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CONNECTED NEAR EQUITABLE DOMINATION IN GRAPHS

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Abstract. Let $G = (V, E)$ be a graph, $D \subseteq V$ and u be any vertex in D. Then the out degree of u with respect to D denoted by $od_D(u)$, is defined as $od_D(u) = |N(u) \cap (V - D)|$. A subset $D \subseteq V(G)$ is called a near equitable dominating set of G if for every $v \in V - D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leqslant 1$. A near equitable dominating set D is said to be a connected near equitable dominating set if the subgraph $\langle D \rangle$ induced by D is connected. The minimum of the cardinality of a connected near equitable dominating set of G is called the connected near equitable domination number and is denoted by $\gamma_{cne}(G)$. In this paper results involving this parameter are found, bounds for $\gamma_{cne}(G)$ are obtained. Connected near equitable domatic partition in a graph G is studied.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m, respectively. For graph theoretic terminology we refer to Chartrand and Lesnaik [3].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

A subset S of V is called a dominating set if $N[S] = V$. The minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$ $(\Gamma(G))$. An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [5]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [6]. Various types of domination have been defined and studied by

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several authors and more than 75 models of domination are listed in the appendix of Haynes et al. [5]. Sampathkumar and Walikar [8] introduced the concept of connected domination in graphs. A dominating set D of a connected graph G is called a connected dominating set if the induced subgraph $\langle D\rangle$ is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$.

A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers.

A subset D of $V(G)$ is called an equitable dominating set if for every $v \in V-D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d(u)-d(v)| \leqslant 1$. The minimum cardinality of such an equitable dominating set is denoted by γ_e and is called the equitable domination number of G. A vertex $u \in V$ is said to be degree equitable with a vertex $v \in V$ if $|d(u)-d(v)| \leqslant 1$. If D is an equitable dominating set then any super set of D is an equitable dominating set [9].

Equitable domination has interesting application in the context of social networks. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society persons with nearly equal status, tend to be friendly.

Let $D \subseteq V(G)$ and u be any vertex in D. The out degree of u with respect to D denoted by $od_D(u)$, is defined as $od_D(u) = |N(u) \cap (V-D)|$. D is called a near equitable dominating set of G if for every $v \in V-D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leqslant 1$. The minimum cardinality of such a dominating set is denoted by γ_{ne} and is called the near equitable domination number of G. A partition $P = \{V_1, V_2, ..., V_l\}$ of a vertex set $V(G)$ of a graph is called near equitable domatic partition of G if V_i is near equitable dominating set for every $1 \leqslant i \leqslant l$. The near equitable domatic number of G is the maximum cardinality of near equitable domatic partition of G and denoted by $d_{ne}(G)$[1].

For a near equitable dominating set D of G it is natural to look at how connected D behaves. For example, for the cycle $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$, $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_1, v_2, v_3, v_4\}$ are near equitable dominating sets, S_1 is not connected and S_2 is connected.

In this paper, we introduce the concept of a connected near equitable domination to initiate a study of a connected near equitable domination number and a connected near equitable domatic number.
We need the following to prove main results.

LEMMA 1.1. ([7]) *For any connected graph G of order n with maximum degree Δ,*

$$\gamma_c \geqslant \lfloor \frac{n}{\Delta+1} \rfloor.$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

DEFINITION 1.1. ([1]) Let $G = (V, E)$ be a graph and D be a near equitable dominating set of G. Then $u \in D$ is a near equitable pendant vertex if $od_D(u) = 1$. A set D is called a near equitable pendant set if every vertex in D is an equitable pendant vertex.

THEOREM 1.1. ([1]) *Let T be a wounded spider obtained from the star $K_{1,n-1}$, $n \geqslant 5$ by subdividing m edges exactly once. Then*

$$\gamma_{ne}(T) = \begin{cases} n, & \text{if } m = n-1\ ; \\ n-1, & \text{if } m = n-2; \\ n-2, & \text{if } m \leqslant n-3. \end{cases}$$

2. Connected Near Equitable Domination In Graphs

DEFINITION 2.1. A near equitable dominating set D of a graph G is said to be a connected near equitable dominating set if the subgraph $\langle D \rangle$ induced by D is connected. The minimum of the cardinalities of a connected near equitable dominating sets of G is called the connected near equitable domination number and is denoted by $\gamma_{cne}(G)$.

