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THE AVERAGE LOWER DOMINATION NUMBER OF GRAPHS

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Abstract. The average lower domination number $\gamma_{av}(G)$ is defined as

$$\frac{1}{|V(G)|}\Sigma_{v\in V(G)}\gamma_v(G)$$

where $\gamma_v(G)$ is the minimum cardinality of a maximal dominating set that contains v. In this paper, the average lower domination number of complete k-ary tree and B_n tree are calculated. Moreover we obtain the $\gamma_{av}(G^*)$ for thorn graph G^*. Finally we compute the $\gamma_{av}(G_1+G_2)$ of G_1 and G_2.

1. Introduction

A network is modelled with graphs in a situation which the centers are equal to the vertex of graphs and connection lines are equal to the edges of a graph. A graph G is denoted by $G=(V(G),E(G))$, where $V(G)$ and $E(G)$ are vertex and edge sets of G, respectively. Let v be a vertex in $V(G)$.

In a graph $G=(V(G),E(G))$, a subset $S\subseteq V(G)$ of vertices is a dominating set if every vertex in $V(G)-S$ is adjacent to at least one vertex of S. The domination number of $\gamma(G)$ is the minimum cardinality of a dominating set. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set.

Henning [**12**] introduced the concept of average domination. The lower domination number, denoted by $\gamma_v(G)$ is the minimum cardinality of a dominating set of (G) that contains v.

The average lower domination number $\gamma_{av}(G)$is defined as

$$\frac{1}{|V(G)|}\Sigma_{v\in V(G)}\gamma_v(G)$$

where $\gamma_v(G)$ is the minimum cardinality of a maximal dominating set that contains v.

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Clearly for a vertex v in a graph G, $\gamma(G) \leqslant \gamma_{av}(G)$ with equality if and only if v belongs to a $\gamma(G)$-set. Consequently, $\gamma_{av}(K_n) = 1$, while for a cycle C_n on $n \geqslant 3$ vertices, $\gamma_{av}(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.

PROPOSITION 1.1 ([**12**]). *For any graph G of order n with domination number* γ, $\gamma_{av}(G) \leqslant \gamma + 1 - \frac{\gamma}{n}$, *with equality if and only if G has a unique* $\gamma(G)$*-set.*

THEOREM 1.1 ([**12**]). *If T is a tree of order* $n \geqslant 4$ *then* $\gamma_{av}(T) \leqslant \frac{n}{2}$ *with equality if and only if T is the corona of a tree.*

In this paper, the average lower domination number of complete k-ary tree and B_n tree are calculated. Moreover we obtain the $\gamma_{av}(G^*)$ for thorn graph G^*. Finally we compute the $\gamma_{av}(G_1 + G_2)$ of G_1 and G_2.

2. Average Lower Domination Number Of Some Graphs

Firstly we give the definition of a complete k-ary tree with depth n. The average lower domination number of complete k-ary tree are calculated. Moreover we obtain $\gamma_{av}(B_n)$ for binomial tree and $\gamma_{av}(G^*)$ for thorn graph G^*.

DEFINITION 2.1. ([**3**]) A complete k-ary tree with depth n is all leaves have the same depth and all internal vertices have degree k. A complete k-ary tree has $\frac{k^{n+1}-1}{k-1}$ vertices and $\frac{k^{n+1}-1}{k-1} - 1$ edges.

THEOREM 2.1. *Let G be a complete k-ary tree with depth n. Then*

$$\gamma_{av}(G) = \begin{cases} \gamma(G) + 1 - \frac{\gamma(G)+k}{|V(G)|} & , \quad n \equiv 0 \ (mod \ 3) \\ \\ \gamma(G) + 1 - \frac{\gamma(G)}{|V(G)|} & , \quad otherwise \end{cases}$$

PROOF. If G is a k-ary tree with depth n then $|V(G)| = \frac{k^{n+1}-1}{k-1}$. We have two cases for n to find the average lower average number of G.

