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BOUNDS FOR OUT DEGREE EQUITABLE DOMINATION NUMBERS IN GRAPHS

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Abstract. Let $G = (V, E)$ be a simple graph. Let D be a dominating set and $u \in D$. The edges from u to $V - D$ are called dominating edges. A dominating set D is called *outdegree equitable* if the difference between the cardinalities of the sets of dominating edges from any two points of D is at most one. The minimum cardinality of an out degree equitable dominating set is called the outdegree equitable domination number and is denoted by γ_e.The existence of an outdegree equitable dominating set is guaranteed. Out degree equitable domination is introduced in this paper. Minimum, minimal,independent outdegree equitable dominating sets,out degree Equitable points and Equitable neighbourhood numbers are defined and also obtained the bounds for γ_e.

1. Introduction

Prof. E. Sampathkumar introduced the concept of equitability in terms of outward influence. In an organization, the executive body which takes all decisions is to be formed on the basis of two criteria.

(i) Members of the body should have contacts with all the members of the organization.
(ii) The influences of members of the executive body over the rest of the organization are to be equitable.

The first criterion is taken care of by the concept of domination in graphs. The second criterion is taken care of by the new concept of Prof. E. Sampathkumar, namely outward equitability.

A graph model for this is suggested as follows: Let S_1 and S_2 be two subsets of the vertex set V of a graph $G = (V, E)$. Members of S_1 are said to be outdegree equitable with respect to S_2 if they have equitable number of neighbours in S_2.This

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yields a new parameter namely outdegree equitable domination number.A study of this concept and the parameter is made in this paper.

2. Out Degree Equitable Dominating sets - Minimal and Minimum Outdegee Equitable Dominating sets

DEFINITION 2.1. Let $G = (V, E)$ be a simple graph. Let D be a dominating set and $u \in D$. The edges from u to $V - D$ are called dominating edges. A dominating set D is called *outdegree equitable* if the difference between the cardinalities of the sets of dominating edges from any two points of D is at most one. The minimum cardinality of an outdegree equitable dominating set is called the outdegree equitable domination number and is denoted by γ_e.The existence of an outdegree equitable dominating set is guaranteed.

DEFINITION 2.2. A subset D of V is a *minimal outdegree equitable dominating set* if no proper subset of D is an outdegree equitable dominating set.

DEFINITION 2.3. An outdegree equitable dominating set is said to be 1-minimal if $D - v$ is not an outdegree dominating set for all $v \in D$.

EXAMPLE 2.1.

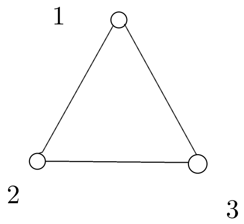

Fig. 1.1

$\{1\}$ is an outdegree equitable dominating set of K_3
$\{2, 3\}$ is an outdegree equitable dominating set of K_3
$\{2, 3\}$ is not a minimal outdegree equitable dominating set, since $\{2, 3\}$ contains $\{2\}$ which is an outdegree equitable dominating set of K_3.

EXAMPLE 2.2.

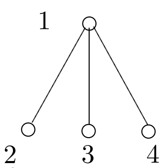

Fig. 1.2

{1} is an outdegree equitable dominating set.
{1,2} is not an outdegree equitable dominating set.
{1,2,3} is an outdegree equitable dominating Set.
Thus {1,2,3} is an outdegree equitable dominating set for which the sub set $\{1, 2\}$ is not an outdegree equitable dominating set but the subset $\{1\}$ is an outdegree equitable dominating set. Thus if outdegree equitable dominating set D is 1-minimal, then D need not be minimal.

DEFINITION 2.4. Let D be a subset of V. Let $u \in D$. The out degree of u is defined as the number of edges from u to $V - D$ (i.e) out degree of u is $|N(u) \cap (V - D)|$. The out degree of u in D is denoted by $d_D^o(u)$. The out degree of D denoted by D_0 is defined as $D_0 = \min_{u \in D}\{d_D^o(u)\}$.

