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# A COMMON FIXED POINT THEOREMS FOR TWO PAIRS OF WEAKLY COMPATIBLE MAPPINGS THROUGH GENERALIZED ALTERING DISTANCE FUNCTION

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Abstract. The main aim of this paper is to extend the results of Nguyen Van Luong and Nguyen Xuan Thuan [4] on fixed point theory. Incidently, we observed some inconsistencies in their proof and example. We extended their results and exhibited a supporting example.

## 1. Introduction

M.S.Khan et.al. [3] introduced the concept of 'Altering distance function' and used it for solving fixed point problems in metric space. Generalizing this, Choudary obtained a fixed point theorem for a pair of self maps in a complete metric space. K.P.R.Rao et.al. [5] extended the main Theorem of [1].

Nguyen Van Luong and Nguyen Xuan Thuan [4] considered two common fixed theorems for pairs of weakly compatible maps by using a generalized altering distance function of five variables.

In this paper, we have extended and generalized their results and also pointed out some of the inconsistencies in the proof of their main theorem.

Now, we here under give the necessary definitions and results for the development of our Theorem. For other standard definitions and results, we refer [2] and [6].

Definition 1.1. ([6]) Self maps $f$ and $g$ on a metric space $(X, d)$ are said to be weakly compatible if and only if $(iff)$ $f$ and $g$ commutes at their coincident points in $X$.

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*Key words and phrases.* common fixed point; weakly compatible mappings; generalized altering distance function.

NOTATION 1.1. For any integer $k \geqslant 2$, let $\Phi_k$ denote the set of all functions $\phi : [0,\infty)^k \to [0,\infty)$ such that

(i) $\phi$ is continuous (on its domain),
(ii) $\phi$ is monotonic increasing in all its variables,
(iii) for any $t_1, t_2, \cdots, t_k \in [0,\infty)$,
$\phi(t_1, t_2, \cdots, t_k) = 0 \Leftrightarrow t_1 = t_2 = \cdots = t_k = 0$.

DEFINITION 1.2. ([1]) Any function $\phi \in \Phi_k$(introduced above) is said to be a generalized altering distance function.

THEOREM 1.1 (Theorem (2.1) of [**4**]). *Let A, B, S and T be self mappings of a metric space $(X,d)$ such that*

(i) $\psi\binom{d(Ax,By),}{max\{d(Ax,Sx),d(By,Ty),d(Sx,Ty)\}}$

$$\leqslant \phi_1(d(Ax,Sx), d(By,Ty), d(Sx,Ty), d(Ax,Ty), d(By,Sx))$$

$$-\phi_2(d(Ax,Sx), d(By,Ty), d(Sx,Ty), d(Ax,Ty), d(By,Sx)),$$

*for all $x, y \in X$, where $\phi_1, \phi_2 \in \Phi_5$, $\psi : [0,\infty)^2 \to \mathbb{R}$ is continuous and $\psi(u,u) = \phi_1(u,u,u,u,2u)$, $\forall u \in [0,\infty)$;*
(ii) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*
(iii) *the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible;*
(iv) *One of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of $X$.*

*Then A, B, S and T have a unique common fixed point, say u. Moreover, u is the unique common fixed point of A & S as well as B & T.*

## 2. Main Results

THEOREM 2.1. *Let A, B, S and T be self mappings of a metric space $(X,d)$ such that*

(i) $\psi\binom{d(Ax,By),}{max\{d(Ax,Sx),d(By,Ty),d(Sx,Ty)\}}$

$$\leqslant \phi_1\binom{d(Ax,Sx),d(By,Ty),d(Sx,Ty),\frac{1}{2}[d(Ax,Ty)+d(By,Sx)],}{\frac{1}{3}[d(Ax,Sx)+d(By,Ty)+d(Sx,Ty)]}$$

$$-\phi_2\binom{d(Ax,Sx),d(By,Ty),d(Sx,Ty),\frac{1}{2}[d(Ax,Ty)+d(By,Sx)],}{\frac{1}{3}[d(Ax,Sx)+d(By,Ty)+d(Sx,Ty)]}$$

