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ISOLATE DOMINATION NUMBER AND MAXIMUM DEGREE

I.Sahul Hamid and S.Balamurugan

ABSTRACT. A subset D of the vertex set $V(G)$ of a graph G is called a *dominating set* of G if every vertex in $V - D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set is called *the domination number* and is denoted by $\gamma(G)$. A dominating set D such that $\delta(< D >) = 0$ is called an *isolate dominating set*. The minimum cardinality of an isolate dominating set is called *the isolate domination number* and is denoted by $\gamma_0(G)$. In this paper we investigate the properties of the graphs for which $\gamma_0 = n - \Delta$

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to the book by Chartrand and Lesniak [**1**]. All graphs in this paper are assumed to be non-trivial. The *open neighbourhood* of a vertex $v \in V$ is $N(v) = \{x \in V : vx \in E\}$, that is the set of all vertices adjacent to v. The *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$. If $S \subseteq V$ and $v \in S$, then a vertex u in V is said to be a *private neighbour* of v with respect to S if $N(u) \cap S = \{v\}$. The subgraph induced by a set $S \subseteq V$ is denoted by $< S >$. A *wounded spider* is a tree obtained from a star $K_{1,t}$ where $t \geqslant 1$, by subdividing at most $t - 1$ edges of the star.

One of the fastest growing areas within graph theory is the study of domination. A *dominating set* in a graph G is a set S of vertices of G such that any vertex in $V - S$ is adjacent to a vertex in S and the minimum cardinality of a dominating set in G is called the *domination number* which is denoted by $\gamma(G)$. In fact, there are scores of graph theoretic concepts involving domination, covering and independence. The

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bibliography in domination maintained by Haynes et. al. [**3**] currently has over 1200 entries in which one can find an appendix listing some 75 different types of domination and domination related parameters which have been studied in the literature. Hedetniemi and Laskar [**4**] edited an issue of Discrete Mathematics devoted entirely to domination, and a survey of advanced topics in domination is given in the book by Haynes et.al. [**2**].

Neverthless, despite the many variations possible, we can so far identify only a limited number of basic domination parameters; 'basic' in the sense that they are defined for every non-trivial connected graph. For instance independent domination, connected domination, total domination, global domination and acyclic domination are some basic domination parameters. In this sequence, the notion of isolate domination was introduced in [**5**] as a new such basic domination parameter. An *isolate dominating* set of a graph G is a dominating set S of G such that $\delta(< S >) = 0$ and the *isolate domination number*, denoted by $\gamma_0(G)$, is the minimum cardinality of an isolate dominating set of G. In this paper, we extend the study of this new variation of parameter by establishing a relation between the isolate domination number and the maximum degree of a graph G.

2. Graphs which satisfy $\gamma_0(G) + \Delta(G) = |V(G)|$

It is obvious that for any vertex v in a graph G, the set $V - N(v)$ is always an isolate dominating set of G which inturn implies that $\gamma_0(G) \leqslant n - \Delta(G)$. In this section, we investigate the properties of graphs for which the equality holds and also characterize the family of triangle-free graphs attaining the bound. For this purpose, we introduce the following notations.

For a vertex v in a graph G, we define N_v to be the set of all vertices in G which are not adjacent to v, that is $N_v = V - N[v]$. A vertex x in $N(v)$ is said to be a *major vertex* if x is adjacent to all the vertices of N_v. A vertex of $N(v)$ which is not a major vertex is called a *non-major vertex.* We also define the following.

(i) $M(v)$ is the set of all major vertices of G with respect to v.
(ii) $M'(v)$ is the set of all non-major vertices of G with respect to v.
(iii) For a vertex $y \in N_v$, $M_v(y) = N(y) \bigcap M(v)$ and $M'_v(y) = N(y) \bigcap M'(v)$.
(iv) $m(v) = |M(v)|$ and $m_v(y) = |M_v(y)|$.
(v) $m'(v) = |M'(v)|$ and $m'_v(y) = |M'_v(y)|$.

Further, for a vertex v in G, we define the properties C_1, C_2, C_3 and C_4 as follows:

C_1: N_v is independent.
C_2: A vertex of $N(v)$ having more than one neighbour in N_v belongs to $M(v)$.
C_3: If $N(v)$ contains at least one pendant vertex of G, then no subset of N_v with fewer than $|N_v| - l$ vertices dominates $M'(v)$, where l denotes the number of pendant vertices of G in $N(v)$.
C_4: Suppose $N(v)$ contains no pendant vertices of G. Then for each $y \in N_v$,
 (i) $m'_v(y) \neq 1$.