OBSERVATION 2.1. *For any connected graph G, $\gamma(G) \leqslant \gamma_{ne}(G) \leqslant \gamma_{cne}(G)$.*

OBSERVATION 2.2. *For any connected graph G, $\gamma_c(G) \leqslant \gamma_{cne}(G)$.*

OBSERVATION 2.3. *For any connected graph G, $\gamma_{cne}(G) = 1$ if and only if $\gamma_{ne}(G) = 1$.*

PROPOSITION 2.1. *A connected near equitable dominating set exists for a graph G if and only if G is connected.*

OBSERVATION 2.4. *There exist graphs for which the four parameters $\gamma(G)$, $\gamma_c(G)$, $\gamma_{ne}(G)$ and $\gamma_{cne}(G)$ are distinct. For graph G given in Fig.1, we have $\gamma(G) = 4$, $\gamma_c(G) = 8$, $\gamma_{ne}(G) = 12$ and $\gamma_{cne}(G) = 14$.*

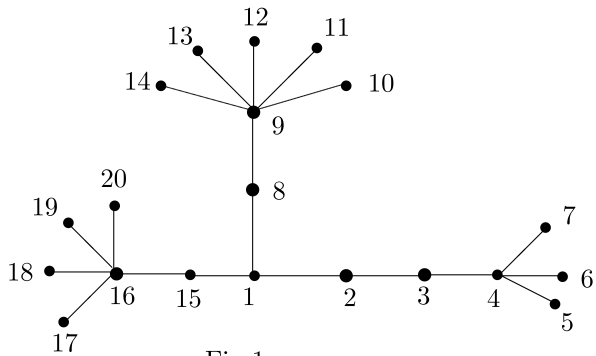

Fig 1

We now proceed to compute $\gamma_{cne}(G)$ for some standard graphs. It can be easily verified that

(1) For any path P_n, $n \geqslant 3$, $\gamma_{cne}(P_n) = \gamma_c(P_n) = n-2$.
(2) For any cycle C_n, $\gamma_{cne}(C_n) = \gamma_c(C_n) = n-2$.

(3) For any complete graph K_n, $\gamma_{cne}(K_n) = \gamma_{ne}(K_n) = \lfloor \frac{n}{2} \rfloor$, where $\lfloor x \rfloor$ is a greatest integer not exceeding x.
(4) For any star $K_{1,n}$, $\gamma_{cne}(K_{1,n}) = \gamma_{ne}(K_{1,n}) = n - 1$.
(5) For the double star $S_{n,m}$,

$$\gamma_{cne}(S_{n,m}) = \gamma_{ne}(S_{n,m}) = \begin{cases} 2, & \text{if } n, m \leqslant 2 \text{ ;} \\ n + m - 2, & \text{if } n, m \geqslant 2 \text{ and } n \text{ or } m \geqslant 3 \text{ .} \end{cases}$$

(6) For the complete bipartite graph $K_{n,m}$ with $1 < m \leqslant n$, we have

$$\gamma_{cne}(K_{n,m}) = \gamma_{ne}(K_{n,m}) = \begin{cases} m - 1, & \text{if } n = m \text{ and } m \geqslant 3 \text{ ;} \\ m, & \text{if } n - m = 1 \text{ ;} \\ n - 1, & \text{if } n - m \geqslant 2 \text{ .} \end{cases}$$

(7) For the wheel W_n on n vertices,

$$\gamma_{cne}(W_n) = \gamma_{ne}(W_n) = \lceil \frac{n-1}{3} \rceil + 1$$

.

THEOREM 2.1. *Let T be a wounded spider obtained from the star $K_{1,n-1}$, $n \geqslant 5$ by subdividing m edges exactly once. Then*

$$\gamma_{cne}(T) = \gamma_{ne}(T) = \begin{cases} n, & \text{if } m = n - 1 \text{ ;} \\ n - 1, & \text{if } m = n - 2; \\ n - 2, & \text{if } m \leqslant n - 3. \end{cases}$$

PROOF. Proof follows from Theorem 1.3 □

A vertex of a graph is said to be pendant if its neighborhood contains exactly one vertex. The vertex which is adjacent to the pendant vertex is called support vertex.