Case 1. If $n \equiv 1$ (mod 3) or $n \equiv 2$ (mod 3) then G has a unique $\gamma(G)$-set. The minimal domination set of G contains the vertices on the levels $(n - 1 - 3i)$ for $0 \leqslant i \leqslant \lfloor \frac{n}{3} \rfloor$. Let vertices set of G be $V(G) = V(G_1) \cup V(G_2)$ where,
$V(G_1)$: The set contains the vertices on the levels $(n - 1 - 3i)$ for $0 \leqslant i \leqslant \lfloor \frac{n}{3} \rfloor$,
$V(G_2)$: The set contains the vertices of $V(G) - V(G_1)$.

i) If $v \in V(G_1)$, then $\gamma_v(G) = \gamma(G)$ since the vertex v is in the dominating set. Since this equality is satisfied for every vertex of $V(G_1)$ we have

$$\Sigma_{v \in V(G_1)} \gamma_v(G) = \gamma(G).\gamma(G) \ .$$

ii) If $v \in v(G_2)$, then $\gamma_v(G) = \gamma(G) + 1$ since the vertex v is not in the dominating set. Since this equality is satisfied for every vertex in $V(G_2)$, we have

$$\Sigma_{v \in V(G_2)} \gamma_v(G) = (|V(G)| - \gamma(G))(\gamma(G) + 1).$$

Consequently,

$$\gamma_{av}(G) = \frac{1}{|V(G)|} \Sigma_{v \in V(G)} \gamma_v(G) = \frac{1}{|V(G)|} (\Sigma_{v \in V(G_1)} \gamma_v(G) + \Sigma_{v \in V(G_2)} \gamma_v(G))$$

$$= \frac{1}{|V(G)|} [(\gamma(G).\gamma(G)) + (|V(G)| - \gamma(G)).(\gamma(G) + 1)]$$

$$= \gamma(G) + 1 - \tfrac{\gamma(G)}{|V(G)|}. \qquad (1)$$

Case 2. If G is a k-ary tree with depth n and $n \equiv 0(mod\ 3)$, then G has $k+1$ domination sets which give the domination number of G. The minimal domination set of G contains the vertices on the levels $(n-1-3i)$ for $0 \leqslant i \leqslant \lfloor \frac{n}{3} \rfloor$. But in this case the vertex on the 0^{th} level cannot be reached. Therefore the vertex on the 0^{th} level or one of the vertices on the 1^{st} level should be taken to the dominating set. Hence there are $k+1$ dominating sets according to the choice of vertices.

i) If $v \in \gamma(G)$-set, then $\gamma_v(G) = \gamma(G)$ since the vertex v is in the dominating set. We have to repeat this process for $k + \gamma(G)$ vertices. Therefore

$$\Sigma_{v \in V(G)} \gamma_v(G) = (\gamma(G) + k).\gamma(G)\ .$$

ii) If $v \notin \gamma(G)$-set, then $\gamma_v(G) = \gamma(G)$ since the vertex v is in the dominating set. We have to repeat this process for $|V(G)| - k - \gamma(G)$ vertices. Hence,

$$\Sigma_{v \in V(G)} \gamma_v(G) = (|V(G)| - (\gamma(G) + k)).(\gamma(G) + 1)\ .$$

As a result

$$\gamma_{av}(G) = \tfrac{1}{|V(G)|}[(\gamma(G) + k).\gamma(G) + (|V(G)| - (\gamma(G) + k)).(\gamma(G) + 1)]$$

$$= \gamma(G) + 1 - \tfrac{\gamma(G)+k}{|V(G)|} \qquad (2)$$

By (1) and (2) the proof is completed. □

Definition 2.2. ([**3**]) The binomial tree of order $n \geqslant 0$ with root R is the tree B_n defined as follows.

1) If $n = 0$, $B_n = B_0 = R$, i.e., the binomial tree of order zero consists of a single node R.

2) If $n > 0$, $B_n = R, B_0, B_1, \ldots, B_{n-1}$, i.e., the binomial tree of order $n > 0$ comprises the root R, and n binomial subtrees, $B_0, B_1, , B_{n-1}$.

Theorem 2.2. *Let B_n be a binomial tree. Then* $\gamma_{av}(B_n) = 2^{n-1}$.

