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SOME REMARKS ABOUT R-LABELINGS OF POSETS

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Abstract. We describe a family of posets with positive flag h-vectors that do not admit an R-labeling. This family contains the example of R. Ehrenborg and M. Readdy presented in [**4**]. Furthermore, for a poset that has an R-labeling, we consider the complex of all rising chains. We show that the f-vector and homotopy type of this complex do not depend of a concrete labeling.

1. Introduction

We shortly review some concepts about partially ordered sets (posets). We refer the reader to Chapter 3 of [**7**] for a detailed overview of poset terminology.

A poset P is *graded* if it has a minimal element $\hat{0}$, maximal element $\hat{1}$ and a rank function ρ such that $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ whenever y covers x. The rank of the poset P is defined to be $\rho(P) = \rho(\hat{1})$. For a graded poset P of rank $n+1$ and $S \subseteq [n] = \{1, 2, \ldots, n\}$ let f_S denote the number of chains $x_1 < x_2 < \ldots < x_k$ in P such that $S = \{\rho(x_1), \rho(x_2), \ldots, \rho(x_k)\}$. The sequence $(f_{S(P)})_{S\subseteq[n]}$ is called the *flag f-vector* of P. The *flag h-vector* of P is the sequence $(h_{S(P)})_{S\subseteq[n]}$ defined by

$$h_S = \sum_{T\subseteq S} (-1)^{|S\setminus T|} f_T.$$

Let $E(P)$ denote the set of all covering relations in P:

$$E(P) = \{(x, y) \in P \times P : x \prec y\}.$$

In other words, $E(P)$ is the set of edges in the Hasse diagram of P.

Definition 1.1. A map $\lambda : E(P) \to \mathbb{Z}$ is called an R-labeling if for every interval $[x, y]$ of P there is a unique rising chain $x = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_k = y$ such that $\lambda(x_0, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_{k-1}, k)$. This unique chain is called rising.

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R. Stanley introduced the concept of labelings of posets in [**5**] and [**6**]. The term "R-labeling" appeared in [**2**]. Some more rigorous types of labeling of a poset (EL-labeling and CL-labeling, see [**3**]) enable us to compute homology of a poset. An R-labeling of a graded poset P can be used for obtaining some important enumerative characteristics of P, such as Möbius function, Euler characteristic, flag h-vector,. . . .

For example, if a graded poset has an R-labeling, every entry of its flag h-vector is non-negative, see Theorem 3.14.2 in [**7**]. Therefore, a poset P with a negative entry in its flag h-vector does not have an R-labeling.

In [**4**], R. Ehrenborg and M. Readdy construct a family of posets where each member has a positive flag h-vector but has no R-labeling.

Let T_n denote the butterfly poset, a unique graded poset of rank n such that there are two elements of rank i for $1 \leqslant i \leqslant n-1$ and every element different from $\hat{0}$ covers all elements of one rank below. It is easy to check that the flag f- and h-vectors of T_n are given by

$$f_S(T_n) = 2^{|S|} \text{ and } h_S(T_n) = 1 \text{ for } S \subseteq [n-1].$$

Let P_n consist of two copies of the T_n where we have identified the minimal elements and the maximal elements. Note that $h_S(P_n) = 2 - (-1)^{|S|} > 0$.

THEOREM 1.1 (Ehrenborg-Readdy, [**4**]). *The poset P_n for $n > 3$ does not have an R-labeling.*

2. Posets without R-labeling

DEFINITION 2.1. Let P be a graded poset with an R-labeling. For every $x \in P$ we can associate the rising tree T_x. Vertices of T_x are the elements of $[x, \hat{1}]$. A pair uv such that $x \leqslant u \prec v$ is an edge of T_x if and only if the unique rising chain from x to v contains u.

The existence and uniqueness of a rising chain in every interval $[x, y]$ guaranties that T_x is an acyclic connected graph. For $x < v \leqslant \hat{1}$ let $T_{x|v}$ denote the subtree of T_x spanned by $[v, \hat{1}]$.

REMARK 2.1. Let P be a graded poset with an R-labeling. Assume that the edge uv of the Hasse diagram is a common edge of T_x and T_y. If pq is an edge in $T_{x|v}$, then we have a rising chain $x \prec x_1 \prec \cdots \prec u \prec v \prec z_1 \prec \cdots \prec p \prec q$. We know that there exists a rising chain $y \prec y_1 \prec \cdots \prec u \prec v$ from y to v. Therefore

$$y \prec y_1 \prec \cdots \prec u \prec v \prec z_1 \prec \cdots \prec p \prec q$$

is a rising chain from y to q. So, we can conclude that $T_{x|v} = T_{y|v}$.