THEOREM 2.2. *Let T be a tree in which every non-pendant vertex is either a support or adjacent to a support and every non- pendant vertex which is support is adjacent to two pendant vertices. Then $\gamma_{cne}(T) = \gamma_{ne}(T) = \gamma_c(T)$.*

PROOF. Let D denote set of all non-pendant vertices of T. Clearly, D is a γ_c-set. Since the out degree of any vertex of D is at most two, it follows that D is a γ_{ne}-set. Since the induced subgraph $\langle D \rangle$ is connected, D a connected near equitable dominating set. Therefore $\gamma_{cne}(T) \leqslant \gamma_{ne}(T)$. But by Observation 2.1, $\gamma_{ne}(T) \leqslant \gamma_{cne}(T)$. Hence $\gamma_{cne}(T) = \gamma_{ne}(T)$. Since D is a γ_c-set, it follows that $\gamma_{cne}(T) \leqslant \gamma_c(T)$. But by Observation 2.2, $\gamma_c(T) \leqslant \gamma_{cne}(T)$. Hence $\gamma_{cne}(T) = \gamma_c(T)$. Thus $\gamma_{cne}(T) = \gamma_{ne}(T) = \gamma_c(T)$. □

THEOREM 2.3. *Let G be a connected graph such that $G = P_n$ or $G = C_n$. Then $\gamma_{cne}(G) = \gamma_{ne}(G)$ if and only if $n \leqslant 4$.*

COROLLARY 2.1. *For any connected graph G of order n, $n \leqslant 4$, $\gamma_{cne}(G) = \gamma_{ne}(G)$.*

THEOREM 2.4. *For any tree T with each support vertex adjacent to at least two pendant vertices, $\gamma_{cne}(T) = \gamma_{ne}(T)$ if and only if every non-pendant vertex is either a support or adjacent to a support.*

PROOF. Let T be a tree, and let D be a γ_{ne}-set. Suppose $\gamma_{cne}(T) = \gamma_{ne}(T)$, then D contains no isolated vertex. Since $\gamma_{cne}(T) = \gamma_{ne}(T)$ and each support vertex is adjacent to at least two pendant vertices, it follows that D contains all non-pendant vertices. Therefore every non-pendant vertex is either a support or adjacent to a support.
Conversely, the converse follows by Theorem 2.2. □

We now proceed to obtain a characterization of trees for which $\gamma_{cne}(T) = \gamma_{ne}(T)$.

THEOREM 2.5. *For any tree T, $\gamma_{cne}(T) = \gamma_{ne}(T)$ if and only if $T = P_n$, $n \leqslant 4$ or every non-pendant vertex of T is either a support or adjacent to a support and each support vertex is adjacent to at least two pendant vertices.*

A graph is said to be complete if the path length between any two distinct vertices is 1. Analogous to this definition we can define the near equitably complete graph as follows.

DEFINITION 2.2. A graph G is called a near equitably complete graph if for any near equitable dominating set D of G, $|od_D(u) - od_{V-D}(v)| \leqslant 1$, for all $u \in D$, and $v \in V - D$. Further, if the subgraph $\langle D \rangle$ induced by D is connected, then G is called a connected near equitably complete graph.

EXAMPLE 2.1. The standard graphs P_n, C_n, K_n and $K_{n,m}$ are near equitably complete and connected near equitably complete. But in the graph in view of Fig 2, $D = \{v_1, v_4, v_6\}$ is a near equitable dominating set and $D_1 = \{v_2, v_3, v_6\}$ is a connected near equitable dominating set. with respect to D and D_1, this graph is neither near equitably complete nor connected near equitably complete.

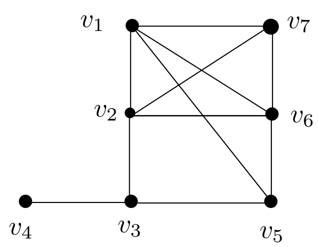

Fig 2

A vertex in a graph G having no edge incident on it is called an isolated vertex and the set of all isolated vertices of G is denoted by I_s. Analogously, we can define the isolated near equitable vertex as follows.

Definition 2.3. Let G be a graph and let D be a near equitable dominating set. $u \in D$ is called an isolated near equitable vertex if $|od_D(u) - od_{V-D}(v)| \geqslant 2$, for every vertex $v \in V - D$. $I_{oe}(D)$ be the set of all isolated near equitable vertices of D.

Example 2.2. The center of wheel W_n, $n \geqslant 8$ is an isolated near equitable vertex, with respect to any near equitable dominating set of W_n.

