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COMPLEMENTARY TREE VERTEX EDGE DOMINATION

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Abstract. The concept of complementary tree vertex edge dominating set (*ctved*-set) of a finite, connected graph G is introduced and characterization result for a non empty proper subset of the vertex set V of G to be a *ctved*-set is obtained. The minimum cardinality of a *ctved*-set is denoted by $\gamma_{ctve}(G)$ and is called as *ctved* number of G. Bounds for this parameter as well, are obtained. Further, the graphs of order n for which the *ctved* numbers are 1, 2, $n-1$ are characterized. Trees having $ctved - numbers$ $n-2$, $n-3$ are also characterized. Exact values of this parameter for some standard graphs are given.

1. Introduction

The concept of domination introduced by Ore [**5**] is an active topic in graph theory and has numerous applications to distributed computing, the web graph and adhoc networks. Haynes *et al.* ([**2**]) gave a comprehensive introduction to theoretical and applied facets of domination in graphs.

For ready reference, we here - under give the necessary notation, definitions used in the subsequent work.

All the graphs considered in this paper are undirected, simple, finite and connected.

2. Preliminaries

We, first give a few definitions, observations and results that are useful for development in the succeeding articles.

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Definition. The girth of a graph G, denoted by $g(G)$ is defined as the length of a shortest cycle in G.

Definition. By a sector graph of order n, we mean a graph obtained by introducing a new vertex and joining it to each vertex of a path of order $n-1$ and is denoted by $\bigwedge_n$.

Definition. A support vertex in G is a non pendant vertex adjacent to a pendant vertex.

Definition ([5]) A subset D of the vertex set V of G is said to be a dominating set of G if each vertex in $V-D$ is adjacent to some vertex of D. The domination number $\gamma(G)$ is the minimum cardinality of the dominating set of G.

Definition ([6]) A subset D of the vertex set V of G is a connected dominating set if it is a dominating set and the subgraph induced by D(i.e.$< D >$) is connected. The connected domination number denoted by $\gamma_c(G)$ is the cardinality of a minimum connected dominating set in G.

Definition ([3]) A dominating set D of a connected graph G is a non split dominating set, if the induced subgraph $< V-D >$ is connected in G. The non split domination number $\gamma_{ns}(G)$ of G is the minimum cardinality of a non split dominating set in G.

Definition ([5]) A subset D of V is said to be a vertex edge dominating set(ved - set) of G if each edge in G has either one of its ends from D or one of its ends is adjacent to a vertex in D. The vertex edge domination number $\gamma_{ve}(G)$ is the minimum cardinality of the vertex edge dominating set of G.

Many variants of vertex - vertex, edge - edge, vertex - edge, edge - vertex dominating sets have been studied. In the present paper, we introduce a new variant of vertex - edge dominating set named as complementary tree vertex edge dominating set.

Definition 1.1. A $ved-set$ D of a (connected)graph G is said to be a complementary tree vertex edge dominating set($ctved-set$) of G iff the subgraph induced by $V-D$(i.e $< V-D >$) is a tree.

A $ctved-set$ of minimum cardinality is called a minimum $ctved-set$($mctved-set$) of G. This minimum cardinality is called the complementary tree vertex edge domination number of G and is denoted by $\gamma_{ctve}(G)$. Any $mctved-set$ of G is referred by $\gamma_{ctve}(G)-set$.

For standard terminology and notation, we refer Bondy & Murthy ([**1**]).

Unless otherwise stated, by G we mean a finite , simple, connected graph with n vertices and e edges.

3. Characterization and other relevant results

In this section, we initially state characterization result for a proper subset of the vertex set of G to be a *ctved*-set of G. There after we give the bounds for this parameter in terms of various other parameters.

THEOREM 3.1. *(Characterization Result) A non empty proper subset D of the vertex set V of a graph G is a ctved − set in G iff the following are satisfied:*
(i) $F = \{xy \in E(G)/atleast\ one\ of\ x, y\ is\ in\ D\}$ is an edge dominating set of G.
(ii) D is not a vertex cut in G
(ii) Any cycle in G has atleast one vertex from D.

