]. The scattering number $s(G)$ of G is defined by

$$s(G) = max\{\omega(G-S) - |S| : S \subset V(G)\ \omega(G-S) > 1\}$$

where $\omega(G-S)$ denote the number of components in $G-S$. A cutset S of a graph G fulfilling $s(G) = \omega(G-S) - |S|$ is said to be a scattering set. Unlike the other measures, the scattering number shows not only the difficulty to break down the network but also the damage that has been caused [7].

The paper is organized as follows. In Section 2, we introduce a new vulnerability parameter, edge scattering number. Also we compute the edge scattering number of some special graphs. In Section 3, we establish relationships between the edge scattering number and some other graph parameters. Conclusions are addressed in Section 4.

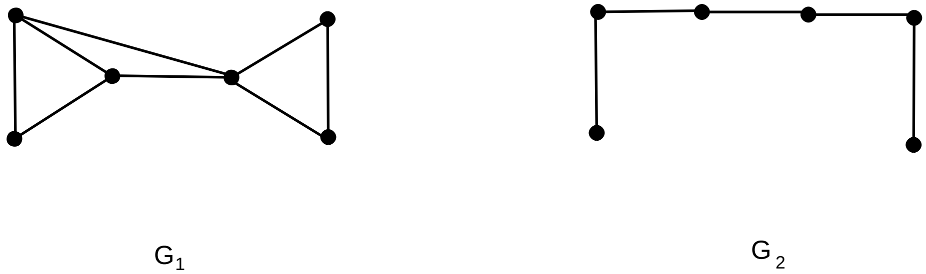

Figure 1: The graphs G_1 and G_2.

2 Edge scattering number

We introduce a new vulnerability parameter, edge scattering number. The edge scattering number of a noncomplete connected graph G is defined to be

$$es(G) = max\{\omega(G-S) - |S| : S \subseteq E(G),\ \omega(G-S) > 1\}$$

where $\omega(G-S)$ denote the number of components in $G-S$. A set $S \subseteq E(G)$, is said to be the es-set of G, if

$$es(G) = \omega(G-S) - |S|.$$

The edge scattering number differs from scattering number in showing the vulnerability of networks. For example, consider the graphs G_1 and G_2 in Figure 1.
It can be easily seen that the scattering number of these graphs are equal.

$$s(G_1) = s(G_2) = 1$$

On the other hand, the edge scattering number of G_1 and G_2 are different.

$$es(G_1) = 0$$

$$es(G_2) = 1$$

Thus, the edge scattering number is better than the scattering number for these two graphs.
We consider the edge scattering number of several special graphs.

Theorem 2.1 *Let T be an tree with order n. The edge scattering number of T is 1.*

Proof. The tree T has n vertices and $n-1$ edges. Let S be an edge cut set of T. If $|S| = r$ then $\omega(T-S) \leq r+1$. From the definition of edge scattering number we have

$$\omega(T-S) - |S| \leq r + 1 - r$$

and when we take the maximum of both sides we have

$$es(T) \leq 1. \tag{1}$$

It is obvious that there is an es-set S^* of T such that $|S^*| = 1$, $\omega(T - S^*) = 2$. By the definition of edge scattering number we have

$$\omega(T - S^*) - |S^*| = 2 - 1$$

and we get

$$es(T) = 1. \tag{2}$$

The proof is completed by (1) and (2). □

The following corollary is easily obtained from Theorem 2.1.

Corollary 2.1 *The edge scattering number of*
a) *the path P_n is 1.*
b) *the star $K_{1,n-1}$ is 1.*
c) *the comet $C_{a,b}$ is 1.*

Theorem 2.2 *The edge scattering number of the cycle C_n is* 0.

Proof. The proof is very similar to that of Theorem 2.1.

Theorem 2.3 *The edge scattering number of the complete graph K_n $(n \geq 3)$ is* $3 - n$.

Proof. The complete graph K_n has n vertices and $\frac{n.(n-1)}{2}$ edges. Let S be an edge cut set of K_n and $|S| = r$. If $n-1 \leq r \leq \frac{n.(n-1)}{2}$ then $\omega(K_n - S) \leq \lfloor \frac{2r}{n-1} \rfloor$. Thus

$$\omega(K_n - S) - |S| \leq \lfloor \tfrac{2r}{n-1} \rfloor - r$$

and when we take the maximum of both sides we have

$$es(K_n) \leq \max_r\{\lfloor \tfrac{2r}{n-1} \rfloor - r\}$$

the function $f(r) = \lfloor \frac{2r}{n-1} \rfloor - r$ takes its maximum value at $r = n-1$ and we get

$$es(K_n) \leq 3 - n. \tag{3}$$

It is obvious that there is an es-set S^* of K_n such that $|S^*| = n-1$, $\omega(K_n - S^*) = 2$. By the definition of edge scattering number we have

$$\omega(K_n - S^*) - |S^*| = 2 - (n-1).$$

So,

(4) $$es(K_n) = 3 - n.$$

The proof is completed by (3) and (4). □

Theorem 2.4 *The edge scattering number of the complete bipartite graph $K_{a,b}$ $(2 \leq a \leq b)$ is $2 - a$.*