(ii) When $M(v) \neq \phi$, if $M'(v) - M'_v(y) \neq \phi$ then no subset of N_v with fewer than $|N_v| - m'_v(y)$ vertices dominate $M'(v) - M'_v(y)$ and if $M'(v) - M'_v(y) = \phi$, then

$$m'_v(y) \geqslant \begin{cases} n - \Delta - 1 & if \text{m(v)=1} \\ n - \Delta - 2 & \text{otherwise} \end{cases}$$

THEOREM 2.1. *Let G be a graph on n vertices with $\gamma_0(G) = n - \Delta$. Then $\Delta \geqslant n/2$ and the conditions C_1 to C_3 hold for any vertex v of degree Δ.*

PROOF. If there exist two adjacent vertices x and y in N_v then $(N_v - \{x\}) \cup \{v\}$ will be an isolate dominating set of G with cardinality $n - \Delta - 1$, which is a contradiction and thus proving C_1.

We now proceed to prove that $\Delta \geqslant n/2$. Suppose $\Delta < n/2$. Let us first claim that any vertex in $N(v)$ has exactly one neighbour in N_v. Suppose there exists a vertex $u \in N(v)$ having at least two neighbours in N_v, say x and y. Since $\Delta < n/2$ it follows that $|N_v| \geqslant \Delta$ so that there exists a vertex z in N_v which is not adjacent to u. Hence the set $D = (N_v - \{x, y\}) \cup \{u, v\}$ is an isolate dominating set of cardinality less than or equal to $n - \Delta - 1$, contradicting $\gamma_0(G) = n - \Delta$ and thus every vertex in $N(v)$ has atmost one neighbour in N_v. Further, the set N_v contains at least Δ vertices as $\Delta < n/2$. In addition, we have also proved that N_v is independent. These observations imply the required claim. Now, it follows from the above claim together with the fact that N_v is independent that the sets N_v and $N(v)$ have equal number of vertices. Hence a vertex in $N(v)$ together with its non-neighbours in N_v form an isolate dominating set with cardinality $n - \Delta - 1$, this arrives at a contradiction. Hence $\Delta \geqslant n/2$.

We now prove C_2. Suppose there exists a vertex $x \in N(v)$ with more than one neighbour in N_v and also having a non-neighbour z in N_v. Then the set $D = (N_v - N(x)) \cup \{x, v\}$ is a dominating set in which z is an isolated vertex with $|D| = |N_v| - |N_v \cap N(x)| + 2 < |N_v| - 1 + 2 = |N_v| + 1 = n - \Delta$. This contradiction proves what we desired.

Let us now prove C_3. Suppose there exists a subset D of N_v with cardinality fewer than $|N_v| - l$ dominating $M'(v)$. Then $M(v) \neq \phi$, otherwise $< N_v - D >$ would be totally disconnected and consequently G is disconnected. So, the set D together with the l pendant vertices of G in $N(v)$ and a vertex in $M(v)$ form an isolate dominating set of cardinality $|D| + l + 1 < |N_v| - l + l + 1 = |N_v| + 1 = n - \Delta$, which is a contradiction. This proves C_3. $\square$

REMARK 2.1. The conditions stated in the above theorem are not sufficient for a graph G to have $n - \Delta$ as its isolate domination number. For example, the graph given in Figure 1 satisfies the conditions C_1 to C_3, however the value of γ_0 fr the graph is $4 < n - \Delta$.

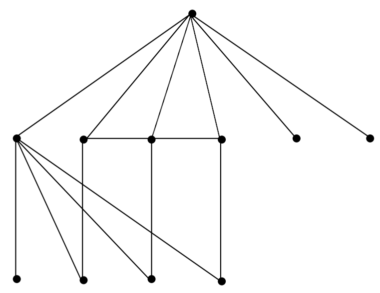

Figure 1: A graph G with $\gamma_0(G) < n - \Delta$ satisfying the conditions C_1 to C_3

Even if necessary and sufficient condition for a graph G for which $\gamma_0(G) = n-\Delta$ is not found, we could settle this for the family of triangle-free graphs which is of course a large family of graphs.

THEOREM 2.2. *Let G be a triangle free graph on n vertices. Then $\gamma_0(G) = n-\Delta$ if and only if $\Delta \geqslant n/2$ and the conditions C_1 to C_4 hold for any vertex v of degree Δ.*

PROOF. Suppose $\gamma_0(G) = n - \Delta$. The conditions C_1 to C_3 have already been proved in theorem 2.1 and therefore it is sufficient to prove the condition C_4 alone. First, let us prove $C_4(i)$. Suppose there is a vertex y in N_v such that $m'_v(y) = 1$. Since G is triangle-free $< N(v) >$ is independent. Also as $N(v)$ contains no pendant vertices of G, every vertex of $N(v)$ must have a neighbour in N_v. Therefore the set $D = (N_v - \{y\}) \cup M'_v(y)$ would be a dominating set of G in which the vertex w of $M'_v(y)$ is an isolated vertex. Hence the set D is an isolate dominating set of G of cardinality $|N_v| < n - \Delta$. This contradiction proves $C_4(i)$.