THEOREM 2.1. *Let P_1 and P_2 be two posets of rank $n > 3$ with just two elements of rank i_j in P_j for some $1 < i_1, i_2 < n-1$. Let Q be a poset obtained by identification of the maximal elements and the minimal elements of P_1 and P_2. The poset Q does not admit an R-labeling.*

Proof. Suppose that Q has an R-labeling λ. In that case, there exists the unique rising chain from $\hat{0}_Q$ to $\hat{1}_Q$. Without lose of generality we may assume that this chain is contained in P_1. Let x and y denote the only two elements of P_2 of rank i_2. Note that any $z \in P_2$, $\rho(z) > i_2$, $z \neq \hat{1}_Q$ is contained in $T_{\hat{0}_Q|x}$ or $T_{\hat{0}_Q|y}$. Now, we consider two possible cases.

1° There exists $u \in P_2$, $\rho(u) = i_2 - 1$ such that ux and uy are both the edges of $T_{\hat{0}_Q}$. As we suppose that λ is an R-labeling there exists a unique rising chain $u = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_{n-i_2-1} = \hat{1}_Q$ from u to $\hat{1}_Q$. Without loss of generality we assume that $x_1 = x$. From Remark 2.1 we conclude that $T_{\hat{0}_Q|x} = T_{u|x}$, and therefore the vertex $\hat{1}_Q$ is contained in $T_{\hat{0}_Q|x}$. So, we obtain that $\hat{0}_Q \prec \cdots \prec u \prec x \prec \cdots \prec \hat{1}_Q$ is another rising chain from $\hat{0}_Q$ to $\hat{1}_Q$, which is a contradiction.

2° There exist vertices u and v in P_2, $\rho(u) = \rho(v) = i_2 - 1$ such that ux and vy are edges of $T_{\hat{0}_Q}$. Now, we consider the unique rising chain $u = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_{n-i_2-1} = \hat{1}_Q$. If $x_1 = x$ from Remark 2.1 we have that $T_{\hat{0}_Q|x} = T_{u|x}$. As before, we obtain another rising chain form $\hat{0}_Q$ to $\hat{1}_Q$ in P_2, a contradiction.

If $x_1 = y$ and edge yx_2 is contained in $T_{\hat{0}_Q}$, we obtain that $T_{\hat{0}_Q|x_2} = T_{y|x_2}$. Again, we know that $\hat{1}_Q \in T_{y|x_2}$, and we can find another rising chain from $\hat{0}_Q$ to $\hat{1}_Q$, a contradiction.

If $x_1 = y$ and edge yx_2 is not contained in $T_{\hat{0}_Q}$, we have that xx_2 is an edge in $T_{\hat{0}_Q}$. Then, $u \prec x \prec x_2$ and $u \prec y \prec x_2$ are two different rising chains in $[u, x_2]$, yielding a contradiction.

□

Note that the result of Theorem 1.1 directly follows from the previous theorem.

Remark 2.2. Let $P_n^{k,i}$ denote the unique graded poset of rank n such that there are two elements of rank i and k elements of rank $0 < j < n$ for $j \neq i$. Every element of $P_n^{k,i}$ different from $\hat{0}$ covers all of the elements of one rank below. It is not complicated to check that $P_n^{k,i}$ has an R-labeling. Let Q consist of $P_n^{k,p}$ and $P_n^{k,q}$, $p \neq q$ where we identified the maximal elements and the minimal elements. For $1 < p, q < n$ we know that Q does not admit an R-labeling. Note that for $S \subseteq [n-1]$, $S \neq \emptyset$ the entry $h_S(Q)$ can be arbitrary large.

3. Complexes of rising chains

The *order-complex* $\Delta(P)$ of a graded poset P is the simplicial complex on vertex set P whose faces are the chains in P. This object is a passage between combinatorics and topology. The study of algebraic and topological properties of these complexes is a standard technique in enumerative combinatorics, see chapter 3 in [**7**].

DEFINITION 3.1. For a graded poset P and an R-labeling $\lambda : E(P) \to \mathbb{Z}$ of P let $\Delta_\lambda(P)$ denote a subcomplex of $\Delta(P)$ spanned by all rising chains in P. We say that $\Delta_\lambda(P)$ is a complex of rising chains.

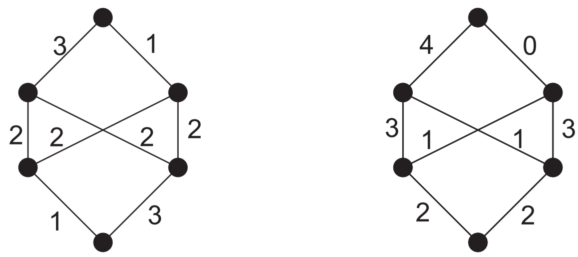

FIGURE 1. Different R-labelings of the same poset

In other words, a chain $C : x = x_1 < x_2 < \cdots < x_k = y$ is a face of $\Delta_\lambda(P)$ if and only if C is contained in the unique rising chain from x to y.

EXAMPLE 3.1. There is an example (see Figure 1) where different R-labelings of the same poset produce different complexes of rising chains.