DEFINITION 2.5. Let D be a subset of v. Let $u_1, u_2 \in D$. u_1 and u_2 are said to be outdegree equitable points (or simply equitable points) if $|d_D^o(u_1) - d_D^o(u_2)| \leqslant 1$. Otherwise u_1 and u_2 are said to be non-equitable points.

3. Bounds for Outdegree Equitable Domination number

THEOREM 3.1. *Let G be a graph containing two points u, v with $N(u)$, $N(v) \neq \phi$ and $N[u] \cap N[v] = \phi$. Then $1 \leqslant \gamma_e(G) \leqslant n - 2$.*

PROOF. Let $D = V - \{u, v\}$. Since $N[u] \cap N[v] = \phi$, u and v are not adjacent. Since $N(u) \neq \phi, N(v) \neq \phi$ and $N[u] \cap N[v] = \phi$ there exist distinct vertices x, y in D such that x is adjacent to u and y is adjacent to v. The out degree of any point in D is either 0 or 1. Hence D is an equitable dominating set of G. Therefore $\gamma_e(G) \leqslant n - 2$. □

THEOREM 3.2. *Let D be an outdegree equitable dominating set. Then $\sum_{u \in D} d(u) = |V - D|$ if and only if (i) D is independent and (ii) For every $u \in V - D$ there exists a unique vertex $v \in D$ such that $N(u) \cap D = \{v\}$.*

PROOF. Suppose (i) and (ii) hold. Then clearly $|V - D| = \sum_{u \in D} d(u)$.

Conversely, suppose $|V - D| = \sum_{u \in D} d(u)$. Suppose D is not independent. Then there exists $u, v \in D$ such that u and v are adjacent. Then $\sum_{u \in D} d(u)$ exceeds $|V - D|$ by at least two, which is a contradiction. Therefore D is independent. Suppose (ii) is false. Then $|N(u) \cap D| \geqslant 2$ for some $u \in V - D$. Let $v, w \in D$ such that $v, w \in N(u)$. Then $\sum_{x \in D} d(x)$exceeds $|V - D|$ by at least one because u is counted twice, once in $d(v)$ and in $d(w)$, a contradiction. Therefore (ii) must hold. □

DEFINITION 3.1. Let D be an outdegree equitable dominating set. Then the outdegree of any point of D is either k or $k + 1$ where k is a non negative integer. The outdegree of D denoted by d_0 is defined as the minimum of the outdegrees of vertices of D namely k.

COROLLARY 3.1. *Let D be an outdegree equitable dominating set such that $|V - D| = \sum_{u \in D} d(u)$. Let d_o be the outdegree of D then $|D| \leqslant \frac{n}{d_o + 1}$*

PROOF. $|V - D| = \sum_{u \in D} d(u) = \sum_{u \in D} d_D^o(u) \geqslant |D|\,(d_0)$
$\Rightarrow n - |D| \geqslant |D|\,(d_0)$. Therefore $|D| \leqslant \frac{n}{d_0 + 1}$. □

Corollary 3.2. *Let D be an outdegree equitable dominating set such that $|V-D| = \sum_{u\in D} d(u)$. Let k be the number of points in D with outdegree d_0. Then $\gamma_e \leqslant \frac{n+k}{2+d_0}$.*

Proof.

$$\begin{aligned} |V-D| &= \sum_{u\in D} d(u) \\ n-|D| &= kd_0 + (|D|-k)(d_0+1) \\ &= kd_0 + |D|\, d_0 + |D| - kd_0 - k \\ &= |D|\, d_o + |D| - k \\ \Rightarrow n+k &= |D|\,(2+d_0) \\ \Rightarrow |D| &= \frac{n+k}{2+d_0}. \end{aligned}$$

Therefore $\gamma_e \leqslant |D| = \frac{n+k}{2+d_0}$. □

Corollary 3.3. *Let D be an outdegree equitable dominating set such that $|V-D| = \sum_{u\in D} d(u)$. Number of edges from D to $V-D$ is $|D|\,(d_0+1)-k$ where k is the number of vertices in D having outdegree d_0.*

Proof follows from Corollary 3.2.