*for all $x, y \in X$, where $\phi_1, \phi_2 \in \Phi_5$, $\psi : [0,\infty)^2 \to [0,\infty)$ is continuous and $\psi(u,u) = \phi_1(u,u,u,u,u)$, $\forall u \in [0,\infty)$;*
(ii) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*
(iii) *One of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of $X$;*
(iv) *the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible.*

*Then A, B, S and T have a unique common fixed point, say z. Further, z is the unique common fixed point of A &S as well as B &T.*

PROOF. Let $x_0 \in X$. By (ii) construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n}(say)$$

and

$$Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}(say), for \quad n = 0, 1, 2, \ldots$$

Let $a_n = d(y_n, y_{n+1})$.
Taking $x = x_{2n}$ and $y = x_{2n+1}$ in $(i)$ we get that

$$\psi\binom{d(y_{2n},y_{2n+1}),}{max\{d(y_{2n},y_{2n-1}),d(y_{2n+1},y_{2n}),d(y_{2n-1},y_{2n})\}}$$

$$\leqslant \quad \phi_1\binom{d(y_{2n},y_{2n-1}),d(y_{2n+1},y_{2n}),d(y_{2n-1},y_{2n}),\frac{1}{2}[d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})],}{\frac{1}{3}[d(y_{2n},y_{2n-1})+d(y_{2n+1},y_{2n})+d(y_{2n-1},y_{2n})]}$$

$$-\phi_2\binom{d(y_{2n},y_{2n-1}),d(y_{2n+1},y_{2n}),d(y_{2n-1},y_{2n}),\frac{1}{2}[d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})],}{\frac{1}{3}[d(y_{2n},y_{2n-1})+d(y_{2n+1},y_{2n}+)+d(y_{2n-1},y_{2n})]}$$

i.e.

$$\psi\binom{a_{2n},}{max\{a_{2n-1},a_{2n},a_{2n-1}\}}$$

$$\leqslant \quad \phi_1\binom{a_{2n-1},a_{2n},a_{2n-1},\frac{1}{2}[d(y_{2n+1},y_{2n-1})],}{\frac{1}{3}[a_{2n-1}+a_{2n}+a_{2n-1}]}$$

$$-\phi_2\binom{a_{2n-1},a_{2n},a_{2n-1},\frac{1}{2}[d(y_{2n+1},y_{2n-1})],}{\frac{1}{3}[a_{2n-1}+a_{2n}+a_{2n-1}]} \qquad (2.1.1)$$

If $a_{2n-1} < a_{2n}$ then

$$\begin{aligned}
\psi(a_{2n}, a_{2n}) \quad &\leqslant \quad \phi_1(a_{2n}, a_{2n}, a_{2n}, a_{2n}, a_{2n}) \\
&\quad -\phi_2\binom{a_{2n-1},a_{2n},a_{2n-1},\frac{1}{2}[d(y_{2n+1},y_{2n-1})],}{\frac{1}{3}[a_{2n-1}+a_{2n}+a_{2n-1}]} \\
&\leqslant \quad \phi_1(a_{2n}, a_{2n}, a_{2n}, a_{2n}, a_{2n}) \\
&\quad -\phi_2(a_{2n-1}, a_{2n}, a_{2n-1}, 0, \tfrac{1}{3}[a_{2n-1} + a_{2n} + a_{2n-1}]) \\
&< \quad \psi(a_{2n}, a_{2n})
\end{aligned}$$

which is a contradiction, since $a_{2n} > 0$. Thus $a_{2n} \leqslant a_{2n-1}$.

Similarly, by taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in $(i.)$ and proceeding se above, we get that $a_{2n+1} \leqslant a_{2n}$. Hence $\{a_n\}$ is a decreasing sequence of non-negative real numbers and so converges to some $a \geqslant 0$.