However, by an easy examination we obtain that:

(1) These complexes of rising chains have the same f-vector.
(2) Both of these complexes are homotopy equivalent to a wedge of circles.
(3) These two complexes have the same homotopy type.

PROPOSITION 3.1. *For any two R-labeling λ and λ' complexes $\Delta_\lambda(P)$ and $\Delta_{\lambda'}(P)$ have the same f-vector.*

PROOF. Assume that $x = x_0 < x_{i_1} < x_{i_2} < \cdots < x_{i_k} = y$ is a k-face of $\Delta_\lambda(P)$. Let $C' : x = y_0 \prec y_1 \prec \cdots \prec y_t = y$ denote the unique rising chain in $[x, y]$ under labeling of P with λ'. Now, let y_j denote the element of C' such that $\rho(y_j) = \rho(x_{i_j})$. Obviously, $x = y_0 < y_1 < \cdots < y_k = y$ is a k-face of $\Delta_{\lambda'}(P)$. It is an easy check that the above correspondence is a bijection between k-faces of $\Delta_\lambda(P)$ and $\Delta_{\lambda'}(P)$. □

THEOREM 3.1. *For any graded poset P and its R-labeling $\lambda : E(P) \to \mathbb{Z}$ the complex $\Delta_\lambda(P)$ is homotopy equivalent to a wedge of $|E(P)| - |P| + 1$ circles.*

PROOF. For $x \in P$ let S_x denote the subcomplex of $\Delta_\lambda(P)$ spanned by all faces in which x is the minimal element. Note that S_x is contractible. Assume that $\hat{0}, x_1, \ldots, x_m, \hat{1}$ is a linear extension of P. We built up the complex $\Delta_\lambda(P)$ by adding subcomplexes $S_{\hat{0}}$, $S_{x_1} \ldots$ one by one. Let Δ_i denote $S_{\hat{0}} \cup S_{x_1} \cup \cdots S_{x_i}$. We will use the induction to show that Δ_i is contractible or a wedge of circles. The complex Δ_0 is contractible and we have that $\Delta_{i+1} = \Delta_i \cup S_{x_{i+1}}$. From Lemma 10.4 in [**1**] we obtain that

$$\Delta_{i+1} \simeq \Delta_i \cup cone(\Delta_i \cap S_{x_{i+1}}).$$

Remark 2.1 guaranteed that $\Delta_i \cap S_{x_{i+1}}$ is the union of disjoint contractible complexes. There is an obvious bijection between connected contractible component of $\Delta_i \cap S_{x_{i+1}}$ that do not contain x and the edges of the rising tree $T_{x_{i+1}}$ that do not appear in some T_{x_j} for $j \leqslant i$. If this intersection has β connected components and if we assume that Δ_i is homotopy equivalent to a wedge of α circles, then Δ_{i+1} is homotopy equivalent to a wedge of $\alpha + \beta - 1$ circles.

Note that there is $|P| - 1$ edges of $E(P)$ contained in the rising tree T_0 and they do not contribute the circles in $\Delta_\lambda(P)$. The edge uv that is not contained in $T_{\hat{0}}$ contributes one connected contractible components in $\Delta_{r-1} \cup S_{x_r}$ (here x_r is the first element in the linear extension of P such that T_{x_r} contains uv). Therefore, we obtain that $\Delta_\lambda(P)$ is homotopy equivalent to a wedge of $|E(P)| - |P| + 1$ circles.

□

Now, we will use the previous theorem to calculate homotopy type of rising complexes of some well-known posets.

Example 3.2. The rising complex of a butterfly poset T_n is homotopy equivalent to a wedge of $2n-3$ circles. For the Boolean algebra B_n there exists a natural R-labeling $\lambda : B_n \to [n]$ defined by $\lambda(A \prec B) = B \setminus A$. As we have that $|B_n| = 2^n$ and $|E(B_n)| = n2^{n-1}$, from Theorem 3.1 we obtain that $\Delta_\lambda(B_n)$ is a wedge of $(n-2)2^{n-1} + 1$ spheres.

If the posets P and Q have R-labelings, say λ' and λ'', it is well known that $P \times Q$ admits an R-labeling λ too. Assume that $\Delta_{\lambda'}(P)$ and $\Delta_{\lambda''}(Q)$ are homotopy equivalent to a wedge of α and β circles respectively. From the previous theorem we obtain that $\Delta_\lambda(P \times Q)$ is homotopy equivalent to a wedge of $|P|\beta + |Q|\alpha + (|P| - 1)(|Q| - 1)$ circles.

We could apply this on the product of two chains $\mathbf{m} = ([m], <)$ and $\mathbf{n} = (n, <)$. The rising complex $\Delta_\lambda(\mathbf{m} \times \mathbf{n})$ is homotopy equivalent to a wedge of $(m-1)(n-1)$ circles.

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