PROOF. Trivial □

THEOREM 3.2. *For a graph G,*

$$\lceil \frac{2(n-1)-e}{2} \rceil \leqslant \gamma_{ctve}(G).$$

($\lceil x \rceil$ denotes the smallest integer $\geqslant x$).

PROOF. Suppose that D is a $\gamma_{ctve}(G) - set$. So, follows that $< V - D >$ is a tree. Hence it has $n - \gamma_{ctve}(G)$ vertices and $n - \gamma_{ctve}(G) - 1$ edges. Clearly each edge in $< V - D >$ is dominated by a vertex in D. This implies corresponding to each edge in $< V - D >$, there is an edge in $G - < V - D >$. Hence,

$$e \geqslant 2(n - \gamma_{ctve}(G) - 1) \Rightarrow \lceil \frac{2(n-1)-e}{2} \rceil \leqslant \gamma_{ctve}(G)$$

□

Note. The bound is attained if $G \cong C_n, n \geqslant 3$.

COROLLARY 3.1. *If G is a tree, then*

$$\lceil \frac{e}{2} \rceil \leqslant \gamma_{ctve}(G).$$

PROOF. The result follows since $e = n - 1$. □

Note. The bound is attained in the case of P_4.

PROPOSITION 3.1. (1) *For any path P_n with $n \geqslant 5$, $\gamma_{ctve}(P_n) = n - 3$.*
(2) *For any cycle C_n with $n \geqslant 5$, $\gamma_{ctve}(C_n) = n - 3$.*
(3) *For any complete bipartite graph $K_{m,p}$ with $m + p \geqslant 4$, $\gamma_{ctve}(K_{m,p}) = m + p - 3$.*
(4) *For the complete bipartite graph $K_{2,1}$, $\gamma_{ctve}(K_{2,1}) = 2$.*
(5) *For any star graph $K_{1,p}$, $\gamma_{ctve}(K_{1,p}) = 1$.*
(6) *For any bistar graph $S_{m,p}$, $\gamma_{ctve}(S_{m,p}) = min\{m + 1, p + 1\}$.*
(7) *For any complete graph $K_n (n \geqslant 3)$, $\gamma_{ctve}(K_n) = n - 2$.*
(8) *$\gamma_{ctve}(C_p o K_1) = p + 1$, where $C_p o K_1$ is the corona of C_p and K_1 and ($p \geqslant 5$).*
(9) *For any Wheel Graph W_p, $\gamma_{ctve}(W_p) = 2$.*

THEOREM 3.3. *For a graph G with $g(G) \geqslant 4$,*

$$\gamma_{ctve}(G) \leqslant n - \Delta(G).$$

PROOF. Let v be a vertex in G such that $d_G(v) = \Delta(G)$. Then $(V - N[v]) \bigcup \{v_i\}$ (v_i is a neighbour of v) is a $ctved - set$ in G. Hence, $\gamma_{ctve}(G) \leqslant n - \Delta(G)$. □

Note. The bound is attained in the case of $< v_1v_2v_3v_4v_1 > \bigcup \{v_1v_5\}$.

COROLLARY 3.2. *For a graph G with $g(G) \geqslant 4$ & $\delta(G) \geqslant 2$,*

$$\gamma_{ctve}(G) \leqslant n - \Delta(G) - 1.$$

PROOF. Let $d_G(v) = \Delta(G)$. Then $(V - N[v])$ is a $ctved - set$ in G. Hence, $\gamma_{ctve}(G) \leqslant n - \Delta(G) - 1$. □

THEOREM 3.4. *For any tree T with $n \geqslant 4$,*

$$\gamma_{ctve}(T) \leqslant n \ - \ max\{d(u) \ : \ u \text{ is a support vertex in } T\}.$$

PROOF. Let v be a support vertex in T. Then $(V - N[v]) \bigcup \{v_i\}$($v_i$ is a non pendant neighbour of v) is a $ctved - set$ in T of cardinality $n - d(v)$. Hence the inequality holds. □

Note. The bound is attained for $P_n, \ n \geqslant 4$.

