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A NOTE ON THE EQUITABLE COVERING AND EQUITABLE PACKING OF A GRAPH

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Abstract. Let $G = (V, E)$ be a simple graph. A subset S of V is called an equitable vertex covering of G if for every equitable edge $e = uv$, either $u \in S$ or $v \in S$. The minimum cardinality of an equitable vertex cover of G ia called the euitable covering number of G and is denoted by $\alpha_o^e(G)$.In this paper results involving in this new parameter are found. Also we introduced the Equitable Packing and Equitable Full sets.

1. Introduction and definitions

New concepts of domination arise from practical considerations. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society, persons with nearly equal status, tend to be friendly. In an industry, employees with nearly equal powers form association and move closely. Equitability among citizens in terms of wealth, health, status etc is the goal of a democratic nation. In order to study this practical concept, a graph model is to be created. Prof. E. Sampathkumar is the first person to recognize the spirit and power of this concept and introduced various types of equitability in graphs like degree equitability, outward equitability, inward equitability, equitability in terms of number of equal degree neighbours, or in terms of number of strong degree neighbours etc. In general, if $G = (V, E)$ is a simple graph and $\phi : V(G) \to N$ is a function, we may define equitability of vertices in terms of ϕ - values of the vertices. A subset D of V is called an equitable dominating set if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d(u) - d(v)| \leqslant 1$, where $d(u)$ denotes the degree of vertex u and $d(v)$ denotes the degree of vertex v. The minimum cardinality of such

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a dominating set is denoted by γ^e and is called the equitable domination number of G. Degree Equitable domination on Graphs were introduced in [7].

Definition 1.1. A subset D of V is called an equitable dominating set if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leqslant 1$. The minimum cardinality of such a dominating set is denoted by γ^e and is called the equitable domination number of G.

Definition 1.2. A vertex $u \in V$ is said to be degree equitable with a vertex $v \in V$ if $|deg(u) - deg(v)| \leqslant 1$.

Remark 1.1. If D is an equitable dominating set then any super set of D is an equitable dominating set.

Definition 1.3. An equitable dominating set D is said to be a minimal equitable dominating set if no proper subset of D is an equitable dominating set.

Definition 1.4. A minimal equitable dominating set of maximum cardinality is called a Γ^e-set and its cardinality is denoted by Γ^e.

Definition 1.5. An equitable dominating set is said to be 1 - minimal if $D - v$ is not an equitable dominating set for all $v \in D$.

Remark 1.2. If a vertex $u \in V$ be such that $|deg(u) - deg(v)| \geqslant 2$ for all $v \in N(u)$ then u is in every equitable dominating set. Such points are called *equitable isolates*. Let I_e denote the set of all equitable isolates. Vacuously isolated points are equitable isolated points. Hence $I_s \subseteq I_e \subseteq D$ for every equitable dominating set D where I_s is the set of all isolated points of G.

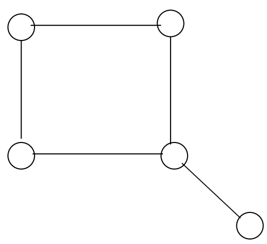
Remark 1.3. An equitable dominating set D is minimal if and only if it is 1 - minimal.

Definition 1.6. Let $G = (V, E)$ be a simple graph. Let H be the graph constructed from G as follows: $V(H) = V(G)$, two points u and v are adjacent in H if and only if u and v are adjacent and degree equitable in G. H is called the *adjacency inherent equitable graph of* G or *equitable associate of* G and is denoted by $e(G)$.

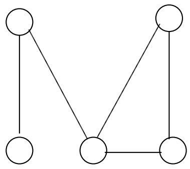
Remark 1.4. (i) $e = uv \in E(e(G))$. Then u and v are adjacent and degree equitable in G. Therefore $e \in E(G)$. $E(e(G)) \subseteq E(G)$.
(ii) An edge $e = uv \in E(G)$ is said to be equitable if $|d(v) - d(u)| \leqslant 1$.
Let $E^e(G)$ be the set of all equitable edges of G. Then clearly,
$E^e(G) = E(e(G))$.

Remark 1.5. $\overline{e(G)}$ need not be equal to $e(\overline{G})$.

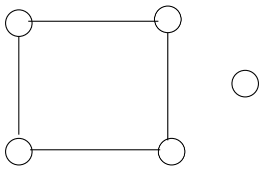
For consider G :



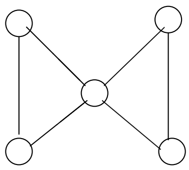
$\overline{G}$:



$e(G)$:



$\overline{e(G)}$:



$e(\overline{G})$:

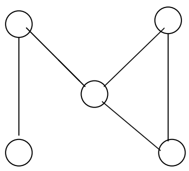


Fig. 1.1

Therefore $\overline{e(G)} \neq e(\overline{G})$. Equitable Associate Graph of a Graph were introduced in [8].