Now, let $M(v) \neq \phi$ and $y \in N_v$. If $M'(v) - M'_v(y) \neq \phi$ and there is a subset S of N_v with fewer then $|N_v| - m'_v(y)$ vertices dominating $M'(v) - M'_v(y)$ then the set $S \cup \{x\} \cup M'_v(y)$, for some $x \in M(v)$ will form an isolate dominating set of cardinality $|S| + 1 + m'_v(y) < |N_v| - m'_v(y) + 1 + m'_v(y) = |N_v| + 1$. This contradiction gives the required result. Now suppose $M'(v) - M'_v(y) = \phi$. If $m(v) = 1$, then the set $D = \{x\} \cup M'_v(y)$, where $x \in M(v)$ will form an isolate dominating set of cardinality $|D| = 1 + m'_v(y) \geqslant n - \Delta$. That is $m'_v(y) \geqslant n - \Delta - 1$, when $m(v) = 1$. If $m(v) > 1$ then D must contain at least one more vertex in addition to the existing vertices, to dominate the major vertices otherthan x. So that $|D| = 2 + m'_v(y) \geqslant n - \Delta$. Hence it follows that $m'_v(y) \geqslant n - \Delta - 2$ when $m(v) > 1$. Thus the condition $C_4(ii)$ is proved.

Conversely, suppose $\Delta \geqslant n/2$ and the conditions C_1 to C_4 hold in G. Let D be an isolate dominating set of G. We prove that D contains at least $n - \Delta$ vertices by considering the following cases.

Case 1. $M'(v) = \phi$

Suppose $D \cap N(v) = \phi$ then it follows from the condition C_1 that $N_v \subseteq D$ and also $v \in D$ so that $D = \{v\} \cup N_v$. Suppose $D \cap N(v) \neq \phi$. Then the vertex v cannot be an isolate in D. Further, since the vertices of $N(v)$ are major, it follows that no vertex of N_v can be an isolate in D. So that the isolates in D must be in $N(v)$. Hence D contains neither v nor the vertices of N_v so that $D = N(v)$. Therefore in either case $|D| \geqslant n - \Delta$.

Case 2. $M(v) \neq \phi$ and $M'(v) \neq \phi$

Suppose $v \in D$. Then $D \cap M(v) = \phi$, for otherwise no vertex of G can be an isolate in D. Hence every vertex in $D - \{v\}$ dominates at most one vertex of N_v and so $|D| \geqslant 1 + |N_v| = n - \Delta$. Assume $v \notin D$. Le l be the number of pendant vertices of G in $N(v)$.

Subcase 2.1. $l \geqslant 1$

If $D \cap M(v) = \phi$ then at least $|N_v|$ vertices are required to dominate N_v. Also, D must contain all the pendant vertices of G in $N(v)$ as $v \notin D$ and so $|D| \geqslant l + |N_v| = l + n - \Delta - 1 \geqslant n - \Delta$.

Now suppose $D \cap M(v) \neq \phi$. Let us now see the minimum number of vertices required to dominate the non-major vertices (in $N(v)$) which are not pendants. Even if we use only the vertices in N_v to dominate such non-major vertices, condition C_3 says, at least $|N_v| - l$ vertices are needed and obviously the other ways of dominating those vertices require more than what we needed when only the vertices in N_v are used. Also, D must contain all the pendant vertices. Thus D consists of a major vertex together with the $|N_v| - l$ vertices of N_v and all the pendant vertices of G in $N(v)$ so that $|D| \geqslant 1 + |N_v| - l + l = n - \Delta$.

Subcase 2.2. $l = 0$

Suppose $D \cap M(v) = \phi$. Then as $v \notin D$, at least one vertex of $M'(v)$, say x, must be selected to dominate the vertex v. Let $N(x) \cap N_v = \{\text{y}\}$. Then $m'_y(v) \geqslant 1$. Further, by the condition $C_4(i)$, $m'_v(y) \neq 1$ and hence it follows that $m'_v(y) \geqslant 2$. Also, since $D \cap M(v) = \phi$, at least $|N_v| - 1$ vertices are needed to dominate $N_v - \{y\}$. Until now, the set D contains the vertex x together with those $|N_v| - 1$ vertices dominating $N_v - \{y\}$. If $y \in D$ then we proved. If $y \notin D$, then all the vertices of $m'_v(y)$ must be selected. Therefore, $|D| \geqslant |N_v| - 1 + m'_v(y) \geqslant |N_v| + 1 \geqslant n - \Delta$.