Observations 3.1. 1. $\gamma_{ctve}(G) \leqslant \gamma_{ctve}(H)$, where H is a spanning subgraph of G.
2. For a graph G with atleast two vertices, $1 \leqslant \gamma_{ctve}(G) \leqslant n - 1$.

THEOREM 3.5. *G be a graph with vertex set $V = \{v_1, v_2, ..., v_n\}$. Then $\gamma_{ctve}(G) = n - 1$ iff $G = P_2$.*

PROOF. Assume that $\gamma_{ctve}(G) = n - 1$. Then $D = V - \{v_n\}$ is a $ctved - set$ in G. If $diam(G) \geqslant 3$, then we have a $ctved - set$ $D^{'} \subset D$ of cardinality atmost $n - 2$. This contradicts our assumption. Hence $diam(G) \leqslant 2$.

Let $diam(G) = 2$. Suppose that G has pendant vertices, say $\{u_1, u_2, ..., u_m\}$. Since $diam(G) = 2$, all the pendant vertices are adjacent to u(say). Clearly all the vertices in $V - \{u, u_1, u_2, ..., u_m\}$ are adjacent to u.

Suppose G has non pendant edges. Let $x_1y_1, x_2y_2, ..., x_ty_t$ be the non pendant edges in G. Then by the nature of u, $\{x_1, x_2, ..., x_t, y_1, y_2, ..., y_t\}$ forms a $ctved - set$ in G of cardinality atmost $n - 2$, a contradiction to our assumption. Hence G has no non pendant edges i.e $G \cong K_{1,p}$. By Proposition.2.4(5), $\gamma_{ctve}(G) = 1 < n - 1$, a contradiction.
Hence follows that $diam(G) = 1$. This implies $G = P_2$.

The converse part is clear. □

THEOREM 3.6. *T be a tree with $n \geqslant 4$. Then $\gamma_{ctve}(T) = 2$ if and only if T is obtained by adding zero or more leaves to exactly one support vertex in P_4.*

PROOF. Assume that $\gamma_{ctve}(T) = 2$.
Let $D = \{v_1, v_2\}$ be a $ctved - set$ in T. By the property of D, $< D >$ is connected and exactly one of v_1, v_2 is a pendant vertex in T. W.l.g assume that v_1 is a

pendant vertex in T. Now $diam(T) = 3$. Let $< v_1v_2v_3v_4 >$ be a diametral path in T. Clearly by the property of D, no vertex other than v_3 can be adjacent to v_2. Since $diam(T) = 3$, any vertex in $V - \{v_1, v_2, v_4\}$ is adjacent to v_3. Hence $T = P_4$ or T is obtained by adding zero or more leaves to exactly one support vertex which is v_3.

The converse part is clear. □

THEOREM 3.7. *For a graph G,*

$$\gamma_{ctve}(G) + \Delta(G) \leqslant 2n - 2.$$

PROOF. Since $\Delta(G) \leqslant n - 1$ and $\gamma_{ctve}(G) \leqslant n - 1$, the result follows. □

THEOREM 3.8. *For any graph G, $\gamma_{ctve}(G) + \Delta(G) = 2n - 2$ if and only if $G = P_2$.*

PROOF. Suppose $\gamma_{ctve}(G) + \Delta(G) = 2n - 2$. This is possible only when $\gamma_{ctve}(G) = n - 1$ and $\Delta(G) = n - 1$. Then by Theorem.2.8, $G = P_2$.
The converse part is clear. □

THEOREM 3.9. *If G is a graph with $\delta(G) > 1$, then $\gamma_{ctve}(G) = 2$ if and only if there is an edge $f = uv$ in G satisfying the following :*
(i) Each edge e' in $E - \{uv\}$ is vertex edge dominated(ve - dominated) by u or v.
(ii) e' lies on a cycle containing the edge uv in G.
(iii) G is not a union of $k - cycles(k \leqslant 4)$ having uv as the common edge.