Proof. Any binomial tree B_n consists of 2^n vertices; 2^{n-1} vertices with degree 1. While the domination set is found, all of the vertices with degree 1 or the vertices adjacent to these vertices should be taken into the set. Therefore the domination number of B_n is $\gamma(B_n) = 2^{n-1}$. Obviously the domination set satisfying the domination number can be obtained for every element of B_n. Since $\gamma_v(B_n) = 2^{n-1}$ for every element v of B_n. Hence

$$\Sigma_{v \in V(B_n)} \gamma_v(B_n) = 2^{n-1}.2^n\ .$$

From the definition of average lower domination number we have

$$\gamma_{av}(B_n) = \tfrac{1}{2^n} 2^{n-1}.2^n = 2^{n-1}\ .$$

□

Definition 2.3. ([**13**]) Let $p_1, p_2, \ldots, p_n$ be non-negative integers and G be such a graph, $V(G) = n$. The thorn graph of the graph, with parameters $p_1, p_2, \ldots, p_n$, is obtained by attaching p_i new vertices of degree 1 to the vertex u_i of the graph G, $i = 1, 2, \ldots, n$. The thorn graph of the graph G will be denoted by G^* or by $G^*(p_1, p_2, \ldots, p_n)$, if the respective parameters need to be specified.

Theorem 2.3. *Let G be a non complete connected graph with order n and G^* be a thorn graph of G with every $p_i = 1$. Then*

$$\gamma_{av}(G^*) = n$$

.

Proof. The number of vertices of G^* is $2n$. While the domination set is found every vertex of degree 1 or the vertex adjacent to it must be taken into the dominating set. Therefore the domination number of G^* is $\gamma(G^*) = n$. Thus the domination set satisfying the domination number can be obtained for every element of G^*. Since $\gamma_v(G^*) = n$ for every element v of G^*, therefore

$$\Sigma_{v \in G^*} \gamma_v(G^*) = 2n.n.$$

From the definition of average lower domination number we have

$$\gamma_{av}(G^*) = \frac{1}{2n}.2n.n = n.$$

□

Theorem 2.4. *Let G^* be a thorn graph of G with every $p_i > 1$. Then*

$$\gamma_{av}(G^*) = |V(G)| + 1 - \frac{|V(G)|}{|V(G^*)|}$$

.

Proof. Let G^* be a thorn graph of G with every $p_i > 1$. Obviously $\gamma(G^*) = |V(G)|$, hence all of the vertices of G should be taken into the dominating set. Let vertices set of G^* be $V(G^*) = V(G_1) \cup V(G_2)$ where,
$V(G_1)$: The set contains the vertices of graph G.
$V(G_2)$: The set contains the vertices of $V(G) - V(G_1)$
Then we have

$$\Sigma_{v \in V(G^*)} \gamma(G^*) = \Sigma_{v \in V(G_1)} \gamma_v(G^*) + \Sigma_{v \in V(G_2)} \gamma_v(G^*)$$

i) If $v \in V(G_1)$, then $\gamma_v(G^*) = |V(G)|$. We have to repeat this process for every vertices of $V(G_1)$. Hence

$$\Sigma_{v \in V(G_1)} \gamma_v(G^*) = |V(G)||V(G)|.$$

(ii) If $v \in V(G_2)$, then $\gamma_v(G^*) = |V(G)| + 1$. We have to repeat this process for every vertices of $V(G_2)$. So,

$$\Sigma_{v \in V(G_2)} \gamma_v(G^*) = (|V(G^*)| - |V(G)|)(|V(G)| + 1).$$

.
From the definition of average lower domination number we have

$$\gamma_{av}(G^*) = \frac{1}{|V(G^*)|}(|V(G)||V(G)| + (|V(G^*)| - |V(G)|)(|V(G)| + 1))$$
$$= |V(G)| + 1 - \frac{|V(G)|}{|V(G^*)|}.$$ □

3. Join Operation

We give some result of average lower domination number of $G_1 + G_2$.