Now, from (2.1.1), we have

$$\begin{aligned}
\psi(a_{2n}, a_{2n-1}) \quad &\leqslant \quad \phi_1\binom{a_{2n-1},a_{2n},a_{2n-1},\frac{1}{2}[a_{2n}+a_{2n-1}],}{\frac{1}{3}[a_{2n-1}+a_{2n}+a_{2n-1}]} \\
&\quad -\phi_2(a_{2n-1}, a_{2n}, a_{2n-1}, 0, \tfrac{1}{3}[a_{2n-1} + a_{2n} + a_{2n-1}]).
\end{aligned}$$

Letting $n \to \infty$, we get that
$\psi(a, a) \leqslant \phi_1(a, a, a, a, a) - \phi_2(a, a, a, 0, a)$. By the property of $\psi$, we have

$\phi_2(a, a, a, 0, a) = 0 \Leftrightarrow a = 0$ (by the property of $\phi_2$).
i.e,

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \qquad (2.1.2)$$

Now, we show that $\{y_n\}$ is a Cauchy sequence (in $X$). In view of (2.1.2), it is sufficient to show that the subsequence $\{y_{2n}\}$ of $\{y_n\}$ is Cauchy. Suppose not; there exists an $\epsilon > 0$ and subsequences $\{y_{2n(k)}\}$ and $\{y_{2m(k)}\}$ such that $n(k) > m(k) \geqslant k$ and

$$d(y_{2m(k)}, y_{2n(k)}) \geqslant \epsilon \qquad (2.1.3)$$

Further, we can assume that

$$d(y_{2m(k)}, y_{2n(k)-2}) < \epsilon \qquad (2.1.4)$$

(by choosing $n(k)$ to be the smallest number exceeding $m(k)$ for which (2.1.3) holds).

Now,

$$\begin{aligned} \epsilon &\leqslant d(y_{2m(k)}, y_{2n(k)}) \\ &\leqslant d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \\ &< \epsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}). \end{aligned}$$

Letting $k \to \infty$ and using (2.1.2), we get that

$$\lim_{n\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \epsilon \qquad (2.1.5)$$

Further, we have

$$d(y_{2n(k)}, y_{2m(k)-1}) \leqslant d(y_{2n(k)}, y_{2m(k)}) + d(y_{2m(k)}, y_{2m(k)-1})$$

and

$$d(y_{2n(k)}, y_{2m(k)}) \leqslant d(y_{2n(k)}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).$$

Letting $k \to \infty$ in the above inequalities and using (2.1.2) and (2.1.5), we get that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) = \epsilon \qquad (2.1.6)$$

Similarly, we get that

$$\lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}) = \epsilon \qquad (2.1.7)$$

and

$$\lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}) = \epsilon \qquad (2.1.8).$$

Taking $x = x_{2m(k)}$ and $y = x_{2n(k)+1}$ in (i), we get that

$$\psi\binom{d(y_{2m(k)}, y_{2n(k)+1}),}{max\{d(y_{2m(k)}, y_{2m(k)-1}), d(y_{2n(k)+1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2n(k)})\}}$$

$$\leqslant \quad \phi_1\binom{\binom{d(y_{2m(k)}, y_{2m(k)-1}), d(y_{2n(k)+1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2n(k)}),}{\frac{1}{2}[d(y_{2m(k)}, y_{2n(k)})+d(y_{2n(k)+1}, y_{2m(k)-1})]},}{\frac{1}{3}[d(y_{2m(k)}, y_{2m(k)-1})+d(y_{2n(k)+1}, y_{2n(k)})+d(y_{2m(k)-1}, y_{2n(k)})]}$$

$$-\phi_2\binom{\binom{d(y_{2m(k)}, y_{2m(k)-1}), d(y_{2n(k)+1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2n(k)}),}{\frac{1}{2}[d(y_{2m(k)}, y_{2n(k)})+d(y_{2n(k)+1}, y_{2m(k)-1})],}}{\frac{1}{3}[d(y_{2m(k)}, y_{2m(k)-1})+d(y_{2n(k)+1}, y_{2n(k)})+d(y_{2m(k)-1}, y_{2n(k)})]}$$