2. Main Results

Equitable Covering and Equitable Packing

Definition 2.1. A subset S of V is called an equitable vertex covering of G if for every equitable edge $e = uv$, either $u \in S$ or $v \in S$. The minimum cardinality of an equitable vertex cover of G is called the equitable covering number of G and is denoted by $\alpha_o^e(G)$

Example 2.1.
(i) $\alpha_o^e(K_n) = n - 1$.
(ii)$\alpha_o^e(P_n) = \lfloor \frac{n}{2} \rfloor$
(iii) $\alpha_o^e(C_n) = \alpha_o(C_n) = \lceil \frac{n}{2} \rceil$
(iv) $\alpha_o^e(W_n) = \lceil \frac{n-1}{2} \rceil$if $n \geqslant 6$.
(v) If G is either regular or bi regular with bi regularity $(k, k+1)$,
then $\alpha_o^e(G) = \alpha_o(G)$.
(vi) If G is a graph in which every vertex is an equitable isolate,
then $\alpha_o^e(G) = 0$. Thus $\alpha_o^e(K_{1,n}) = 0$. But $\alpha_o(K_{1,n}) = 1$.
(vii) $\alpha_o^e(K_{m,n}) = 0$, if $|m - n| \geqslant 2$.
Observe that $\alpha_o^e(K_{m,n}) = \alpha_o(K_{m,n}) = min\{m, n\}$, if $|m - n| \leqslant 1$

Remark 2.1. Let G have no equitable isolates. Then every equitable vertex cover of G is an equitable dominating set of G. Therefore for a graph G without equitable isolates, $\gamma^e(G) \leqslant \alpha_o^e(G)$. Note that $\alpha_o^e(G) \leqslant \alpha_o(G)$. Therefore $\gamma^e(G) \leqslant \alpha_o^e(G) \leqslant \alpha_o(G)$.

Theorem 2.1. $\alpha_o^e(G) + \beta_o^e(G) = n$.

Proof. Let S be an α_o^e -set. Consider $V - S$. Let $u, v \in V - S$. If u and v are adjacent and degree equitable, then the equitable edge uv has no end in S, a contradiction. Therefore $V - S$ is an equitable independent set. Therefore $|V - S| \leqslant \beta_o^e(G)$. That is $n - |S| \leqslant \beta_o^e(G)$, which means $n - \alpha_o^e(G) \leqslant \beta_o^e(G)$. Hence $n \leqslant \alpha_o^e(G) + \beta_o^e(G)$.

Let S be a $\beta_o^e(G)$-set. Consider $V - S$. Let $e = uv$ be an equitable edge in G. If both u and v are in S, then S is not equitable independent, a contradiction. Therefore either u or $v \in V - S$. Therefore $V - S$ is an equitable vertex cover of G. Therefore $|V - S| \geqslant \alpha_o^e(G)$. That is $n - |S| \geqslant \alpha_o^e(G)$ That is $n - \beta_o^e(G) \geqslant \alpha_o^e(G)$. Therefore $n \geqslant \alpha_o^e(G) + \beta_o^e(G)$.

Therefore $n = \alpha_o^e(G) + \beta_o^e(G)$. □

Definition 2.2. A set of equitable lines in G is set to be independent, if no two lines in that set have a common point. The equitable line independence number denoted by $\beta_1^e(G)$ is the maximum cardinality of an independent set of equitable lines.

Example 2.2.

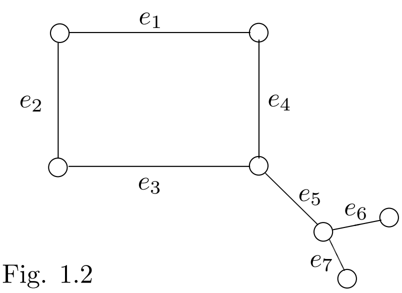


Fig. 1.2

$\{e_1, e_3, e_6\}$ is a maximum line independent set and hence $\beta_1(G) = 3$. But $\beta_1^e(G)$ is 2, since $\{e_1, e_3\}$ is a maximum equitable line independent set of G.

REMARK 2.2. For any graph G $\beta_1^e(G) = \beta_1(e(G))$

THEOREM 2.2. *If G is bipartite, then* $\beta_1^e(G) = \alpha_0^e(G)$

PROOF. Let G be bipartite. Then $e(G)$ is also bipartite. Therefore $\beta_1(e(G)) = \alpha_0(e(G))$. Therefore $\beta_1^e(G) = \alpha_0^e(G)$. □

DEFINITION 2.3. A subset S is called an equitable 2 - packing if
$N^e[u] \cap N^e[v] = \phi$, for all $u, v \in S$.

DEFINITION 2.4. The equitable packing number $\rho^e(G)$ is the cardinality of a maximum equitable 2-packing.

THEOREM 2.3. *For any graph G, $\rho^e(G) \leqslant \gamma^e(G)$.*

PROOF. Let $S = \{u_1, u_2, ..., u_{\rho^e}\}$ be a maximum equitable 2 - packing. Let $D = \{v_1, v_2, \ldots, v_{\gamma^e}\}$ be a minimum equitable dominating set of G. Then $G = \bigcup\limits_{i=1}^{\gamma^e} N^e[v_i] = V(G)$. If $\rho^e(G) > \gamma_e(G)$ then there exists u_i, u_j such that u_i, $u_j \in N[v_i]$ for some i, $1 \leqslant i \leqslant \gamma^e$. Therefore $N^e[u_i] \cap N^e[u_j]$ contains u_i and hence non-empty. But $N^e[u_i] \cap N^e[u_j] = \phi$, as S is an equitable 2-packing. Therefore, we get that $\rho^e(G) \leqslant \gamma^e(G)$. □

REMARK 2.3. If S is a 2-packing, Then S is an equitable 2-packing. Thus $\rho(G) \leqslant \rho^e(G)$.