THEOREM 3.1. *If G_1 and G_2 are two graphs with domination numbers different from 1, then $\gamma_{av}(G_1 + G_2) = 2$.*

PROOF. The domination set of $G_1 + G_2$ is formed by the pairs of (x, y) such that x is any vertex of the graph G_1 and y is any vertex of G_2. Since a domination set can be formed by every element v in $G_1 + G_2$, we have $\gamma_v(G_1 + G_2) = 2$. Then by the definition

$$\gamma_{av}(G_1 + G_2) = \frac{1}{|V(G_1+G_2)|}.2|V(G_1 + G_2)| = 2.$$

□

THEOREM 3.2. *Let G_1 and G_2 be two graphs with orders m and n, respectively, and let $\gamma(G_1) = 1$ or $\gamma(G_2) = 1$. Let a be the number of the domination sets satisfying $\gamma(G_1) = 1$ and b be the number of the domination sets satisfying $\gamma(G_2) = 1$, then*

$$\gamma_{av}(G_1 + G_2) = \begin{cases} 2 - \frac{a}{m+n} & , \quad \gamma(G_1) = 1 \quad and \quad \gamma(G_2) \neq 1 \\ 2 - \frac{b}{m+n} & , \quad \gamma(G_1) \neq 1 \quad and \quad \gamma(G_2) = 1 \\ 2 - \frac{a+b}{m+n} & , \quad \gamma(G_1) = 1 \quad and \quad \gamma(G_2) = 1 \end{cases}$$

PROOF. The proof is done in three cases according to the domination number of the graphs G_1 and G_2.

Case 1: Let $\gamma(G_1) = 1$ and $\gamma(G_2) \neq 1$. In this case,

(i) If $v \in V(G_1)$ and an element of one of the a sets satisfying $\gamma(G_1) = 1$ then $\gamma_v(G_1 + G_2) = 1$ and this equality is satisfied for a vertices.

(ii) If $v \in V(G_1 + G_2)$ which doesn't satisfy $\gamma(G_1) = 1$, then $\gamma_v(G_1 + G_2) = 2$, and this equality is satisfied for $m + n - a$ vertices. Therefore,

$$\gamma_{av}(G_1 + G_2) = \frac{1}{m+n}.(a + (m + n - a).2) = 2 - \frac{a}{m+n} \tag{3}$$

Case 2: Let $\gamma(G_1) \neq 1$ and $\gamma(G_2) = 1$.

(i) If $v \in V(G_2)$ and an element of one of the b sets satisfying $\gamma(G_2) = 1$ then $\gamma_v(G_1 + G_2) = 1$ and this equality is satisfied for b vertices.

(ii) If $v \in V(G_1 + G_2)$ which doesn't satisfy $\gamma(G_2) = 1$, then $\gamma_v(G_1 + G_2) = 2$, and this equality is satisfied for $m + n - b$ vertices. Hence we have,

$$\gamma_{av}(G_1 + G_2) = \frac{1}{m+n}.(b + (m + n - b).2) = 2 - \frac{b}{m+n} \tag{4}$$

Case 3: Let $\gamma(G_1) = 1$ and $\gamma(G_2) = 1$. Then

(i) If $v \in V(G_1)$ and an element of one of the a sets satisfying $\gamma(G_1) = 1$ then $\gamma_v(G_1 + G_2) = 1$ and this equality is satisfied for a vertices.

(ii) If $v \in V(G_2)$ and an element of one of the b sets satisfying $\gamma(G_2) = 1$ then $\gamma_v(G_1 + G_2) = 1$ and this equality is satisfied for b vertices.

(iii) If $v \in V(G_1 + G_2)$ which doesn't satisfy $\gamma(G_1) = 1$ and $\gamma(G_2) = 1$, then $\gamma_v(G_1 + G_2) = 2$, and this equality is satisfied for $m + n - a - b$ vertices.

Therefore,

$$\gamma_{av}(G_1 + G_2) = \tfrac{1}{m+n}.(a + b + (m + n - a - b).2) = 2 - \tfrac{a+b}{m+n} \qquad (5)$$

By (3), (4) and (5) the proof is completed. □

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