Letting $k \to \infty$ and using (2.1.2),(2.1.6)(2.1.7) and (2.1.8), we get that

$$\begin{aligned}\psi(\epsilon, \epsilon) = \psi(\epsilon, max\{0, 0, \epsilon\}) &\leqslant \phi_1(0, 0, \epsilon, \epsilon, \tfrac{1}{3}\epsilon) - \phi_2(0, 0, \epsilon, \epsilon, \tfrac{1}{3}\epsilon) \\ &\leqslant \phi_1(\epsilon, \epsilon, \epsilon, \epsilon, \epsilon) - \phi_2(0, 0, \epsilon, \epsilon, \tfrac{1}{3}\epsilon) \\ &< \psi(\epsilon, \epsilon)\end{aligned}$$

which is a contradiction, since $\epsilon > 0$. Thus $\{y_{2n}\}$ is a Cauchy sequence and hence $\{y_n\}$ is a Cauchy sequence (in $X$).

**Case I:** Suppose $A(X)$ or $T(X)$ is a complete subspace of $X$.

Since $\{y_{2n}\} \subseteq A(X)(\subseteq T(X))$, there is a $z \in X$ such that $\{y_{2n}\} \to z$ as $n \to \infty$. $\Rightarrow \{y_n\} \to z$ as $n \to \infty$. (further, it follows that $\{y_{2n+1}\} \to z$ as $n \to \infty$).
Since $A(X) \subseteq T(X)$, there is a $v \in X$ such that $z = Tv$.

By taking $x = x_{2n}$ and $y = v$ in $(i.)$ we get that

$$\psi\binom{d(y_{2n}, Bv),}{max\{d(y_{2n}, y_{2n-1}), d(Bv,z), d(y_{2n-1}, z)\}}$$

$$\leqslant \quad \phi_1\binom{d(y_{2n}, y_{2n-1}), d(Bv, z), d(y_{2n-1}, z), \frac{1}{2}[d(y_{2n}, z)+d(Bv, y_{2n-1})],}{\frac{1}{3}[d(y_{2n}, y_{2n-1})+d(Bv,z)+d(y_{2n-1},z)]}$$

$$-\phi_2\binom{d(y_{2n},y_{2n-1}),d(Bv,z),d(y_{2n-1},z),\frac{1}{2}[d(y_{2n},z)+d(Bv,y_{2n-1})],}{\frac{1}{3}[d(y_{2n}, y_{2n-1})+d(Bv, z)+d(y_{2n-1}, z)]}$$

Letting $n \to \infty$, we get that

$$\begin{aligned}\psi\binom{d(z, Bv),}{max\{0, d(Bv, z), 0\}} &\leqslant \phi_1(0,\ d(Bv,\ z),\ 0,\ \tfrac{1}{2}d(Bv,\ z),\ \tfrac{1}{3}d(Bv,\ z)) \\ &\quad -\phi_2(0,\ d(Bv,\ z),\ 0,\ \tfrac{1}{2}d(Bv,\ z),\ \tfrac{1}{3}d(Bv,\ z)) \\ \psi(d(z, Bv), d(Bv, z)) &\leqslant \phi_1(d(Bv, z), d(Bv, z), d(Bv, z), d(Bv, z), d(Bv, z)) \\ &\quad -\phi_2(0, d(Bv, z), 0, \tfrac{1}{2}d(Bv, z), \tfrac{1}{3}d(Bv, z))\end{aligned}$$

$\Rightarrow Bv = z$. Thus $Bv = z = Tv$.

Since $\{B,T\}$ is weakly compatible, $BTv = TBv$. i.e, $Bz = Tz$.