PROOF. Assume that $\gamma_{ctve}(G) = 2$. Let $D = \{u, v\}$ be a $ctved - set$ in G. Clearly $< D >$ is connected i.e uv is an edge in G. Let $e' = xy$ be an edge in $E - \{uv\}$. By the definition of D, $e' = xy$ is ve - dominated by a vertex in D. Now, we have two possibilities.

Case:1 $x = u$ or $y = v$.
W.l.g assume that $x = u$ i.e f, e' are adjacent. Then $< yxv >$ is a path in G. Since G is connected there is a $y - v$ path in G. If all the $y - v$ paths in G are through x then $< V - D >$ is disconnected, a contradiction since D is a $ctved - set$ in G. Hence there is a $y - v$ path in G edge disjoint with the path $< yxv >$. Now the union of the former path with the later gives a cycle that contains the edges xy, uv.

Case:2 $x \neq u$ and $y \neq v$.
If xy is ve - dominated by both u and v, then there is a cycle containing both the edges xy, uv. If not, then as in the Case:1, we get a contradiction to the fact that D is a $ctved - set$ in G.

Hence condition (ii) holds. Clearly condition (iii) holds.

In the converse case, clearly $D = \{u, v\}$ is a (connected)ve - dominating set in G of cardinality two and obviously a $ctved - set$. Hence $\gamma_{ctve}(G) \leqslant 2$. If $\gamma_{ctve}(G) = 1$, then we get a contradiction to (iii). Hence $\gamma_{ctve}(G) = 2$. □

Note. 1. Any $ctve$ - dominating set in G is a ve - dominating set in G. Hence $\gamma_{ve}(G) \leqslant \gamma_{ctve}(G)$.
2. A non split dominating set for G is a $ctved - set$ in G. Hence $\gamma_{ctve}(G) \leqslant \gamma_{ns}(G)$.

The following two are consequences.

PROPOSITION 3.2. *$\gamma_{ve}(P_n) = \gamma_{ctve}(P_n)$ iff $n \leqslant 3$.*

PROPOSITION 3.3. *$\gamma_{ve}(C_n) = \gamma_{ctve}(C_n)$ iff $n \leqslant 5$.*

THEOREM 3.10. *Let T be a tree and D be the set of all pendant vertices in T. Then D is a $ctved-set$ of G iff each edge of degree atleast two is a support edge in T.*

PROOF. Assume that each edge of degree two in T is a support edge. Then D is a $ved-set$ of T. Since $< V - D >$ is a tree follows that D is a $ctved-set$ in T. Conversely, let e' be an edge in T such that $deg(e') \geqslant 2$. If e' is not a support edge in T, then none of the ends of e' is adjacent to a vertex in D, which is a contradiction.

Thus the result is proved. □

THEOREM 3.11. *Let T be a tree, then $\gamma_{ctve}(T) = n - 2$ if and only if $T = P_3$ or $T = P_4$.*

PROOF. Assume that $\gamma_{ctve}(T) = n-2$. Clearly T cannot have adjacent pendant vertices. So any support vertex cannot be adjacent to more than one pendant vertex. If T has a path $< v_1v_2v_3v_4v_5 >$, then $V - \{v_2, v_3, v_4\}$ is a $ctved-set$ in T of cardinality at most $n - 3$, a contradiction to our assumption. So $diam(G) \leqslant 3$.

Suppose $diam(T) = 3$. Let $< v_1v_2v_3v_4 >$ be a diametral path in T. If there are pendant vertices adjacent to v_2, v_3, other than v_1, v_4, then $\gamma_{ctve}(T) \leqslant n - 3$, a contradiction to our assumption. So $T = < v_1v_2v_3v_4 > = P_4$.

Suppose $diam(T) = 2$. If T has more than two pendant vertices, then $\gamma_{ctve}(T) \leqslant n - 2$, a contradiction to our assumption. Hence T has exactly two pendant vertices. So $T = P_3$.