Corollary 3.4. *Let D be an outdegree equitable dominating set such that $|V-D| = \sum_{u\in D} d(u)$. Number of edges of $G \leqslant \frac{(n-|D|)(n-|D|-1)}{2} + |D|\,(d_0+1) - k$ and maximum is attained when $< V-D >$ is complete.*

Proof is obvious.

Corollary 3.5. *If G has no isolates and $|V-D| = \sum_{u\in D} d(u)$ for some outdegree equitable dominating set D, then $V-D$ is a 1- minimal outdegree equitable dominating set of G.*

Proof. Since G has no isolates and $|V-D| = \sum_{u\in D} d(u)$ we get that every point of D is adjacent to some point of $V-D$. Therefore $V-D$ is a dominating set. Also for every $u \in V-D$, there exists a unique vertex $v \in D$ such that $N(u)\cap D = \{v\}$. Therefore $V-D$ is an out degree equitable dominating set. Therefore 1-minimal out degree equitable dominating set we get that $V-D$ is a 1-minimal out degree equitable dominating set. □

Theorem 3.3. *Let G be a graph of order n such that $\gamma_e(G) = \left[\frac{n}{\Delta+1}\right]$. Then $\Delta+1$ divides n.*

Proof. Let S be a minimum out degree equitable dominating set of G. Suppose S is not independent, then by previous theorem,

$$\begin{aligned}
|V-S| &< \sum_{u\in S} d(u) \leqslant |S|\,\Delta = \gamma_e(G)\Delta \\
\Rightarrow n-\gamma_e &< \gamma_e\Delta \\
\Rightarrow n &< \gamma_e(\Delta+1) \\
\Rightarrow \frac{n}{\Delta+1} &< \gamma_e \\
\text{(i.e) } \gamma_e(G) &> \frac{n}{\Delta+1}
\end{aligned}$$

which is a contradiction. Therefore S is independent.

Claim For every vertex $u \in V-S$, there is a unique vertex $v \in S$ such that $N(u)\cap S = \{v\}$.

Suppose this is not true. Then proceeding as above, we get a contradiction. Therefore the claim holds. Therefore from an earlier proposition, $|V-S| = \sum_{u\in S} d(u)$.

Claim For every vertex $u \in S, d(u) = \Delta$. Suppose not. Then there exists a vertex $w \in S$ such that $d(w) < \Delta$

Therefore $\sum_{u\in S} d(u) < |S|\,\Delta$.

$$\begin{aligned}
\Rightarrow |V-S| &< |S|\,\Delta \\
\Rightarrow n-\gamma_e &< \gamma_e\Delta \\
\Rightarrow n &< \gamma_e(\Delta+1) \\
\Rightarrow \gamma_e &> \frac{n}{\Delta+1}
\end{aligned}$$

which is a contradiction. Therefore every vertex of S is of degree Δ.

Therefore $|V-S| = \sum_{u\in S} d(u)$

$$\begin{aligned}
\Rightarrow n-\gamma_e &= |S|\,\Delta \\
\Rightarrow n-\gamma_e &= \gamma_e\Delta \\
\Rightarrow n &= \gamma_e(\Delta+1).
\end{aligned}$$

Therefore $(\Delta+1)$ divides n. □

Remark 3.1. Converse of the above theorem is false.

Example 3.1. Consider the following graph.

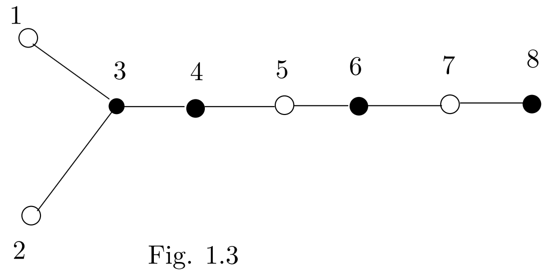

Fig. 1.3

Here $\Delta = 3$, $n = 8$, $\gamma_e = 4$. That is $(\Delta + 1)$ divides n. $\frac{n}{\Delta+1} = 8/4 = 2$. Therefore $\gamma_e \neq \frac{n}{\Delta+1}$.

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