THEOREM 2.4. *For any tree T such that $e(T)$ is connected, $\gamma^e(T) = n - \Delta^e(T)$ if and only if $e(T)$ is a wounded spider with utmost two legs.*

PROOF. Let T be a tree with $e(T)$ connected and $\gamma^e(T) = n - \Delta^e(T)$. Then $\gamma(e(T)) = n - \Delta(e(T))$. Therefore by a theorem in [9], $e(T)$ is a wounded spider with utmost two legs.

The converse is obvious. □

3. Equitable Full Sets

DEFINITION 3.1. Let $G = (V, E)$ be a graph and $D \subset V$. Then
(i) D is *full* if every $u \in D$ is adjacent to some $v \in V - D$.
(ii) D is *equitably full* if every $u \in D$ equitably dominates some $v \in V - D$.

THEOREM 3.1. *D is equitably full if and only if $V - D$ is an equitable dominating set. Proof is obvious.*

THEOREM 3.2. *Let G be a graph of order n. If there exists an equitable dominating set D which is equitably full then $\gamma^e \leqslant \frac{n}{2}$.*

PROOF. $\gamma^e \leqslant |D|, \gamma^e \leqslant |V - D| = n - |D|$. Therefore $\gamma^e \leqslant n - \gamma^e$.
Therefore $\gamma^e \leqslant \frac{n}{2}$. □

REMARK 3.1. A minimal equitable dominating set need not be equitably full.

Example 3.1. In G :

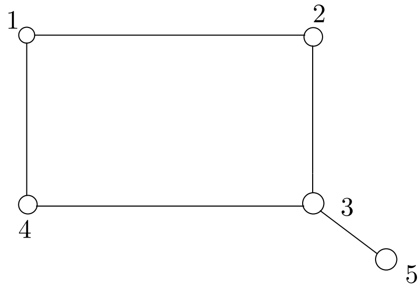


Fig. 1.3

$D = \{1, 3, 5\}$ is minimal equitable dominating set. $V - D = \{2, 4\}$. $5 \in D$ does not equitably dominate any point of $V - D$. Therefore D is not equitably full.

If G is a graph without equitable isolated points then any minimal equitable dominating set is equitably full.

Definition 3.2. The equitable full number $f^e(G)$ is the maximum cardinality of an *equitably full set* of G.

Theorem 3.3. *Let G be a graph of order n then $f^e(G) + \gamma^e(G) = n$.*

Proof. Let S be a γ^e-set. Then $V - S$ is equitably full. Therefore $|V - S| \leqslant f^e(G)$. That is $n - |S| \leqslant f^e(G)$ That is $n - \gamma^e \leqslant f^e(G)$. Let D be a maximum equitably full set. Then $V - D$ is an equitable dominating set. Therefore $|V - D| \geqslant \gamma^e$. That is $|V| - |D| \geqslant \gamma^e$. That is $n - f^e \geqslant \gamma^e$. Therefore $n - \gamma^e \geqslant f^e$. Therefore $f^e(G) + \gamma^e(G) = n$. □

Theorem 3.4. *Let G be a graph without isolates. Then $\gamma^e(G) \geqslant n - m(G)$.*

Proof. It is well known that $\gamma(G) \geqslant n - m$. But $\gamma^e(G) \geqslant \gamma(G)$. Therefore $\gamma^e(G) \geqslant n - m$. □

Theorem 3.5. *Let G be a graph. Then $\gamma^e(G) \geqslant n - m^e(G)$ where $m^e(G)$ is the number of equitable edges of G.*

Proof. Let G be a graph. Let D be a γ^e-set. Therefore there exists at least $|V - D|$ equitable edges that go from (V - D) to D. Let m_o^e be the number of equitable edges in $< D >$ and these are all distinct from the $|V - D|$ or more edges from $(V - D)$ to D. Therefore $m^e \geqslant |V - D| + m_o^e = n - |D| + m_o^e = n - \gamma^e + m_o^e$. Therefore $\gamma^e \geqslant n - m^e + m_o^e$. Therefore $\gamma^e \geqslant n - m^e$. □

Theorem 3.6. *For any (n, m) graph G , $\beta_o^e(G) \leqslant f^e(G) \leqslant m$.*

Proof. $\beta_o^e(G) = n - \alpha_o^e(G) \leqslant n - \gamma^e(G) = f^e(G)$. But $f^e(G) = n - \gamma^e(G) \leqslant m$ (since $n - m \leqslant \gamma(G) \leqslant \gamma^e(G)$). Therefore $\beta_o^e(G) \leqslant f^e(G) \leqslant m$. □

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