The converse part is clear. □

COROLLARY 3.3. *For a tree T with $n \geqslant 3$,*

$$\gamma_{ctve}(T) + \Delta(T) \leqslant 2n - 3.$$

Furthermore, $\gamma_{ctve}(T) + \Delta(T) = 2n - 3$ if and only if $T = K_{1,1}$ or $T = S_{1,1}$.

PROOF. The proof follows by the above result and the fact that $\Delta(T) \leqslant n - 1$. □

COROLLARY 3.4. *For a tree T with $n \geqslant 3$, $\gamma_{ctve}(T) \leqslant n - 3$.*

PROOF. The proof follows by the above result and the fact that $\delta(T) = 1$. □

THEOREM 3.12. *For a tree T with $n \geqslant 5$, $\gamma_{ctve}(T) = n - 3$ if and only if any of the following holds:*
(i) There is a support vertex v adjacent to atmost two pendant vertices such that $\Delta(G) = d(v) = 3$.
(ii) $T = P_n$

PROOF. Assume that $\gamma_{ctve}(T) = n-3$. Let $V - \{v_1, v_2, v_3\}$ be a $\gamma_{ctve}(T) - set$. By definition of $ctved - set$, atmost two of $\{v_1, v_2, v_3\}$ can be (adjacent)pendant vertices and adjacent with the third vertex.

Case: 1 Suppose that two of $\{v_1, v_2, v_3\}$ are pendant vertices.
W.l.g assume the vertices to be v_1, v_2. Then they are adjacent with v_3. Clearly v_3 is a support vertex of degree atleast three. If $d(v_3) > 3$, then $(V - N[v_3]) \bigcup \{v\} (v$ *is a non pendant neighbour of* $v_3)$ is a $ctved - set$ of cardinality atmost $n-4$, a contradiction to our assumption. So $\Delta(G) \geqslant 3$.
Suppose that there is a vertex v of degree k, where $k \geqslant 4$. Clearly $(V - N[v]) \bigcup \{u\} (u$ *is a non pendant neighbour of* $v)$ is a $ctved - set$ of cardinality atmost $n-4$, a contradiction to our assumption. Therefore $\Delta(G) = 3 = d(v_3)$, where v_3 is a support vertex in T.

Case: 2 Suppose exactly one of v_1, v_2, v_3 is a pendant vertex.
W.l.g assume that v_1 is a pendant vertex. By definition, one of v_2, v_3 is a support vertex. W.l.g assume that v_2 is the support vertex. Since $n \geqslant 4$, $d(v_2) \geqslant 3$. If $d(v_2) > 3$, then as in the case:1,we get a contradiction to our assumption. Hence in this case also claimant holds.

Case: 3 Suppose that none of $\{v_1, v_2, v_3\}$ is a pendant vertex. By definition of $ctved - set$, one of v_1, v_2, v_3 is a common neighbour of the remaining two. W.l.g assume that v_2 is a common neighbour of v_1, v_3. By our supposition and by proposition.2.4(i),$T = P_n$.

The converse part is clear. □

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References

[1] J.A.Bondy & U.S.R Murthy, *Graph theory with Applications*, The Macmillan Press Ltd, (1976).
[2] Haynes T.W., Hedetneimi S.T. and Slater P.J. (eds.), *Fundamentals of Dominations in Graphs,*, Marcel Dekker, New York,1998.
[3] Kulli V.R, Janakiram B., *The non split domination number of a graph.*, Indian J. Pure Appl. Math., **vol. 31(5)** (2000), 545 - 550.
[4] Laskar R., Peters K., *Vertex and edge dominating parameters in Graphs.*, Congr. Numer. **vol. 48** (1985), 291-305
[5] Ore O., *Theory of Graphs.* Amer. Math. Soc. Colloq. Publ., **Vol. 38**, Providence (1962).
[6] Sampathkumar E., Walikar H.B., The connected domination number of a graph. *J. Math. Phy. Sci.,* **13(6)**, (1979), 607-613.

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