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REFINEMENT OF INEQUALITY INVOLVING RATIO OF MEANS FOR FOUR POSITIVE ARGUMENTS

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Abstract. In this paper using Simpson's quadrature formula for convex function, the inequality involving ratio of means for four positive arguments proved by Rooin and Hassni (2007) is refined.

1. Introduction

The contributions of eminent researchers on the Arithmetic mean, Geometric mean, Logarithmic mean and Identric means are highlighted in the books [**1**,**3**] are respectively given by;

$$A(a,b) = \tfrac{a+b}{2}, \qquad G(a,b) = \sqrt{ab},$$
$$L(a,b) = \tfrac{a-b}{lna-lnb}, \qquad I(a,b) = e^{\left[\frac{alna-blnb}{a-b}-1\right]}.$$

In [**4**], Rooin and Hassni constructed some new inequalities between important means and applications to Ky Fan types inequalities as follows:

$$\frac{G(a,b)}{G(c,d)} \leqslant \frac{L(a,b)}{L(c,d)} \leqslant \frac{I(a,b)}{I(c,d)} \leqslant \frac{A(a,b)}{A(c,d)}, \text{where } a,b,c,d>0. \tag{1.1}$$

In this paper, the above inequality due to Rooin and Hassani [**4**] is refined by using Simpson's quadrature formula for convex function.

2. Application to refinement of ratio of means and its inequality

Let a,b,c,d are positive real numbers such that $b > a \geqslant c > d$, consider the double integral

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \frac{1}{xy}dxdy,$$

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and for $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ which is equivalent to (by the properties of definite integrals),

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \frac{1}{xy}dxdy = \frac{1}{(b-a)}\int_a^b \frac{1}{x}dx\frac{1}{(d-c)}\int_c^d \frac{1}{y}dy. \tag{2.1}$$

This paper is based on certain inequalities satisfied by the 4-convex functions [**1**–**3**, **5**]. That is the functions which are differentiable 4-times and $f^{(4)}(x) \geqslant 0$ for all values of x. Now recall the Simpson's quadrature formula in the form of lemma as below:

LEMMA 2.1. *If $f \in C^{(4)}([a,b])$ and $f^{(4)}(x) \geqslant 0$, then the mean value of f*

$$M(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

does not exceed the sum

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

that is

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^4}{2880}f^4(c),$$

for some $c \in (a,b)$.

Applying Lemma 2.1 in Eqn 2.1 gives that the value of the product of integral

$$\frac{1}{(b-a)}\int_a^b f(x)dx\frac{1}{(d-c)}\int_c^d f(y)dy$$

can not exceed the product of sums as given below:

$$\frac{1}{36}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\left[f(c) + 4f\left(\frac{c+d}{2}\right) + f(d)\right] \tag{2.2}$$

(**i**) Take $f(x) = \frac{1}{x^2}$ and $f(y) = \frac{1}{y^2}$. Then

$$\tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d \tfrac{1}{x^2y^2}dxdy,$$

and for $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ which is equivalent to

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \frac{1}{x^2y^2}dxdy = \frac{1}{(b-a)}\int_a^b \frac{1}{x^2}dx\frac{1}{(d-c)}\int_c^d \frac{1}{y^2}dy. \tag{2.3}$$

Applying Lemma 2.1 in Eqn 2.3 and on simple computations gives

(2.4)

$$\triangle_1 = \left[9\frac{G^4(a,b)A^2(a,b)A^2(a,b)G^2(c,d)A^2(c,d)}{[A^2(a,b)A(a^2,b^2) + 2G^4(a,b)]\,[A^2(c,d)A(c^2,d^2) + 2G^4(c,d)]}\right]^{\frac{1}{2}} \leqslant \frac{G(a,b)}{G(c,d)}$$

and

(2.5)
$$\frac{G(a,b)}{G(c,d)} \leqslant \left[\frac{1}{9}\frac{A^2(a,b)A(a^2,b^2)+2G^4(a,b)}{G^2(a,b)A^2(a,b)}\frac{A^2(c,d)A(c^2,d^2)+2G^4(c,d)}{G^4(c,d)A^2(c,d)}\right]^{\frac{1}{2}} = \triangle_2.$$

