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A NOTE ON MINIMAL DOMINATING AND COMMON MINIMAL DOMINATING GRAPHS

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Abstract. In 1998, V.R.Kulli and B.Janakiram posed three problems in Graph Theory Notes of New York, XXXIV. Among the three, solution to one problem is published in Graph Theory Notes of New York LIII, 37-38(2007). In this paper the remaining two problems are solved.

1. Introduction

The graphs considered here are finite, undirected without loops or multiple edges. Any undefined term in this paper, may be found in Harary [1].

A graph in which every pair of its p vertices are adjacent is called a complete graph. It is denoted by K_p. The complement $\overline{G}$ of a graph G is a graph where vertex set is the same as G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The line graph of a graph G denoted by $L(G)$, is the graph in which the vertices are edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent. The null graph K_0 is the graph with no vertices or edges.

A clique of a graph is a maximal complete subgraph. The clique number $\omega(G)$ of G is the number of vertices present in maximal complete subgraph of G. The clique graph $K(G)$ of a graph G is the intersection graph of the family of cliques of G. The maximum number of independent set of vertices in a graph G is called the vertex *independence number* $\beta_0(G)$ of G.

A set $S \subseteq V$ is a *dominating set* of G if each vertex in $V - S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ is the smallest cardinality of a dominating set. A dominating set is said to be minimal, if no proper subset of S is a dominating set of G. It is well known that, a maximal independent set of G is a minimal dominating set of G.

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The *common minimal dominating graph* $CD(G)$ of G, is the graph having the same vertex set as G, with two vertices adjacent in $CD(G)$ if and only if there exists a minimal dominating set in G containing them. This concept was introduced by Kulli and Janakiram [3].

The *minimal dominating graph* $MD(G)$ of G, is the intersection graph defined on the family of all minimal dominating sets of vertices in G. This concept is also introduced by Kulli and Janakiram [2].

In [4], Kulli and Janakiram posed three problems as follows:

Problem 1: Characterize the graphs G, for which $MD(G) = CD(G)$.

Problem 2: Characterize the graphs G, for which $N(G) = CD(G)$.

Problem 3: Characterize the graphs G, for which $MD(G) = L(\overline{G})$.

The solution to Problem 2 is given by Swaminathan and Wilson Baskar [5].

Here the solutions of Problems 1 and 3 are established.

2. Preliminary Results

We need the following Theorems for our further results.

Theorem A ([1]). For any graph G, $G \cong L(G)$ if and only if G is a cycle.

Theorem B ([2]). For any graph G, $K(\overline{G}) \subseteq MD(G)$. Furthermore, the equality is attained if and only if every minimal dominating set of G is independent.

3. Main Results

Theorem 3.1. *Let $G = (V, E)$ be any connected graph and X be the set of all minimal dominating sets of G. Then $MD(G) = CD(G)$ if and only if $G = K_p$ or G satisfies the following conditions:*

(i) $|X| = |V|$ and $\Delta(G) < p - 1$

(ii) each vertex v_i; $1 \leqslant i \leqslant p$ must be present in exactly n $(\geqslant 2)$ number of minimal dominating sets of G.

Proof. Suppose $MD(G) = CD(G)$, then $|X| = |V|$. If each vertex present in exactly $(n = 1)$ minimal dominating set, then $\delta(G) = \Delta(G) = p - 1$. Hence $G = K_p$. Also now, we need to show that G satisfies the conditions (i) and (ii). If possible, suppose condition (i) is not satisfied, then $|X| \neq |V|$ and $\Delta(G) = p - 1$. This implies $MD(G) \neq CD(G)$, a contradiction.

Suppose G does not satisfy the condition (ii), then there exists at least one vertex v_i; $1 \leqslant i \leqslant p$ which will be present in either $n - 1$ or $n + 1$ minimal dominating sets of G.Therefore either $|X| < |V|$ or $|X| > |V|$. This implies $MD(G) \neq CD(G)$, a contradiction.

Conversely, suppose $G = K_p$, then it is easy to verify the result. If G satisfies the conditions (i) and (ii), we need to show that $MD(G) = CD(G)$. If possible, suppose $MD(G) \neq CD(G)$. Then $|X| \neq |V|$, which contradicts the condition (i). Assume $MD(G) \neq CD(G)$ and $|X| = |V|$, then either each vertex becomes a minimal dominating set or it is present in exactly $n(\geqslant 2)$ minimal dominating sets of G. Thus $MD(G) = CD(G)$, a contradiction. □

THEOREM 3.2. *For any graph G with $\Delta(G) < p-1$, $MD(G) = L(\overline{G})$ if and only if G satisfies the following conditions:*
(*i*) $\beta_0(G) = 2$
(*ii*) *every minimal dominating set of G is independent.*

PROOF. Suppose $MD(G) = L(\overline{G})$. We will prove that G satisfies conditions (i) and (ii). If possible, G does not satisfy the conditions (i) and (ii), then we consider the following cases:

Case 1. If $\beta_0(G) = 1$, then $G = K_p$ and $MD(G) = \overline{K}_p = \overline{G}$. Therefore $L(\overline{G})$ is a null graph. Hence $MD(G) \neq L(\overline{G})$, a contradiction.

Case 2. If $\beta_0(G) \geqslant 3$ then $\omega(\overline{G}) \geqslant 3$. In $L(\overline{G})$ there exist at least three vertices of degree three. Therefore $L(\overline{G}) \nsubseteq K(\overline{G})$ also by Theorem B, $K(\overline{G}) \subseteq MD(G)$. Hence $MD(G) \neq L(\overline{G})$, a contradiction. Thus G satisfies the conditions (i) and (ii).

Conversely, let G satisfy the conditions (i) and (ii). To prove $MD(G) = L(\overline{G})$. Since G is any graph with $\Delta(G) < p-1$ and $\beta_0(G) = 2$, therefore $\omega(\overline{G}) = 2$. Hence $L(\overline{G}) = K(\overline{G})$. Since every minimal dominating set of G is independent, therefore by Theorem B, $K(\overline{G}) = MD(G)$. Hence $MD(G) = L(\overline{G})$. □

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