Suppose $u \in D \cap M(v)$. If one of the major vertices with respect to v in D is an isolate in D then $D = N(v)$, so that $|D| = \Delta \geqslant n - \Delta$ as $\Delta \geqslant n/2$. Now assume that no vertex of $M(v) \cap D$ is an isolate in $< D >$. Let x be an isolate in D. Then $x \in M'(v)$ and it has exactly one neighbour, say y, in N_v being $l = 0$. Obviously $y \notin D$. Therefore $M'_v(y) \subseteq D$ and hence $|D| \geqslant 1 + m'_v(y)$. If $M'(v) - M'_v(y) \neq \phi$, it follows from the condition $C_4(ii)$ that at least $|N_v| - m'_v(y)$ vertices are needed to dominate the vertices of $M'(v) - M'_v(y)$. Thus, $|D| \geqslant 1 + m'_v(y) + |N_v| - m'_v(y) = 1 + |N_v| \geqslant n - \Delta$. Suppose $M'(v) - M'_v(y) = \phi$. Now, if $m(v) = 1$, the condition $C_4(ii)$ implies that $m'_v(y) \geqslant n-\Delta-1$ so that $|D| \geqslant 1+m'_v(y) \geqslant 1+n-\Delta-1 = n-\Delta$. If $m(v) \geqslant 2$, again by $C_4(ii)$ we have $m'_v(y) \geqslant n-\Delta-2$. Also as $m(v) \geqslant 2$, we need at least one more vertex to dominate the vertices of $M(v)-\{u\}$, where $u \in D \cap M(v)$. Hence $|D| \geqslant 2 + m'_v(y) \geqslant n - \Delta$.

Case 3. $M(v) = \phi$

It follows from the condition C_1 that N_v is independent. Moreover by the condition C_2 a vertex of $N(v)$ can dominate atmost one vertex of N_v. Therefore at least $|N_v|$ vertices are necessary to dominate N_v. Hence if $v \in D$ then $|D| \geqslant |N_v| + 1 = n - \Delta$. Suppose $v \notin D$. If $N(v)$ has a pendant vertex then D must contain the pendant vertices of $N(v)$ together with vertices dominating N_v. Thus $|D| \geqslant |N_v|+1 = n-\Delta$. Suppose $N(v)$ has no pendant vertices. As $v \notin D$, D must contain a vertex of $N(v)$, say x, to dominate v. Let $N(x) \bigcap N_v = \{y\}$. Since $M(v) = \phi$, it follows from the conditions C_1 and C_2 that at least $|N_v|-1$ vertices are required to dominate $N_v - \{y\}$. Now D contains the vertex x together with those $|N_v|-1$ vertices dominating $N_v-\{y\}$. If $y \in D$ then $|D| \geqslant |N_v|-1+2 = n-\Delta$. Suppose $y \notin D$. Since G is triangle free and N_v is independent, $M'_v(y) \subseteq D$. Also by the condition $C_4(i)$, $m'_v(y) \neq 1$ and therefore $|D| \geqslant |N_v|-1+m'_v(y) \geqslant |N_v|+1 = n-\Delta$.

Thus every isolate dominating set of G contains at least $n - \Delta$ vertices and so $\gamma_0(G) \geqslant n - \Delta$. □

Remark 2.2. The following points can easily be observed from the above theorem.

(i) As G is triangle-free, $N(v)$ is independent, for any vertex v of degree Δ.
(ii) By Condition C_1, N_v is independent.

Hence the set of graphs characterized in the above theorem are the subclass of connected bipartite graphs.

Corollary 2.1. *Let T be a tree on n vertices. Then $\gamma_0(T) = n - \Delta$ if and only if T is either a wounded spider or a double star whose supports are of degree Δ and $n - \Delta - 1$ respectively.*

Proof. Suppose T is a tree with $\gamma_o(T) = n - \Delta$. Then for vertex v of degree Δ in T, $N(v)$ is independent. Also by the condition C_1, N_v is independent. If $N_v = \phi$ then T is a star which is obviously a wounded spider. Suppose $N_v \neq \phi$. Then by the condition C_1, every vertex of N_v has a neighbour in $N(v)$. Since T is acyclic, $m(v) \leqslant 1$. If $m(v) = 0$, then T is a wounded spider, being $\Delta \geqslant \frac{n}{2}$. Further, if $m(v) = 1$, then all the non-major vertices with respect to v are pendant vertice and hence T is a double star whose supports are of degree Δ and $n - \Delta - 1$ respectively. The Converse is just a verification. □

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DEPARTMENT OF MATHEMATICS, THE MADURA COLLEGE, MADURAI-625 011, INDIA
E-mail address: sahulmat@yahoo.co.in

DEPARTMENT OF MATHEMATICS, ST. XAVIER'S COLLEGE, PALAYAMKOTTAI - 627 002, INDIA
E-mail address: balamaths@rocketmail.com