Proposition 2.2. *Let G be a graph, and let D be a near equitable dominating set. Then $I_s \subseteq I_{oe} \subset D$ if and only if for every vertex $v \in V - D$, $od_{V-D}(v) \geqslant 2$.*

Proof. Let D be a near equitable dominating set. Suppose for every vertex $v \in V - D$, $od_{V-D}(v) \geqslant 2$, then any isolated vertex of G is an isolated near equitable vertex. Therefor $I_s \subseteq I_{oe} \subset D$.
The converse is obvious. □

Proposition 2.3. *A near equitably complete graph G contains no isolated near equitable vertex.*

Remark 2.1. The converse of above Proposition is not true. The graph in view of Fig 2 is neither near equitably complete nor contains isolated near equitable vertex.

Proposition 2.4. *Let G be a connected graph such that the connected near equitable dominating set is a connected near equitable pendant dominating set. Then G is a connected near equitably complete.*

Theorem 2.6. *A tree is connected near equitably complete.*

Proof. Let T be a tree and let D be a connected near equitable dominating set of T, then every vertex $u \in D$ and $v \in V - D$, $od_D(u) \leqslant 2$ and $od_{V-D}(v) = 1$. Therefore $|od_D(u) - od_{V-D}(v)| \leqslant 1$. Hence T is a near connected equitably complete. □

Corollary 2.2. *A tree need not be near equitably complete, in view of Fig* 3.

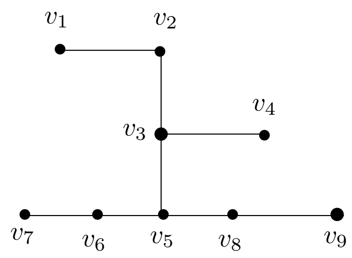

Fig 3

$D = \{v_2, v_3, v_6, v_8\}$ is a near equitable dominating set and $D_1 = \{v_2, v_3, v_5, v_6, v_8\}$ is a connected near equitable dominating set. The graph in Fig 3 is a connected near equitably complete but is not near equitably complete.

Definition 2.4. Let G be a graph. Then the near equitable dominating set D of G is called a 1- near equitable dominating set if for any $v \in V - D$ there exists exactly one vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leqslant 1$.

Example 2.3. A near equitable dominating set D of lK_2, $l \geqslant 1$ is a 1- near equitable dominating set.

Proposition 2.5. *A connected near equitable dominating set of a tree is a 1- near equitable dominating set.*

Proposition 2.6. *A near equitable dominating set of a near equitably complete graph G is not a 1- near equitable dominating set.*

3. Bounds

In this section, we present sharp bounds for $\gamma_{cne}(G)$.

Theorem 3.1. *Let G be a connected graph of order n, $n \geqslant 3$. Then $\gamma_{cne}(G) \leqslant n - 2$.*

Proof. It is enough to show that for any minimum connected near equitable dominating set D of G, $|V - D| \geqslant 2$. Since G is a connected graph, it follows that $\delta(G) \geqslant 1$. Suppose $v \in V - D$ and is adjacent to $u \in D$. Since $od_{V-D}(v) \geqslant 1$, then $od_D(u) \geqslant 2$. Hence the Theorem. □

The bound is sharp for $K_{1,n}$.

Theorem 3.2. *For any connected graph G of order n with maximum degree Δ,*

$$\lfloor \frac{n}{\Delta + 1} \rfloor \leqslant \gamma_{cne}(G) \leqslant n - 2.$$

Proof. Proof follows from Observation 2.2, Lemma 1.2 and Theorem 3.1. □

The bound is sharp for P_3.

Theorem 3.3. *For any graph G, $\gamma_{cne}(G) \leqslant 2m - n$.*

Proof. Let G be any (n, m) graph. Then by Theorem 3.1, $\gamma_{cne}(G) \leqslant n - 2 = 2(n-1) - n \leqslant 2m - n$. □

Theorem 3.4. *For any tree T, $\gamma_{cne}(T) \geqslant n - e$, where e is the number of pendant vertex.*

Proof. Let D be a γ_{cne}-set of T, then D contains all a non-pendant vertices of T and all a pendant vertices except two for each support vertex. Therefore $\gamma_{cne}(T) \geqslant n - e$. □

Corollary 3.1. *For any tree T, $\gamma_{cne}(T) = n - e$ if and only if any support vertex is adjacent to at most two pendant vertices.*

4. Connected Near Equitable Domatic Number

The maximum order of a partition of the vertex set V of a graph G into dominating sets is called the domatic number of G and is denoted by $d(G)$. For a survey of results on domatic number and their variants we refer to Zelinka [10]. In this section we present a few basic results on the connected near equitable domatic number of a graph.