(**ii**) Takeing $f(x)=\frac{1}{x}$ and $f(y)=\frac{1}{y}$ on simple computations gives

$$\frac{1}{(b-a)}(lnb-lna)\frac{1}{(d-c)}(lnd-lnc) \leqslant \frac{1}{6}\left[\frac{a+b}{ab}+\frac{4}{A(a,b)}\right]\frac{1}{6}\left[\frac{c+d}{cd}+\frac{4}{A(c,d)}\right]$$
$$\frac{1}{L(a,b)L(c,d)} \leqslant \frac{1}{9}\left[\frac{A(a,b)}{G^2(a,b)}+\frac{2}{A(a,b)}\right]\left[\frac{A(c,d)}{G^2(c,d)}+\frac{2}{A(c,d)}\right]$$

further simplified to:

(2.6)
$$\triangle_3 = 9\left[\frac{G^2(a,b)A(a,b)}{A^2(a,b)+2G^2(a,b)}\right]\left[\frac{G^2(c,d)A(c,d)}{L^2(a,b)A^2(a,b)+2L^2(a,b)G^2(a,b)}\right] \leqslant \frac{L(a,b)}{L(c,d)}$$

and

(2.7)
$$\frac{L(a,b)}{L(c,d)} \leqslant \frac{1}{9}\left[\frac{L^2(a,b)A^2(a,b)+2L^2(a,b)G^2(a,b)}{G^2(a,b)A(a,b)}\right]\left[\frac{A^2(c,d)+2G^2(c,d)}{G^2(c,d)A(c,d)}\right] = \triangle_4.$$

(**iii**) Take $f(x)=lnx$ and $f(y)=lny$. Then

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d lnx \quad lny \quad dxdy$$

and for $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ which is equivalent to:

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d lnxlnydxdy = \frac{1}{(b-a)}\int_a^b lnxdx\frac{1}{(d-c)}\int_c^d lnydy. \tag{2.8}$$

Applying Lemma 2.1 in Eqn 2.8 and on simple computations gives,

$$\triangle_5 = \left[9\frac{I^2(a,b)}{A^2(a,b)G(a,b)A^2(c,d)G(c,d)}\right]^{\frac{1}{3}} \leqslant \frac{I(a,b)}{I(c,d)} \tag{2.9}$$

and

$$\frac{I(a,b)}{I(c,d)} \leqslant \left[\frac{1}{9}\frac{A^2(a,b)G(a,b)A^2(c,d)G(c,d)}{I^2(c,d)}\right]^{\frac{1}{3}} = \triangle_6 \tag{2.10}$$

On rearranging the Equations 2.9 and 2.10 gives,

$$\triangle_7 = \left[9\frac{I(a,b)I(c,d)}{G^{\frac{1}{3}}(a,b)G^{\frac{1}{3}}(c,d)A^{\frac{4}{3}}(a,b)}\right]^{\frac{3}{2}} \leqslant \frac{A(a,b)}{A(c,d)} \tag{2.11}$$

and

$$\frac{A(a,b)}{A(c,d)} \leqslant \left[\frac{1}{9}\frac{G^{\frac{1}{3}}(a,b)G^{\frac{1}{3}}(c,d)A^{\frac{4}{3}}(a,b)}{I(a,b)I(c,d)}\right]^{\frac{3}{2}} = \triangle_8. \tag{2.12}$$

From the above observations the following inequalities holds:

THEOREM 2.1. *From the above notations,the following are holds.*

(1) $\triangle_1 \leqslant \frac{G(a,b)}{G(c,d)} \leqslant Max\{\triangle_2, \triangle_3\}$

(2) $Min\{\triangle_2, \triangle_3\} \leqslant \frac{L(a,b)}{L(c,d)} \leqslant Max\{\triangle_4, \triangle_5\}$

(3) $Min\{\triangle_4, \triangle_5\} \leqslant \frac{I(a,b)}{I(c,d)} \leqslant Max\{\triangle_6, \triangle_7\}$

(4) $Min\{\triangle_6, \triangle_7\} \leqslant \frac{A(a,b)}{A(c,d)} \leqslant \triangle_8$.

The Theorem 2.1 proves the refinement and sharpening of the inequality 1.1.

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