Proof. The proof is very similar to that of Theorem 2.3. □

We now consider the cartesian product of two graphs.

Definition 2.1 *[4] The cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 also has $V(G_1) \times V(G_2)$ as its vertex set, but here (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.*

Theorem 2.5 *Let $n \geq 3$ be positive integer. Then,*

$$es(K_2 \times P_n) = 0.$$

Proof. The graph $K_2 \times P_n$ has $2n$ vertices and $3n - 2$ edges. Let S be an edge cut set of $K_2 \times P_n$ and $|S| = r$. If $2 \leq r \leq 3n - 2$ then $\omega((K_2 \times P_n) - S) \leq r$. Thus,

$$\omega((K_2 \times P_n) - S) - |S| \leq r - r$$

and when we take the maximum of both sides we have

(5) $$es(K_2 \times P_n) \leq 0.$$

It is obvious that there is an es-set S^* of $K_2 \times P_n$ such that $|S^*| = 2$, $\omega((K_2 \times P_n) - S^*) = 2$. By the definition of edge scattering number we have

$$\omega((K_2 \times P_n) - S^*) - |S^*| = 2 - 2$$

and we get

(6) $$es(K_2 \times P_n) = 0.$$

The proof is completed by (5) and (6). □

Theorem 2.6 *Let $n \geq 4$ be positive integer. Then,*

$$es(K_2 \times C_n) = -1.$$

Proof. The proof is very similar to that of Theorem 2.5. □

Theorem 2.7 *Let $n \geq 4$ be positive integer. Then,*

$$es(K_2 \times K_n) = 2 - n.$$

Proof. The proof is very similar to that of Theorem 2.3. □

3 Bounds for edge scattering number

In this section, we present the related graph parameters and some basic properties.

Theorem 3.1 *Let G be a connected graph. Then,*

$$es(G) \leq 1.$$

Proof. Let S be an edge set of G. If $|S| = r$ then $\omega(G - S) \leq r + 1$. Therefore

$$\omega(G - S) - |S| \leq r + 1 - r$$

and when we take the maximum of both sides we have

$$es(G) \leq 1.$$

The proof is completed. □

Theorem 3.2 *Let G be a connected graph. Then,*

$$es(G) \geq n - m.$$

Proof. Let S^* be an es-set of G and S' be an edge cut set of G. It can be easily seen that there is an edge set S' of G such that $|S'| = m$ then $\omega(G - S') = n$. Thus we get

$$\omega(G - S') - |S'| = n - m.$$

It follows from the definition of edge scattering number that

$$es(G) = \omega(G - S^*) - |S^*| \geq \omega(G - S') - |S'| = n - m$$

and we have

$$es(G) \geq n - m.$$

The proof is completed. □

Theorem 3.3 *Let G be a connected graph and $\delta(G)$ be the minimum degree of G. Then,*

$$es(G) \geq 2 - \delta(G).$$

Proof. Let S be an edge cut set of G and $|S| = r$. If $r = \delta(G)$ then $\omega(G - S) \geq 2$. Hence

$$\omega(G - S) - |S| \geq 2 - \delta(G)$$

and we get

$$es(G) \geq 2 - \delta(G).$$

We complete the proof. □

Theorem 3.4 *Let G be a graph. If G is λ-edge-connected then,*

$$es(G) \geq 2 - \lambda.$$

Proof. Let S be an edge cut set of G and $|S| = r$. If $r = \lambda$ then $\omega(G - S) \geq 2$. Thus

$$\omega(G - S) - |S| \geq 2 - \lambda$$

and when we take the maximum of both sides, the proof is completed. □

4 Conclusion

Many graph-theoretical parameters have been used in the past to describe the stability of communication networks. Most of these parameters do not take into account what remains after the graph is disconnected [9]. The edge scattering number represents a trade-off between the amount of work done to damage the network and how badly the network is damaged.We can say that the disruption is more successful if the disconnected network contains more components.

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