Definition 4.1. Let G be a connected. A connected near equitable domatic partition of G is a partition $\{V_1, V_2, ..., V_k\}$ of $V(G)$ in which each V_i is a connected near equitable dominating set of G. The maximum order of a connected near equitable domatic partition of G is called the connected near equitable domatic number of G and is denoted by $d_{cne}(G)$.

Example 4.1. $\{\{1, 2\}, \{3, 4\}\}$ is a connected near equitable domatic partition of G.

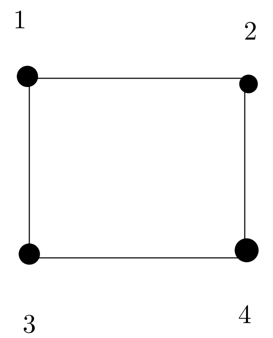

Fig 4

We now proceed to compute $d_{cne}(G)$ for some standard graphs. It can be easily verified that

(1) For any complete graph K_n, $n \geqslant 4$, $d_{cne}(K_n) = d_{ne}(K_n) = 2$.
(2) For any cycle C_n, $n \geqslant 5$, path P_n and star $K_{1,n}$, we have

$$d_{cne}(C_n) = d_{cne}(P_n) = d_{cne}(K_{1,n}) = 1$$

.

(3) For the complete bipartite graph $K_{n,m}$, we have

$$d_{cne}(K_{n,m}) = d_{ne}(K_{n,m}) = \begin{cases} 2, & \text{if } |n-m| \leqslant 2 \text{ ;} \\ 1, & \text{if } |n-m| \geqslant 3,\ n,m \geqslant 2. \end{cases}$$

It is obvious that any partition of V into connected near equitable dominating sets is also a partition of V into connected dominating set, and thus we obtain the obvious bound $d_{cne}(G) \leqslant d_c(G)$. Furthermore, the difference $d_c(G) - d_{cne}(G)$ can be arbitrarily large in a graph G as for the complete graph K_n, $n \geqslant 4$. It can be easily checked that $d_{cne}(K_n) = 2$, while $d_c(K_n) = n$.

THEOREM 4.1. *For any graph G, $d_{cne}(G) \leqslant d_{ne}(G) \leqslant d(G)$.*

PROOF. Let $G = (V, E)$ be a graph. Since any partition of V into connected near equitable dominating sets is also a partition of V into near equitable dominating set and any partition of V into near equitable dominating sets is also a partition of V into dominating sets, it follows that, $d_{cne}(G) \leqslant d_{ne}(G) \leqslant d(G)$. □

REMARK 4.1. Let $v \in V(G)$ and $deg(v) = \delta$. Since any connected near equitable dominating set of G must contain either v or a neighbour of v and $d_{cne}(G) \leqslant d_c(G)$, it follows that $d_{cne}(G) \leqslant \delta(G)$.

REMARK 4.2. Since every member of any connected near equitable domatic partition of a graph G on n vertices has at least $\gamma_{cne}(G)$ vertices, it follows that $d_{cne}(G) \leqslant \frac{n}{\gamma_{cne}(G)}$. This inequality can be strict for C_4.

THEOREM 4.2. *For any connected graph G of order n, $d_{cne}(G) \leqslant \frac{n}{2}$.*

PROOF. Let G be a connected graph of order n, $n \geqslant 2$. If $d_{cne}(G) = 1$, then $d_{cne}(G) \leqslant \frac{n}{2}$. If $\gamma_{cne}(G) \geqslant 2$, by Remark 4.2, $d_{cne}(G) \leqslant \frac{n}{2}$. □

THEOREM 4.3. *For any connected graph G of order n, $n \geqslant 3$,*

$$\gamma_{cne}(G) + d_{cne}(G) \leqslant n + \delta(G) - 2$$

.

PROOF. Since $d_{cne}(G) \leqslant d_c(G) \leqslant \delta(G)$, then by Theorem 3.1,

$$\gamma_{cne}(G) + d_{cne}(G) \leqslant n + \delta(G) - 2$$

. □

The bound is sharp for $K_{1,n}$.

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