By taking $x = x_{2n}$ and $y = z$ in $(i.)$ we get that

$$\psi\big({}^{d(y_{2n},Bz),}_{max\{d(y_{2n},y_{2n-1}),d(Bz,Bz),d(y_{2n-1},Bz)\}}\big)$$

$$\leqslant \quad \phi_1\big({}^{d(y_{2n},y_{2n-1}),d(Bz,Bz),d(y_{2n-1},Bz),\frac{1}{2}[d(y_{2n},Bz)+d(Bz,y_{2n-1})],}_{\frac{1}{3}[d(y_{2n},y_{2n-1})+d(Bz,Bz)+d(y_{2n-1},Bz)]}\big)$$

$$-\phi_2\big({}^{d(y_{2n},y_{2n-1}),d(Bz,Bz),d(y_{2n-1},Bz),\frac{1}{2}[d(y_{2n},Bz)+d(Bz,y_{2n-1})],}_{\frac{1}{3}[d(y_{2n},y_{2n-1})+d(Bz,Bz)+d(y_{2n-1},Bz)]}\big)$$

Letting $n \to \infty$, we get that

$$\begin{aligned}\psi(d(z,Bz),d(Bz,z)) &\leqslant \phi_1(0,0,d(Bz,z),d(Bz,z),\tfrac{1}{3}d(Bz,z))\\ &\quad -\phi_2(0,0,d(Bz,z),d(Bz,z),\tfrac{1}{3}d(Bz,z))\\ &\leqslant \phi_1(d(Bz,z),d(Bz,z),d(Bz,z),d(Bz,z),d(Bz,z))\\ &\quad -\phi_2(0,0,d(Bz,z),d(Bz,z),\tfrac{1}{3}d(Bz,z))\end{aligned}$$

$\Rightarrow d(Bz,z) = 0 \Rightarrow Bz = z$. Thus $Bz = Tz = z$.

Since $B(X) \subseteq S(X)$, there is a $w \in X$ such that $z = Sw$. By taking $x = w$ and $y = x_{2n+1}$ in $(i.)$ we get that

$$\psi\big({}^{d(Aw,y_{2n+1}),}_{max\{d(Aw,z),d(y_{2n+1},y_{2n}),d(z,y_{2n})\}}\big)$$

$$\leqslant \quad \phi_1\big({}^{d(Aw,z),d(y_{2n+1},y_{2n}),d(z,y_{2n}),\frac{1}{2}[d(Aw,y_{2n})+d(y_{2n+1},z)],}_{\frac{1}{3}[d(Aw,z)+d(y_{2n+1},y_{2n})+d(z,y_{2n})]}\big)$$

$$-\phi_2\big({}^{d(Aw,z),d(y_{2n+1},y_{2n}),d(z,y_{2n}),\frac{1}{2}[d(Aw,y_{2n})+d(y_{2n+1},z)],}_{\frac{1}{3}[d(Aw,z)+d(y_{2n+1},y_{2n})+d(z,y_{2n})]}\big)$$

Letting $n \to \infty$, we get that

$$\begin{aligned}\psi(d(Aw,z),d(Aw,z)) &\leqslant \phi_1(d(Aw,z),0,0,\tfrac{1}{2}d(Aw,z),\tfrac{1}{3}d(Aw,z))\\ &\quad -\phi_2(d(Aw,z),0,0,\tfrac{1}{2}d(Aw,z),\tfrac{1}{3}d(Aw,z))\\ &\leqslant \phi_1(d(Aw,z),d(Aw,z),d(Aw,z),d(Aw,z),d(Aw,z))\\ &\quad -\phi_2(d(Aw,z),0,0,\tfrac{1}{2}d(Aw,z),\tfrac{1}{3}d(Aw,z))\end{aligned}$$

Then $d(Aw,z) = 0 \Rightarrow Aw = z$. Thus $Aw = z = Sw$. Since $\{A,S\}$ is weakly compatible, $ASw = SAw$. i.e, $Az = Sz$.

By taking $x = z$ and $y = x_{2n+1}$ in $(i.)$ we get that

$$\psi\big({}^{d(Az,y_{2n+1}),}_{max\{d(Az,Az),d(y_{2n+1},y_{2n}),d(Az,y_{2n})\}}\big)$$

$$\leqslant \quad \phi_1\left(\substack{d(Az,Az),d(y_{2n+1},y_{2n}),d(Az,y_{2n}),\frac{1}{2}[d(Az,y_{2n})+d(y_{2n+1},Az)],\\ \frac{1}{3}[d(Az,Az)+d(y_{2n+1},y_{2n})+d(Az,y_{2n})]}\right)$$

$$-\phi_2\left(\substack{d(Az,Az),d(y_{2n+1},y_{2n}),d(Az,y_{2n}),\frac{1}{2}[d(Az,y_{2n})+d(y_{2n+1},Az)],\\ \frac{1}{3}[d(Az,Az)+d(y_{2n+1},y_{2n})+d(Az,y_{2n})]}\right)$$

Letting $n \to \infty$, we get that

$$\begin{aligned}\psi(d(Az,z),d(Az,z)) &\leqslant \phi_1(0,0,d(Az,z),d(Az,z),\tfrac{1}{3}d(Az,z))\\ &\quad -\phi_2(0,0,d(Az,z),d(Az,z),\tfrac{1}{3}d(Az,z))\\ &\leqslant \phi_1(d(Az,z),d(Az,z),d(Az,z),d(Az,z),d(Az,z))\\ &\quad -\phi_2(0,0,d(Az,z),d(Az,z),\tfrac{1}{3}d(Az,z))\end{aligned}$$

Then $d(Az,z) = 0 \Rightarrow Az = z$. Thus $Az = Sz = z$. Hence $z = Az = Bz = Sz = Tz$.

**Case II:** Suppose $B(X)$ or $S(X)$ is a complete subspace of $X$. In this case, we first show that $Az = Sz = z$ and then $Bz = Tz = z$. Thus $z$ is a common fixed point of $A$, $B$, $S$ and $T$ in $X$.

**Uniqueness:** If $z'$ is also a common fixed point of $A$, $B$, $S$ and $T$ in $X$. By taking $x = z$ and $y = z'$ in $(i.)$ we get that $z' = z$.

If $w$ is also a common fixed point of $A$ and $S$, then by taking $x = w$ and $y = z$ in $(i.)$ we get that $d(w,z) = 0 \Rightarrow w = z$. Similar is the case with $B$ and $T$.

The following results are just extensions of Theorem (2.1) and their proofs run on similar lines. □

THEOREM 2.2. *Let $A$, $B$, $S$ and $T$ be self mappings of a metric space $(X,d)$ such that*

(i) $\psi\left(\substack{d(Ax,By),\\ max\{d(Ax,Sx),d(By,Ty),d(Sx,Ty)\}}\right)$

$$\leqslant \quad \phi_1\left(\substack{d(Ax,Sx),d(By,Ty),d(Sx,Ty),\frac{1}{2}d(Ax,Ty),\frac{1}{2}d(By,Sx),\\ \frac{1}{3}[d(Ax,Sx)+d(By,Ty)+d(Sx,Ty)]}\right)$$

$$-\phi_2\left(\substack{d(Ax,Sx),d(By,Ty),d(Sx,Ty),\frac{1}{2}d(Ax,Ty),\frac{1}{2}d(By,Sx),\\ \frac{1}{3}[d(Ax,Sx)+d(By,Ty)+d(Sx,Ty)]}\right)$$

*for all $x, y \in X$, where $\phi_1, \phi_2 \in \Phi_6$, $\psi : [0,\infty)^2 \to [0,\infty)$ is continuous and $\psi(u,u) = \phi_1(u,u,u,u,u,u)$, $\forall u \in [0,\infty)$;*

(ii) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*

(iii) *One of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of $X$;*

(iv) *the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible.*

*Then $A$, $B$, $S$ and $T$ have a unique common fixed point, say $z$. Further, $z$ is the unique common fixed point of $A$ & $S$ as well as $B$ & $T$.*

Proof. Similar to Theorem (2.1). □

Theorem 2.3. *Let $A$, $B$, $S$ and $T$ be self mappings of a metric space $(X, d)$ such that*

(i) $\psi\Big(\begin{smallmatrix} d(Ax, By), \\ max\{d(Ax, Sx), d(By, Ty), d(Sx, Ty)\}\end{smallmatrix}\Big)$

$$\leqslant \phi_1\Big(\begin{smallmatrix}\big(\begin{smallmatrix} d(Ax,Sx),\, d(By,Ty),\, d(Sx,Ty),\, \frac{1}{2}d(Ax,Ty),\, \frac{1}{2}d(By,Sx), \\ \frac{1}{3}[d(Ax,Sx)+d(By,Ty)+d(Sx,Ty)], \end{smallmatrix}\big) \\ \frac{1}{5}[d(Ax,Sx)+d(By,Ty)+d(Sx,Ty)+\frac{1}{2}(d(Ax,Ty)+d(By,Sx))] \end{smallmatrix}\Big)$$

$$-\phi_2\Big(\begin{smallmatrix}\big(\begin{smallmatrix} d(Ax,Sx),\, d(By,Ty),\, d(Sx,Ty),\, \frac{1}{2}d(Ax,Ty),\, \frac{1}{2}d(By,Sx), \\ \frac{1}{3}[d(Ax,Sx)+d(By,Ty)+d(Sx,Ty)], \end{smallmatrix}\big) \\ \frac{1}{5}[d(Ax,Sx)+d(By,Ty)+d(Sx,Ty)+\frac{1}{2}(d(Ax,Ty)+d(By,Sx))] \end{smallmatrix}\Big)$$

*for all $x, y \in X$, where $\phi_1, \phi_2 \in \Phi_7$, $\psi : [0,\infty)^2 \to [0,\infty)$ is continuous and $\psi(u,u) = \phi_1(u,u,u,u,u,u)$, $\forall u \in [0,\infty)$;*

(ii) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*

(iii) *One of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of $X$;*

(iv) *the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible.*

*Then $A$, $B$, $S$ and $T$ have a unique common fixed point, say $z$. Further, $z$ is the unique common fixed point of $A$ & $S$ as well as $B$ & $T$.*

Proof. Similar to Theorem (2.1). □

We conclude our paper with the following example in support of our Theorem (2.1).

Example 2.1. Let $X = \mathbb{Q}^+ \cup \{0\}$, the set of all non-negative rational numbers and with the usual metric. Define $A$, $B$, $S$ and $T$ be the self maps on $X$ by $Ax = 0$,

$$B(x) = \begin{cases} 0 & \text{if } x \leqslant 3, \\ 1 & \text{if } x > 3. \end{cases}$$

$Sx = x$ and $Tx = x^2$ for all $x \in X$.
Define $\phi_1, \phi_2 : [0,\infty)^5 \to [0,\infty)$ by $\phi_1(t_1,t_2,t_3,t_4,t_5)$=$\max\{t_1,t_2,t_3,t_4,t_5\}$ and $\phi_2 = \frac{1}{2}\phi_1$.
Define $\psi : [0,\infty)^2 \to [0,\infty)$ by $\psi(t_1,t_2) = \frac{2t_1+t_2}{3}$.

**Case(i):** $x \in X$ and $y \leqslant 3$.
L.H.S=$\frac{1}{3}$ $\max\{x, y^2, | x - y^2 |\}$
R.H.S=$\max\{x, y^2, | x - y^2 |, \frac{1}{2}(y^2 + x), \frac{1}{3}(x + y^2 + | x - y^2 |)\}$
=$\frac{1}{2}\max\{x, y^2, | x - y^2 |, \frac{1}{2}(y^2 + x), \frac{1}{3}(x + y^2 + | x - y^2 |)\}$
L.H.S $\leqslant$ R.H.S.

**Case(ii):** $x \in X$ and $y > 3$.
L.H.S=2+$\max\{x, y^2 - 1, | x - y^2 |\}$.
R.H.S=$\frac{3}{2}$ $\max\{x, y^2 - 1, | x - y^2 |, \frac{1}{2}(y^2 + | x - 1 |), \frac{1}{3}(x + y^2 - 1 + | x - y^2 |)\}$.

**Subcase(i):** Let $x > y^2 (> 9)$
L.H.S=2+$\max\{x, y^2 - 1, x - y^2\} = 2 + x$.
R.H.S$\geqslant \frac{3}{2}x$.

Since $2 < \frac{9}{2} < \frac{x}{2}$, follows that $2 + x < \frac{3x}{2}$.
Thus L.H.S $\leqslant$ R.H.S.

**Subcase(ii):** $0 \leqslant x \leqslant y^2$.

**Subcase(ii)(a):** $0 \leqslant x \leqslant y^2 - 1$

L.H.S=2+max$\{x, y^2 - 1, y^2 - x\} = 2 + y^2 - x$.
R.H.S$\geqslant \frac{3}{2} \mid x = y^2 \mid = \frac{3}{2}(y^2 - x)$.
Since $y^2 - x \geqslant 9 - 1 = 8$, follows that $\frac{y^2-x}{2} \geqslant 4 > 2$.
$\therefore 2 + y^2 - x < \frac{y^2-x}{2} + y^2 - x = \frac{3}{2}(y^2 - x)$. Thus L.H.S $\leqslant$ R.H.S.

**Subcase(ii)(b):** $1 < x \leqslant y^2 - 1$

L.H.S=2+max$\{x, y^2 - 1, y^2 - x\} = 2 + y^2 - 1 = y^2 + 1$.
R.H.S$\geqslant \frac{3}{2}(y^2 - 1)$.

$$\begin{aligned} y^2 + 1 &= \tfrac{3}{2}y^2 - \tfrac{1}{2}y^2 + 1 \\ &\leqslant \tfrac{3}{2}y^2 - \tfrac{9}{2} + 1 \\ &= \tfrac{3}{2}y^2 - \tfrac{7}{2} \\ &\leqslant \tfrac{3}{2}y^2 - \tfrac{3}{2} \\ &= \tfrac{3}{2}(y^2 - 1) \end{aligned}$$

Thus L.H.S $\leqslant$ R.H.S.

**Subcase(ii)(c):** $8 < y^2 - 1 \leqslant x \leqslant y^2$

L.H.S=2+max$\{x, y^2 - 1, y^2 - x\} = 2 + x$.
R.H.S$\geqslant \frac{3}{2}x$.
Since $2 + x < 4 + x \leqslant \frac{x}{2} + x = \frac{3x}{2}$. Thus L.H.S $\leqslant$ R.H.S.

The other conditions of the Theorem are trivially satisfied. Clearly $'0'$ is the unique common fixed point of $A$, $B$, $S$ and $T$ (in $X$) as well as $A$ & $S$ and $B$ & $T$. (Observe that $X$ is not complete.) ■

Remark 2.1. Comments on the main result (Theorem(2.1)) of [4].

(a) For $\phi_1, \phi_2 \in \Phi_5$, the variables they have taken are $d(Ax, Sx), d(By, Ty)$, $d(Sx, Ty), d(Ax, Ty)$ & $d(By, Sx)$. Thereby they assume that $\psi(u, u) = \phi_1(u, u, u, u, 2u),\ \forall u \in [0, \infty)$.

(b) In the proof, they have taken

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = b$$

and claimed that $b = a(= 0)$. In fact, in some cases limit may not exist. For example, $y_n = n,\ \forall n$ in the usual metric $d(y_n, y_{n+1}) = 1,\ \forall n$ and $d(y_n, y_{n+2}) = 2,\ \forall n$. Thus their claim is not valid. In our proof using triangle inequality and monotonicity of $\phi_1$ and $\phi_2$, we avoided the introduction of $b$.

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